Origami manifolds

by

Ana Rita Pissarra Pires

Licenciada em Matemática Aplicada e Computação,
Instituto Superior Técnico (2005)

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Author . . . .
Department of Mathematics
April 30, 2010

Certified by . . . .
Victor Guillemin
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Bjorn Poonen
Chairman, Department Committee on Graduate Students

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Abstract

An origami manifold is a manifold equipped with a closed 2-form which is symplectic everywhere except on a hypersurface, where it is a folded form whose kernel defines a circle fibration. In this thesis I explain how an origami manifold can be unfolded into a collection of symplectic pieces and conversely, how a collection of symplectic pieces can be folded (modulo compatibility conditions) into an origami manifold. Using equivariant versions of these operations, I show how classic symplectic results of convexity and classification of toric manifolds translate to the origami world. Several examples are presented, including a complete classification of toric origami surfaces. Furthermore, I extend the results above to the case of nonorientable origami manifolds.

Thesis Supervisor: Victor Guillemin
Title: Professor of Mathematics

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Chapter 1

Introduction

1.1 Symplectic and folded symplectic background

Symplectic geometry is the theory of even-dimensional manifolds M^{2n} equipped with a closed differential 2-form ω that is nondegenerate, i.e., $\omega^n \neq 0$. In particular, because ω^n is never vanishing, symplectic manifolds are necessarily orientable. The Darboux theorem states that all symplectic manifolds are locally isomorphic to even-dimensional Euclidean space with form $\sum_i dx_i \wedge dy_i$.

Symmetries of a manifold can be described by group actions. Let G be a compact connected Lie group that acts by symplectomorphisms on a symplectic manifold M, and let $\mathfrak{g}$ and $\mathfrak{g}^*$ be its Lie algebra and corresponding dual. We say that the action is a *hamiltonian action* if there exists a map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$d\mu^X = \omega(X^\#, \cdot),$$

where $\mu^X = \langle \mu, X \rangle$ is the component of μ along X and $X^\sharp$ is the vector field on M generated by X. Furthermore we require that μ be equivariant with respect to the given action of G on M and the coadjoint action on $\mathfrak{g}^*$. This map is called the *moment map*, and it encodes the symmetry of M that is captured by the G-action, as well as geometric information about the manifold and the action.

V. Guillemin and S. Sternberg [7], and independently M. Atiyah [1], showed that when M is compact connected and G is a torus group, the moment image $\mu(M)$ is convex, and furthermore, it is the convex hull of the image of its fixed point set. For *toric manifolds*, i.e., compact symplectic $2n$-manifolds with an effective hamiltonian action of an n-dimensional torus, we get a stronger result: T. Delzant [5] proved that toric manifolds are in one-to-one correspondence with their moment images, which are convex polytopes of *Delzant type*. Thus, a toric manifold M can be recovered from the combinatorial data of its moment polytope $\mu(M)$.

The next best case to nondegenerate symplectic is that of a folded symplectic structure, that is, a 2-form that is symplectic everywhere except on an exceptional hypersurface, where it vanishes in the transverse direction. These were studied in [4] by A. Cannas da Silva, V. Guillemin and C. Woodward.

Consider a closed 2-form ω on an oriented manifold M^{2n}: If ω^n intersects the zero section of $\Lambda^{2n}(T^*M)$ transversally, then $Z = \left\{p \in M : \omega_p^n = 0\right\}$ is a codimension one embedded submanifold of M. Furthermore, if the restriction $\omega|_Z$ is of maximal rank, i.e., if $i^*\omega$ has a one-dimensional kernel at each point, where $i : Z \hookrightarrow M$ is the inclusion, we say that ω is a *folded symplectic form* and Z is the *folding hypersurface* or *fold*. These structures occur more frequently than their symplectic counterpart: For example, all even dimensional spheres can be given a folded symplectic structure with folding hypersurface the equator, whereas only $\mathbb{S}^2$ admits a symplectic structure.

