FLEXIBILITY AND RIGIDITY IN SYMPLECTIC AND CONTACT GEOMETRY

Idea: some phenomena in symplectic and contact geometry are rigid (similar in spirit to algebraic geometry), and some are flexible (similar to differential topology). Often hard to tell where the boundary is.

1. Symplectic and contact manifolds

Definition. *A* symplectic structure *on a rank $2k$ vector bundle $E \to M$ is a symplectic form (anti-symmetric and non-degenerate) ω_p on each fiber E_p, depending smoothly on p.*

The existence of a symplectic structure is equivalent to each of the following two properties:

- the existence of a reduction of the structure group of E from $GL(2k)$ to $Sp(2k, \mathbb{R})$;
- the existence of an (almost) complex structure on E: $J \in End(E)$ s.t. $J^2 = -Id$ (secretly, this equivalence is the fact that $Sp(2k, \mathbb{R})$ deformation retracts onto $U(k, \mathbb{C})$).

Note: Rank $2k$ symplectic vector bundles on M are classified by $\check{H}^1(M; Sp(2k, \mathbb{R}))$.

Example. $\check{H}^1(M; Sp(2, \mathbb{R})) \equiv \check{H}^1(M; U(1; \mathbb{C})) \equiv H^2(M; \mathbb{Z})$ *given by the first Chern class.* $H^2(M; \mathbb{Z}) \equiv [M, B\mathbb{Z}] \equiv [M, \mathbb{C}P^\infty] \equiv Vect^1_{\mathbb{C}}(M) = H^1(M; U(1))$

Definition. *An* almost symplectic structure *on a manifold M^{2n} is a symplectic structure on $TM \to M$ (ie, a non-degenerate $\omega \in \Omega^2(M)$ that is* not necessarily closed*).*

Note: Every symplectic manifold is almost symplectic.

Definition. *An* almost contact strucutre *on an oriented manifold V^{2n+1} is a hyperplane distribution $\xi^{2n} \subset TV$ and a symplectic structure on the vector bundle $\xi \to V$.*

Note: This is slightly incorrect. This is only (kind of) the definition of a *cooriented* almost contact structure. This is analogous to the difference between a contact structure ξ and a contact form α (s.t. $\ker \alpha = \xi$). Apart from this point, every contact manifold is almost contact.

Note: This is equivalent to being able to reduce structure group of TV from $GL^+(2n+1, \mathbb{R})$ to $U(n, \mathbb{C}) \times Id$.

Theorem (Gromov)**.** *Every almost symplectic structure on M^{2n} is homotopic through almost symplectic structures to a symplectic structure (in any chosen cohomology class in $H^2(M; \mathbb{R})$),* if M is open*. Every almost contact structure on V^{2n+1} is homotopic through almost contact structures to a contact structure,* if V is open*.*

Note: This is an example of an h-principle (for *homotopy equivalence*). It morally says that the existence of symplectic and contact structures on open manifolds is *flexible*.

Note: Idea of proof: dimensional reduction: construct symplectic/contact structure on neighborhood of a codimension 1 skeleton (by induction on dimension of cells). Use inverse of deformation retraction to extend to the whole manifold.

If the manifolds are closed, then the existence questions are much more subtle. Often there are no h-principles (think of as a *rigidity* phenomenon). The following result was obtained using Seiberg–Witten theory:

Theorem (Taubes). *The connected sum of an odd number of copies of $\mathbb{C}P^2$ does not admit a symplectic structure (even though it admits an almost symplectic structure and a cohomology class $\beta \in H^2(M)$ such that $\beta^2 \neq 0$.)*

Unrelated open question: S^6 admits an almost complex structure. Does it admit a complex structure?

Closed contact 3-manifolds.

Definition. *A contact 3-manifold (V^3, ξ) is* overtwisted *if there is an embedding $D \hookrightarrow V$ of a closed disk, which is tangent to ξ along ∂D.*

Note: Model: $\alpha = \cos r \mathrm{d}z + r \sin r \mathrm{d}\theta$ in cylindrical coordinates (on disk of radius π, ξ does a half twist along each ray; characteristic foliation has rays from origin in all directions and singular points at origin and along perimeter of disk).

In what follows, V is a *closed* 3-manifold.

Theorem (Eliashberg). *Every homotopy class of 2-plane fields in TV is homotopic to an overtwisted contact structure on M (unique up to homotopy).*

Note: This is an h-principle for overtwisted contact manifolds.

Corollary (Lutz–Martinet). *Every 3-manifold admits a contact structure.*

Note: There has been a lot of recent work on trying to extend this to higher dimensions. There are 5-dimensional analogues by Casals–Pancholi–Presas and by Etnyre.

Definition. *Contact 3-manifolds that are not overtwisted are called* tight.

Example. (S^3, ξ_{std})*; more generally,* symplectically fillable *contact 3-manifolds are tight.*

Note: There is no h-principle for tight contact manifolds. Their classification is harder (and more rigid) – Colin, Eliashberg, Etnyre, Giroux, Honda, ... (convex surfaces):

Theorem (Eliashberg). *Every tight contact structure on S^3 is homotopic to ξ_{std}.*

Theorem (Colin, Giroux, Honda). *For every manifold V^3, there are only finitely many homotopy classes of plane fields that contain tight contact structures.*

2. Lagrangian and Legendrian submanifolds

Let (M^{2n}, ω) be symplectic.

