Nonsqueezing Theorem and Symplectic Capacities

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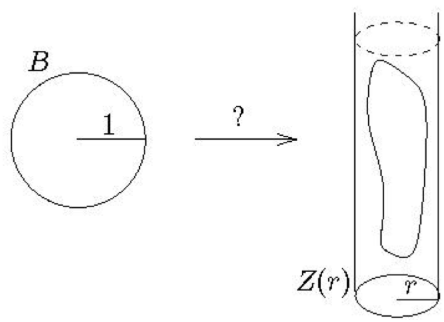
1 Nonsqueezing Theorem

We are interested in symplectic embedding so what is symplectic embedding? Recall that an **embedding** is an immersion which is a homeomorphism onto its image. A **symplectic embedding** is an embedding that preserves the symplectic structures, i.e. $\phi : (M, \omega) \to (M', \omega')$ is a symplectic embedding if and only if $\phi : M \to M'$ is an embedding and $\phi^*(\omega') = \omega$.



Theorem 1.1 (Nonsqueezing theorem)**.** *If there is a symplectic embedding*

$$B^{2n}(r) \hookrightarrow Z^{2n}(R),$$

then

$$r \leq R$$

where $Z^{2n} = B^2 \times \mathbb{R}^{2n-2}$.

Proof. (sketch) We are interested in proving this for dimension 4, which can be generalized in higher dimension. Suppose that there is a symplectic embedding

$$\phi : B^4(1) \to Z^4(r) = D^2(r) \times \mathbb{R}^2.$$

Then we need to show that $r \geq 1$. Equivalently, by slightly increasing r, we may suppose that the image of the ball lies inside the cylinder, and then we need to show that $r > 1$. We need the following ingredients:

Slicing cylinders: Let $(Z(r), \omega_0)$ be the cylinder in $(\mathbb{R}^4, \omega_0)$, and let J be any ω_0-tame almost complex structure (tame means $\omega_0(u, Ju) \neq 0$, $\forall u \neq 0$) on $Z(r)$ that equals the usual structure outside a compact subset of the interior of $Z(r)$. Then there is a J-holomorphic disc $f : (D^2, \partial D^2) \to (Z(r), \partial Z)$ of symplectic area πr^2 through every point of $Z(r)$.

Minimal surfaces We say $f : \sum \to (M, J)$ is J-holomorphic if it satisfies $\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y} = 0$. We think of J-holomorphic curves as minimal surfaces because the area of $f(\sum)$ with respect to the associated metric g_J equals its symplectic area. Since the symplectic area of a surface is invariant under deformations of the surface that fix its boundary, the metric area can only increase under such deformations, so J-holomorphic curves is so-called g_J-minimal surfaces.

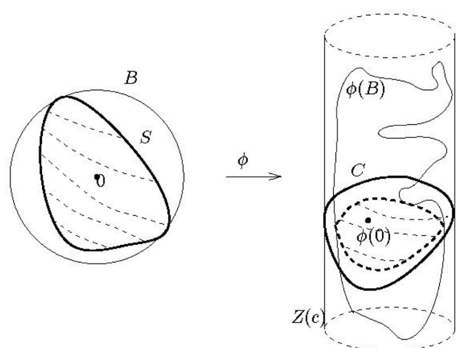
Let g_0 be the usual Euclidean metric on $\mathbb{R}^4$. Suppose that S is a g_0 minimal surface in the ball B of radius 1 that goes through the center of the ball and has the property that its boundary lies on the surfaces of the ball. Then *the g_0-area of S is $\geq \pi$*. Note that the g_0-minimal surface of least area that goes through the center of a unit ball is a flat disc of area π.

We shall prove $r > 1$ by using J-holomorphic slices as described above, but where J is chosen very carefully. In order to make the slicing arguments work, we need our J to equal the standard Euclidean structure J_0 near the boundary of $Z(r)$ and also outside a compact subset of $Z(r)$. But because the image $\phi(B)$ of the ball is strictly inside the cylinder, we can also make J equal to any specified ω_0-tame almost complex structure on $\phi(B)$. In particular, we may assume that J equals the pushforward of the standard structure $\phi_*(J_0)$ on $\phi(B)$, i.e., inside the embedded ball J is "standard".

Then, the statement above about slicing cylinders says that there is a J-holomorphic disc

$$f : D^2 \to Z(r),\ f(0) = \phi(0),$$

that goes through the image $\phi(0)$ of the center of the ball and also has boundary on the boundary of the cynlinder. Further, the symplectic area of the slice $C = f(D^2)$ is πr^2.



Now consider the intersection $C_B := C \cap \phi(B)$ of the slice with the embedded ball $\phi(B)$. By construction, this goes through the image $\phi(0)$ of the center 0 of the ball B. We now look at this situation from the vantage point of the original ball B. Consider the inverse image $S := \phi^{-1}(C_B)$. The rest of our argument involves understanding the properties of this curve S in B.

S is holomorphic with respect to the usual complex structure J_0 on $\mathbb{R}^4 = \mathbb{C}^2$. This follows from our choice of J: by construction, J equals the pushforward of J_0 on the image of $\phi(B)$ of the ball, and so, because C_B lies in $\phi(B)$ and is J-holomorphic, it pulls back to a curve S that is holomorphic with respect to the pullback structure J_0.