Similarly to the Darboux theorem for symplectic manifolds, a local normal form for a folded ω in a neighborhood of a point of Z is

$$x_1 dx_1 \wedge dy_1 + \sum_{i \geq 2} dx_i \wedge dy_i.$$

Furthermore, a semi-global normal form in a tubular neighborhood of (a compact) Z is

$$p^* i^* \omega + d(t^2 p^* \alpha), \tag{1.1}$$

where $p : Z \times (-\epsilon, \epsilon) \to Z$ is the projection $(z,t) \mapsto z$ and $\alpha \in \Omega^1(Z)$ is a one-form dual to a nonvanishing section of the kernel of $i^*\omega$. These results from [4] are proved in Section 2.1.

In [4] it is shown that if the distribution $\ker(i^*\omega)$ on Z integrates to a principal $\mathbb{S}^1$-fibration over a compact base B, then the manifold obtained by *unfolding* M can be endowed with a symplectic structure. This manifold consists of the disjoint union of the closures of the connected components of $M \setminus Z$, with points that are on the same leaf of the distribution on Z being identified (thus producing two copies of B). The symplectic form on the complement of a neighborhood of the copies of B coincides pointwise with ω on the complement of a neighborhood of Z in M.

1.2 Origami results

In this thesis I study the geometry of *origami manifolds*, the class of folded symplectic manifolds whose nullfoliation integrates to a principal $\mathbb{S}^1$-fibration over a compact base B. The fibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$ is called the *nullfibration* and the folded symplectic form is called *origami form*. Much of this work was developed in a joint project with Ana Cannas da Silva and Victor Guillemin [3].

Chapter 2 shows how to move between the symplectic and origami worlds. In section 2.2, the unfolding operation introduced in [4] is slightly modified, but with a similar approach: working in the coordinates of the semi-global normal form (1.1), one can perform symplectic cutting on a symplectified half neighborhood of Z. This yields naturally *symplectic cut pieces* in which B embeds as symplectic submanifold with projectivised normal bundle isomorphic to $Z \to B$ and on which the induced symplectic form coincides with ω up to B and Z respectively.

For the converse construction, in order to build an origami manifold M from two symplectic manifolds M_1 and M_2, one must have symplectomorphic embedded submanifolds $B_1 \hookrightarrow M_1$ and $B_2 \hookrightarrow M_2$ of codimension 2 and a symplectomorphism between neighborhoods of B_1 in M_1 and B_2 in M_2. This construction is detailed in Section 2.3 and requires that a suitable Z be created: Let Z be the radially projectivised normal bundle of B_1 in M_1. A *blow-up model* is a map from $Z \times (-\epsilon, \epsilon)$ to a tubular neighborhood of B_1 in M_1 that pulls back the symplectic form to an origami form, which creates an origami collar neighborhood of Z. By attaching the remainder of the manifolds M_1 and M_2 to this collar neighborhood we obtain an origami manifold.

In Sections 2.4 and 2.5 I show how this *radial blow-up* construction is truly the converse of the new unfolding construction, in the sense that performing one and then the other, or vice-versa, yields manifolds symplectomorphic, or origami-symplectomorphic (the origami analogue of that notion), to the original ones. Thus, an origami manifold is essentially determined by its symplectic cut pieces.

Chapter 3 takes up the theme of torus actions on origami manifolds. An origami manifold endowed with a hamiltonian G-action can be unfolded equivariantly, yielding symplectic G-manifolds to which the machinery of classic symplectic geometry applies. This is used to prove origami analogues of the Atiyah-Guillemin-Sternberg convexity theorem in Section 3.1 and of Delzant's classification theorem in Section 3.2.

The moment image of an origami manifold is the superposition of the (convex) moment polytopes of its symplectic cut pieces. More specifically, if (M, ω, G, μ) is a compact connected origami manifold with nullfibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$ and a hamiltonian action of an m-dimensional torus G with moment map $\mu : M \to \mathfrak{g}^*$, then:

(a) The image $\mu(M)$ of the moment map is the union of a finite number of convex polytopes Δ_i, $i = 1, \ldots, N$, each of which is the image of the moment map restricted to the closure of a connected component of $M \setminus Z$;

(b) Over each connected component Z' of Z, $\mu(Z')$ is a facet of each of the two polytopes corresponding to the neighboring components of $M \setminus Z$ if and only if the nullfibration on Z' is given by a circle subgroup of G. In that case, the two polytopes agree near that facet.