Definition. *An* almost (or formal) Lagrangian immersion *is a commutative diagram*

$$\begin{array}{ccc} TL & \xrightarrow{F} & TM \\ \downarrow & & \downarrow \\ L & \xrightarrow{f} & M \end{array}$$

such that

(1) *F covers f and is a linear injection on every fiber;*
(2) *for every $p \in L$, $F(T_pL) \subset T_{f(p)}M$ is a Lagrangian subspace;*
(3) *$[f^*\omega] = 0 \in H^2(L;\mathbb{R})$.*

Theorem (Gromov)**.** *Every almost Lagrangian immersion is homotopic through almost Lagrangian immersions to a Lagrangian immersion.*

Note:

- this a an h-principle for Lagrangian immersions (which are thus *flexible*);
- there is a similar h-principle for *Legendrian immersions*;
- there is also an h-principle for *subcritical isotropic embeddings*;
- there is also an h-principle for *ϵ-Lagrangian* and *ϵ-Legendrian embeddings*;
- there is NO (general) h-principle for *Lagrangian and Legendrian) embeddings*!!
  (Murphy: h-principle for *loose* Legendrian embeddings.)
- rigidity results for Lagrangian embeddings are often obtained using pseudo-holomorphic curves and Floer homology.

Theorem (Gromov)**.** *If $H_1(L;\mathbb{R}) = 0$ (e.g. if $L = S^n$, $n > 1$), then there is no Lagrangian embedding of L into $(\mathbb{R}^{2n}, \omega_{std})$.*

Note: The theorem is a *rigidity* result.

Note: There are Lagrangian immersions of S^n into $(\mathbb{R}^{2n}, \omega_{std})$ with only one double point (a self-intersection of two branches).

3. Other rigidity results

Definition. $B^{2n}(r) := \left\{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \mid \sum_{i=1}^n (x_i^2 + y_i^2) \leq r\right\}$.

One of the first symplectic rigidity results, using pseudo-holomorphic curves:

Theorem (Gromov's non-squeezing)**.** *If $B^{2n}(r)$ embedds symplectically into $\mathbb{R}^{2n-2} \times B^2(R)$, then $r \leq R$.*

A non-trivial consequence:

Theorem (Eliashberg–Gromov)**.** *Let (M^{2n}, ω) be symplectic. The space* $\mathrm{Symp}(M, \omega)$ *of symplectomorphisms is C^0-closed in the space* $\mathrm{Diff}(M)$ *of diffeomorphisms.*

Theorem (Arnold conjecture)**.** *Let* (M^{2n}, ω) *be symplectic. Let* L *be a Lagrangian submanifold such that* $[\omega]$ *vanishes on* $\pi_2(M, L)$*. Let* $\psi : M \to M$ *be a Hamiltonian diffeomorphism. Then*

- (Gromov)*.* $\psi(L) \cap L \neq \varnothing$*;*
- (Floer)*. if* $\psi(L) \pitchfork L$*, then* $|\psi(L) \cap L| \geq \sum_{i=0}^{n} \dim H_i(L; \mathbb{Z}_2)$*.*

4. Summing up

- flexibility: h-principles (differential topology);
- rigidity: pseudo-holomorphic curves, symplectic invariants (symplectic topology);
- usually, when there is flexibility, any relevant symplectic invariants are trivial (eg: overtwisted contact manifolds have trivial SFT invariants, surgery on loose Legendrians does not change symplectic homology, ...).

Symplectic and contact manifolds

- define almost symplectic manifold and almost contact manifold
- give simple example of classification of almost symplectic or contact manifolds
- state h-principle for open symplectic and contact manifolds. This is a case of *flexibility*.
- talk about existence problem for closed symplectic manifolds. Mention examples from Salamon's notes. Mention alsofollowing: $\mathbb{C}P^4$, S^6. Emphasize on *rigidity*.

Closed contact manifolds.

- overtwisted 3-manifolds
- Yasha's classification. Case of *flexibility*.
- Tight contact 3-manifolds: classification results using dividing surfaces. Case of *rigidity*
- Higher dimensions: There seems to be analogue of overtwisted flexibility

Lagrangian and Legendrian submanifolds

- state h-principle for subcritical isotropic embeddings
- state h-principle for Lagrangian immersions and ϵ-embeddings. Case of *flexibility*
- implies h-principle for Legendrian immersions
- mention Gromov's inexistence of exact tori in R^{2n}. Uses pseudo-holomorphic curves. Case of *rigidity*
- mention Arnold conjectures about Lagrangian intersections. Floer homology. Case of *rigidity*
- Murphy's loose Legendrians. Source of *flexibility*

Gromov's non-squeezing

Case of rigidity. Uses pseudo-holomorphic curves. Implies rigidity of $Symp$ in $Diff$.

Summing up

- flexibility: h-principles (differential topology)
- rigidity: pseudo-holomorphic curves, symplectic invariants (symplectic topology)
- usually, when there is flexibility, the relevant symplectic invariants are trivial (eg: overtwisted disks, loose Legendrians)

Flexibility: - open symplectic and contact manifolds (h-Principle) - overtwisted contact 3-manifolds (classified by homotopy classes of plane fields) - contact 5-manifolds - upcoming: overtwisted contact manifolds in any dimension!! - Lagrangian and Legendrian immersions - ϵ-Lagrangian embeddings
Rigidity - closed symplectic manifolds (look through Salamon's text) - tight contact 3-manifolds - Lagrangian embeddings (Floer homology) - holomorphic curves: non-squeezing and rigidity of symp in diffeo
Think: open problems (existence problem for closed symplectic manifolds), useful examples (S^6)
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