This means that S is a minimal surface with respect to the standard metric g_0 on $\mathbb{R}^4$ associated to ω_0 and J_0. So the area of S with respect to g_0 is at least π. Since S is holomorphic, this metric area is the same as its symplectic area $\omega_0(S)$. Since ϕ preserves ω_0, this equals the ω_0-area of the image curve $\phi(S) = C_B$.

By construction, C_B is just part of the J-holomorphic slice C through $\phi(0)$. It follows that C_B has strictly smaller ω_0-area than C. But our basic theorem about slices say that $\omega_0(C) = \pi r^2$. Putting this all together, we have the following:

$$\pi \leq g_0\text{-area } S = \omega_0\text{-area } S = \omega_0\text{-area } \phi(S) < \omega_0\text{-area } C = \pi r^2.$$

Thus, $\pi < \pi r^2$ so $r > 1$. □

Remark 1.2. There is no analogue of this result when the splitting $\mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2}$ is not symplectic. For example consider the Lagrangian splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Then for every $\delta < 0$, the map $(x, y) \to (\delta x, \delta^{-1} y)$ is a symplectic embedding of $B^{2n}(1)$ into $B^n(\delta) \times \mathbb{R}^n$.

Remark 1.3. Recently, this theorem has been extended Lalonde and McDuff to arbitrary symplectic manifold (M, ω) of dimension $2n$. If there is a symplectic embedding

$$B^{2n+2}(r) \hookrightarrow B^2(R) \times M$$

then $r \leq R$.

Example 1.4. Polterovich observed that, given any r, once can construct a symplectic embedding

$$B^{2n+2}(r) \hookrightarrow \mathbb{T}^2(1) \times \mathbb{R}^{2n}$$

in the following way. Find a linear Lagrangian subspace L of $\mathbb{R}^2 \times \mathbb{R}^{2n}$ whose ϵ-neighbourhood L_ϵ projects injectively onto $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^{2n} = \mathbb{T}^2(1) \times \mathbb{R}^{2n}$. Then consider a composite map

$$B^{2n+2}(r) \hookrightarrow L_\epsilon \hookrightarrow \mathbb{T}^2(1) \times \mathbb{R}^{2n}.$$

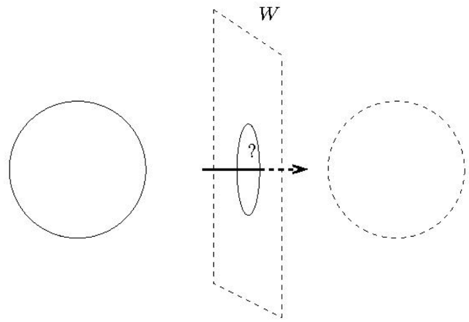
Symplectic Camel Principle: [Due to Gromov] It is impossible to move a ball of radius ≥ 1 symplectically from one side of the wall W to the other. The wall with a hole removed is given by

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, x_1 = 0, x_2^2 + x_3^2 + x_4^2 \geq 1\}.$$

But for volume preserving, you can by imaging it in 3-dimensional case and a sufficiently flexible ballon through any small hole while preserving its volume.

2 Symplectic Capacities

Gromov's nonsqueezing theorem gave rise to the following definition which is due to Ekeland and Hofer. A **symplectic capacity** is a functor c which assigns to every symplectic manifold (M, ω) a nonnegative (possible infinite) number $c(M, \omega)$ and satisfies the following condition.



- **(monotonicity)** If there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.

- **(conformality)** $c(M, \lambda\omega) = \lambda c(M, \omega)$.

- **nontriviality** $c(B^{2n}(1), \omega_0) > 0$ and $c(Z^{2n}(1), \omega_0) < \infty$.

From this definition, it isn't clear that capacities exist. However, the existence of a symplectic capacity with

$$c(B^{2n}(1), \omega_0) = c(Z^{2n}, \omega_0) = \pi$$

is equivalent to Gromov's nonsqueezing theorem. How so? If there is such a capacity then, by the monotonicity, the ball $B^{2n}(1)$ cannot be symplectically embedded in $Z^{2n}(R)$ unless $R \geq 1$. Conversely, define the **Gromov width** of a symplectic manifold (M, ω) by

$$w_G(M, \omega) = w_G(M) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ embeds symplectically in } M\}.$$

The associated invariant $\sup\{\pi r^2 \mid B^{2n}(r) \text{ embeds symplectically in } M\}$ is often called the symplectic radius. The Gromov width clearly satisfies the monotonicity and conformality axioms and satisfies the nontriviality axion by the Nonsqueezing theorem.

Example 2.1. $w_G(B^{2n}(1)) = \pi$.

Example 2.2. $w_G(S^2(\pi r^2) \times \mathbb{T}^{2n}(\pi R^2)) = \pi r^2$.

There are other important capacities such as the Hofer-Zehnder capacity c_{HZ}, the Ekeland-Hofer capacity c_{EH} and the embedded contact homology capacity c_{ECH}. But I think we run out of time....

References

[DM06] Dusa McDuff, Sept 2006, Univ. Minnesota: IT Distinguished Woman Speaker, Department colloquium: What is Symplectic Topology? and talk at the Yamabe conference Symplectomorphism groups an introduction.

[MS95] Dusa McDuff and Dietmar Salamon, Introduction to Symplectic Topology, Oxford Univ. Press, (1995), 2nd edition (1998).