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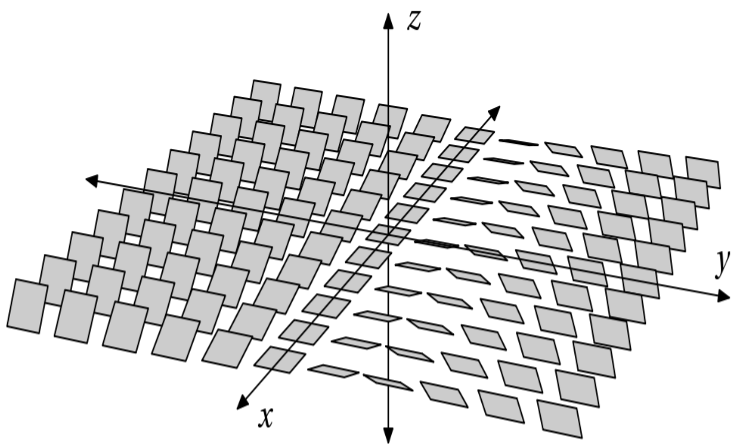
Legendrian knots and differential algebras

Legendrian Knots

A knot is a smooth embedding $L : S^1 \to \mathbb{R}^3$. Consider the the space $\mathbb{R}^3 = \{(p, q, u)\}$ equipped with the standard contact form $\alpha = du - pdq$. A smooth knot L in that space is called Legendrian if the restriction of α to L vanishes.



An oriented Legendrian knot L has three classical invariants

- Its isotopy class in the space of smooth embeddings. Two Legendrian knots are said to be Legendrian isotopic if they can be connected by a path in the space of Legendiran embeddings (there exists some diffeomorphism g of $\mathbb{R}^3$ such that $g * \alpha = \phi\alpha$, where $\phi > 0$.) Every smooth knot is isotopic to a Legendrian one. Two different Legendrian knots that are smooth isotopic may be not Legendrian isotopic. (It's still an open problem to classify the Legendrian knots up to Legendrian isotopy)
- The Bennequin number $\beta(L)$ of L is the linking number between L and $s(L)$ with respect to the orientation defined by $\alpha \wedge d\alpha = -dp \wedge dq \wedge du$, where s is a small shift along the u-direction.
- The Maslov number $m(L)$ is twice the rotation number of the projection of L to the $p - q$ plane. Think about the reason why m and β are invariants under Legendrian isotopy. When the orientation on L changes, the sign of $m(L)$ changes while $\beta(L)$ preserves.

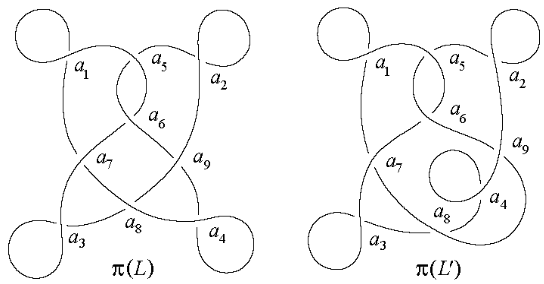
We define two projections. $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $(p, q, u) \mapsto (p, q)$ and $\sigma : \mathbb{R}^3 \to \mathbb{R}^2$, $(p, q, u) \mapsto (q, u)$. We say L is generic if all self-intersections of $\pi(L)$ are transverse double points. We draw the π-diagram, the projection $\pi(L)$, in a way that at every double point the branch with larger u stay the upper one. Then

$\beta(L) = \#\Big($  $\Big) - \#\Big($  $\Big)$,

(Legendrian isotopic of unknots in the contact structure)

Are there Legendrian knots with the same m and β invariants but not Legendrian isotopic to each other?

Theorem 1. *Legendrian knots L_1, L_2 whose $\pi-$diagram are given as follows have the same classical invariants, but are not Legendrian isotopic.*



Differential algebra

We construct a differential graded algebra (A, d) over $\mathbb{Z}_2$ where $d : A \to A$ satisfies $d(ab) = d(a)b + ad(b)$ for $a, b \in A$ and $d^2 = 0$. Denote by $T(a_1, \ldots, a_n)$ the free associative unitary algebra over $\mathbb{Z}_2$ with generators $a_1, \ldots, a_n$, $n \geq 0$. Then $T(a_1, \ldots, a_n) = \oplus_{l=0}^{\infty} A_l$ where A_l is spanned by l generators and A_0 is just $\mathbb{Z}_2$.