Two polytopes Δ_1 and Δ_2 in $\mathbb{R}^n$ agree near a facet $F_1 = F_2$ when there is a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$. When the nullfibration on all components of Z is given by a subgroup of G, two-dimensional origami polytopes resemble paper origami, with the folding hypersurface mapping to folded edges, hence the name. It is then possible to produce origami polytopes that are not convex, not simply connected, or not k-connected, for any choice of k.

The symplectic cut pieces of a toric origami manifold are toric manifolds, hence classified by Delzant polytopes. These symplectic pieces, together with information on how to assemble them, determine the origami manifold. Therefore, a collection of Delzant polytopes, together with information on which pairs of facets "fold together", should determine the original origami manifold. Indeed, toric origami manifolds are classified by their moment data, which can be summarized in the form of an *origami template*: a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a finite collection of oriented n-dimensional Delzant polytopes and $\mathcal{F}$ is a collection of pairs of facets of polytopes in $\mathcal{P}$ satisfying the following properties:

(a) for each pair $\{F_1, F_2\} \in \mathcal{F}$, the corresponding polytopes $\Delta_1 \ni F_1$ and $\Delta_2 \ni F_2$ agree near those facets and have opposite orientations;

(b) if a facet F occurs in a pair in $\mathcal{F}$, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;

(c) the topological space constructed from the disjoint union $\sqcup \Delta_i$, $\Delta_i \in \mathcal{P}$ by identifying facet pairs in $\mathcal{F}$ is connected.

An application of this theorem is a complete listing of origami surfaces: Up to difeomorphism, they are either spheres or tori, with the folding curve the disjoint union of a variable number of circles.

As a counterpoint to the rigid structure of the moment data of origami manifolds, in Section 3.4 I briefly discuss the (non-origami) folded symplectic case. A combinatorial classification as in the origami case is impossible in this situation, as practically any set can be realized as a moment image. Some advances in particular cases have been made by C. Lee in [8].

Chapter 4 deals with the fact that the definition of origami form still makes sense when M is not an orientable manifold. Section 4.1 points out that, thus far, a tubular neighborhood of a connected component of the folding hypersurface has always been an oriented open submanifold of M. Indeed, all folds on an oriented origami manifold are of this type, which we call *coorientable folds*. The other possibility is that of *non-coorientable folds*, any tubular neighborhood of which is non-orientable. All folds on an oriented origami manifold are coorientable, but non-orientable manifolds may have both types of folds, only non-coorientable folds, or even only coorientable folds. For example, the Klein bottle admits origami structures covering all these possibilities.

Section 4.2 shows how the cutting and radial blow-up construction can be extended to accommodate both orientable and non-orientable origami manifolds, partly by working with orientable double covers.

The moment image of a tubular neighborhood of a coorientable fold is an open neighborhood of a facet shared by two superimposing agreeing polytopes. For a noncoorientable fold, it is the neighborhood of a facet of a single polytope. In Section 4.3, the convexity and classification results of Sections 3.1 and 3.2 are modified and extended to include general origami manifolds, whether orientable or nonorientable. In particular, in the definition of origami template, $\mathcal{F}$ becomes a collection of facets and pairs of facets, and the orientability requirement is dropped. The listing of origami surfaces can now be completed with the nonorientable manifolds, diffeomorphic either to the projective plane or the Klein bottle, the folding curve being the disjoint union of a variable number of circles.

Chapter 2

Origami manifolds

2.1 Folded symplectic manifolds

A **symplectic form** on a smooth $2n$-dimensional manifold M^{2n} is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. This nondegeneracy condition means that the top power ω^n does not vanish, and hence is a volume form on M. In particular, ω^n induces an orientation on M.