Let $Y \subset \mathbb{R}^2$ be the $\pi-$diagram of a Legendrian knot L, we define the following items respectively

- $W_k(Y)$ the collection of smooth orientation-preserving immersions $f : \Pi_k \to \mathbb{R}^2$ such that $f(\partial \Pi_k \subset Y)$ where Π_k is a k-gon with vertices numbered counterclockwise.
- $\tilde{W}_k(Y) = W_k(Y)/\{g \in \mathrm{Diff}_+ | g(x_i^k) = x_i^k\}$
- An immersion $f \in \tilde{W}_k(Y)$ is admissible if the first vertex x_0^k is positive and all the others are negative. The set of such f's is denoted by $W_k^+(Y)$.
- $d(a_j) = \displaystyle\sum_{k \geq 1} \sum_{f \in W_k^+(Y, a_j, \ldots, a_{j_{k-1}})} a_{j_1} \cdots a_{j_{k-1}}$ and we extend d to a linear map $A_Y \to A_Y$

Theorem 2. $d^2 = 0$.

Theorem 3. *Let $(A_{L_1}, d), (A_{L_2}, d)$ be the differential graded algebras associated with Legendrian isotopic generic Legendrian knots L_1, L_2. Then the homology rings of (A_{L_1}, d) and (A_{L_2}, d) are isomorphic.*

Example 4. Let's look at three examples.

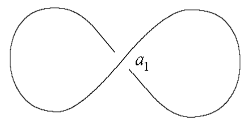


Figure 4:

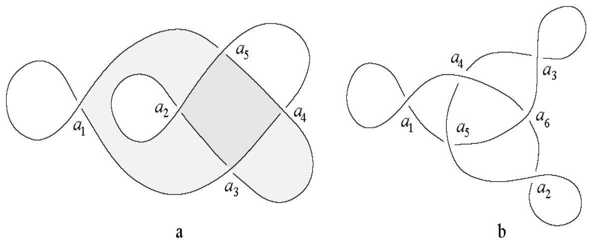


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Figure 5:

We have $m(L_1) = m(L_2) = 0$, $\beta(L_1) = \beta(L_2) = 1$.

- $d(a_1) = 1 + 1 = 0$
- $d(a_1) = a_4 + a_3a_4 + a_4a_5 + a_4a_5a_3a_4$
  $d(a_2) = 1 + a_3 + a_5 + a_3a_4a_5$
  $d(a_3) = d(a_4) = d(a_5) = 0$
- $d(a_1) = 1 + a_4a_6$
  $d(a_2) = 1 + a_5a_4$
  $d(a_3) = 1 + a_6a_5$
  $d(a_4) = d(a_5) = d(a_6) = 0$

We have $m(L_1) = m(L_2) = 0$, $\beta(L_1) = \beta(L_2) = 1$. For L_1, we have $d(a_1) = 1 + a_7 + a_7a_6a_5$, $d(a_2) = 1 + a_9 + a_5a_6a_9$, $d(a_3) = 1 + a_8a_7$, $d(a_4) = 1 + a_8a_9$, $d(a_5) = d(a_6) = d(a_7) = d(a_8) = d(a_9) = 0$; for L_2, we have $d(a_1) = 1 + a_7 + a_7a_6a_5 + a_5$, $d(a_2) = 1 + a_9 + a_5a_6a_9$, $d(a_3) = 1 + a_8a_7$, $d(a_4) = 1 + a_8a_9$, $d(a_5) = d(a_6) = d(a_7) = d(a_8) = d(a_9) = 0$.

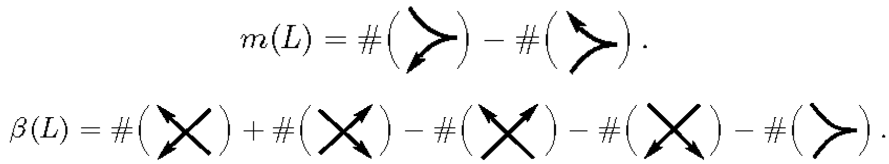
Let $A = T(a_1, \ldots, a_n)$, $\bar{A} = \oplus_{l=1}^{\infty} A_l$. The differential algebra (A, d) is called augmented if $d((\bar{A})) \subset \bar{A}$. Let $d = \sum_{l=0}^{\infty} d_l$ where $d_l(a_i) \subset A_l$ for every $i \in \{1, 2, \ldots, n\}$. Suppose (A, d) is augmented, then $d_0 = 0$ and $d(\bar{A}^m) \subset \bar{A}^m = \oplus_{i=m}^{\infty} A_i$ for every m. So d induces a linear operator $d_{(1)}$ on the quotient vector space $\bar{A}/\bar{A}^2$. $d_{(1)}^2 = 0$. $d_{(1)}$ coincide with the restriction of d_1 to A_1.

Consider the cohomology of $d_{(1)}$. Let $i(A, d) = \dim(\ker d_{(1)}) - \dim(\mathrm{im} d_{(1)}) = n - 2\dim(\mathrm{im} d_{(1)})$. Define the invariant $I(L) = \{i\}$ for $i(T(a_1, \ldots, a_n), d^g)$ where $d^g = gdg^{-1}$ over all $g \in \mathrm{Aut}(A)$ such that (A, d^g) is augmented.