Assume now that a smooth oriented manifold M^{2n} is endowed with a closed 2-form ω such that ω^n vanishes transversally on a set Z. This implies that $Z \stackrel{i}{\hookrightarrow} M$ is an embedded codimension one submanifold of M. This leads to the *folded symplectic* case:

Definition 2.1. A **folded symplectic form** on a smooth oriented $2n$-dimensional manifold M^{2n} is a closed 2-form $\omega \in \Omega^{2n}(M)$ whose top power ω^n vanishes transversally on a submanifold $Z \stackrel{i}{\hookrightarrow} M$ and is such that $i^*\omega$ is of maximal rank (equivalently, $(i^*\omega)^{n-1}$ does not vanish). The manifold (M, ω) is called a **folded symplectic manifold** and Z is called the **folding hypersurface** or **fold**.

We say that two folded symplectic manifolds M_1 and M_2 are **folded-symplectomorphic** if there exists an orientation preserving diffeomorphism $\rho : M_1 \to M_2$ such that $\rho^*\omega_2 = \omega_1$. Note that ρ will necessarily map the fold Z_1 onto the fold Z_2.

Example 2.2. The form $\omega_0 = x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n$ is a folded symplectic form on the Euclidean space $\mathbb{R}^{2n}$, with the fold being the hyperplane $\{x_1 = 0\}$. Furthermore, a folded analogue of the Darboux theorem (Corollary 2.10) states that for each point on the folding hypersurface of a folded symplectic manifold, there is a neighborhood that is folded-symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. $\diamondsuit$

Since M is oriented, the set $M \setminus Z$ splits as M^+, where $\omega^n > 0$, and M^-, where $\omega < 0$. This induces a coorientation and hence an orientation on Z. As $i^*\omega$ is a 2-form of maximal rank on the odd-dimensional manifold Z, it has a one-dimensional kernel at each point. This gives rise to the line field $V \subset TZ$, which we call the **nullfoliation** on Z. Let E be the rank 2 vector bundle over Z whose fiber at each point is the kernel of ω; then $V = E \cap TZ$. The $(2n-2)$-form ω^{n-1} gives an orientation of $(i^*TM)/E$, which induces an orientation on E. Finally, the orientations on the vector bundles E and TZ induce an orientation on the nullfoliation V.

Let v be an oriented non-vanishing section of V and $\alpha \in \Omega^1(M)$ a one-form such that $\alpha(v) = 1$.

Proposition 2.3. *[4] Assume that Z is compact. Then there exists a tubular neighborhood $\mathcal{U}$ of Z in M and an orientation preserving diffeomorphism $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ mapping $Z \times \{0\}$ onto Z such that*

$$\varphi^*\omega = p^*i^*\omega + d(t^2 p^*\alpha), \tag{2.4}$$

where $p : Z \times (-\varepsilon, \varepsilon) \to Z$ is the projection onto the first factor and t is the real coordinate on $(-\varepsilon, \varepsilon)$.

Proof. We follow the proof given in [4].

Let w be a vector field on M such that for all $z \in Z$, the ordered pair (w_z, v_z) is an oriented basis of E_z. Let $\mathcal{U}$ be a tubular neighborhood of Z in M and $\rho : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ the map that takes $Z \times \{0\}$ onto Z and the lines $\{z\} \times (-\varepsilon, \varepsilon)$ onto the integral curves of w. We use ρ to identify $\mathcal{U}$ with $Z \times (-\varepsilon, \varepsilon)$ and w with $\frac{\partial}{\partial t}$. Moreover ρ allows us to extend the vector field v to all of $\mathcal{U}$ via the inclusion of $T_z Z$ into $T_{(z,t)}\mathcal{U}$.