Theorem 5. *If L is Legendrian isotopic to L', then $I(L) = I(L')$.*

Compute the differential algebra (A, d) for L_1, L_2. Let $g \in \mathrm{AUT}_0(A)$ be given by $g(a_i) = a_i + c_i$, $i \in \{1, \ldots, 9\}$. $d^g(a_i) = g(d(a_i))$. After computation, we have $I(L_1) = \{3\}$, $I(L_2) = \{1\}$.

Decompositions of Fronts For a Legendrian knot $L \subset \mathbb{R}^3$, its $\sigma-$projection, or front projection, $\sigma(L) \subset \mathbb{R}^2$ is a singular curve with nowhere vertical tangent vectors. (q-axis horizontal u-axis vertical) Redefine the Maslov and Bennequin number under this projection. L is σ-generic if all self-intersections of $\sigma(L)$ are transverse double points with different q coordinates. Since the overpassing branch (the one with the greater value of p) is always the one with the greater slope, so there is no need to show the type of a crossing of $\sigma(L)$. The Maslov and Bennequin numbers can be computed as follows,

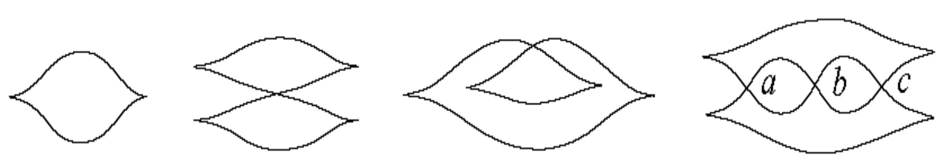


The four conditions of admissible decompositions

Suppose $\Sigma = \sigma(L)$ is a union of closed curves $X_1, \ldots, X_n$ that have finitely many intersections, then $\{X_1, \ldots, X_n\}$ is called a decomposition of Σ. A decomposition $\{X_1, \ldots, X_n\}$ is called admissible if it satisfies the four conditions as follows,

- Each curve X_i bounds a topological disk $X_i = \partial B_i$.
- For each $i \in \{1, \ldots, n\}$, $q \in \mathbb{R}$, the set $B_i(q) = \{u \in \mathbb{R} \mid (q, u) \in B_i\}$ is either a segment or a single point u such that (q, u) is a cusp of Σ, or is empty. (switching/non-switching crossing point)
- If $(q_0, u) \in X_i \cap X_j$ $(i \neq j)$ is switching then for each $q \neq q_0$ sufficiently close to q_0 the set $B_i(q) \cap B_j(q)$ either conincide with $B_i(q)$ or $B_j(q)$, or is empty.
- Every switching crossing is Maslov (if r takes the same value on both its branches)

Example 6. Let's look at the four examples



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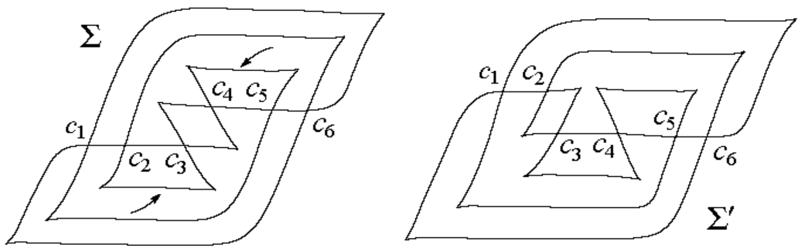
Denote by $Adm(\Sigma)$ the set of admissible decompositions of Σ. Given $D \in Adm(\Sigma)$, denote by $Sw(D)$ the set of its switching points. Define $\theta(D) = \#(D) - \#(Sw(D))$.

Theorem 7. *If $\sigma-$generic Legendrian knots $L_1, L_2 \subset \mathbb{R}^3$ are Legendrian isotopic then there exists a one-to-one mapping $g : Adm(\sigma(L_1)) \to Adm(\sigma(L_2))$ such that $\theta(g(D)) = \theta(D)$ for* *all* $D \in Adm(\sigma(L))$. *In particular, the number* $\#(Adm(\sigma(L)))$ *is an invariant of Legendrian isotopy.*



The fronts Σ_1, Σ_2 correspond to the Legendrian knots L_1, L_2. We show that $\#(Adm(\Sigma_1)) \neq \#(Adm(\Sigma_2))$.

References

[1] Yuri Chekanov, *New Invariants of Legendrian Knots.*

[2] Yuri Chekanov, *Differential algebras of Ledendrian Links.*

[3] John Etnyre, *Legendrian and Transversal Knots.*