We will apply the "Moser trick" to the forms $\omega_0 := p^*i^*\omega + d(t^2 p^*\alpha)$ and $\omega_1 := \omega$ by setting $\omega_s := (1-s)\omega_0 + s\omega_1$ and finding a vector field v_s on M such that

$$\mathcal{L}_{v_s}\omega_s + \frac{d\omega_s}{ds} = 0. \tag{2.5}$$

We must first prove the following:

Lemma 2.6. *The linear combination $\omega_s := (1-s)\omega_0 + s\omega_1$ is a folded symplectic form, with fold Z.*

We begin by proving the following criteria for foldedness:

Lemma 2.7. *Let μ be a closed 2-form on $\mathcal{U}$. Then $p^*i^*\omega + t\mu$ is a folded symplectic form (on a possibly smaller tubular neighborhood of Z) if and only if $\mu(w, v)$ is nonvanishing on Z.*

Proof of Lemma 2.7. We must check that the top power of $p^*i^*\omega + t\mu$ vanishes transversally on Z and that $i^*(p^*i^*\omega + t\mu)$ is of maximal rank.

In order for $(p^*i^*\omega + t\mu)^n = (n-1)t(p^*i^*\omega)^{n-1} \wedge \mu + O(t^2)$ to vanish transversally at $t = 0$ we must have $(p^*i^*\omega)^{n-1} \wedge \mu$ is nonvanishing on Z. Since the kernel of $(p^*i^*\omega)_z$ is spanned by w_z and v_z, this happens if and only if $\mu(w, v)$ is nonvanishing on Z. The rank maximality is satisfied because $i^*(p^*i^*\omega + t\mu) = i^*\omega$. □

Proof of Lemma 2.6. Let us see that both ω_0 and ω_1 are of the form above: We have $\omega_0 = p^*i^*\omega + t\mu_0$, where $\mu_0 = 2dtp^*\alpha + td(p^*\alpha)$ with $\mu_0(w, v) = 2$ on Z. As for ω_1, note that $\iota_u(\omega - p^*i^*\omega) = 0$ for any vector field u in TZ and furthermore $\iota_w(\omega - p^*i^*\omega) = 0$, since $\iota_w\omega = 0$ and $\iota_w(p^*i^*\omega) = 0$. Thus we have $\omega - p^*i^*\omega = 0$ on Z and consequently $\omega - p^*i^*\omega = t\mu_1$ for some $\mu_1 \in \Omega^2(\mathcal{U})$. Since ω is folded, we get for free that $\mu_1(w, v)$ is nonvanishing on Z, and the choices made above furthermore guarantee that it is positive.

We can now write $\omega_s = p^*i^*w + t\mu_s$, where $\mu_s := (1-s)\mu_0 + s\mu_1$. Since $\mu_s(w, v)$ is positive on Z, the form ω_s is folded symplectic. □

We now return to our purpose of finding a suitable vector field v_s: Note that equation 2.5 simplifies to

$$d\iota_{v_s}\omega_s = \omega_0 - \omega_1. \tag{2.8}$$

Since $\omega_0 - \omega_1$ is closed and vanishes on Z, which is a deformation retract of $\mathcal{U}$, there exists a 1-form $\eta \in \Omega^1(\mathcal{U})$ that vanishes to second order on Z and such that $d\eta = \omega_0 - \omega_1$. Then 2.8 is satisfied if

$$\iota_{v_s}\omega_s = \eta.$$

Because ω_s is a folded symplectic form, there exists a unique such vector field, and it vanishes to first order on Z. Integrating v_s we get an isotopy φ_s that satisfies $\frac{d\varphi_s}{ds} \circ \varphi_s^{-1} = v_s$ with $\varphi_0 = \mathrm{id}$, and thus $\varphi_s^*\omega_s = \omega_0$ and φ_s maps Z to Z. □

For Z not compact, replace $\varepsilon \in \mathbb{R}^+$ by an appropriate continuous function $\varepsilon : Z \to \mathbb{R}^+$ in the statement and proof of Proposition 2.3.

Remark 2.9. Let G be a compact connected Lie group that acts on the manifold M and preserves ω. Averaging the oriented nonvanishing section v of the nullfoliation makes it G-invariant, thus making α invariant as well. The open set $\mathcal{U}$ can be chosen G-invariant and the Moser map φ equivariant with respect to the G-action on $Z \times (-\varepsilon, \varepsilon)$, which acts only on Z.

The following Corollary locally classifies folded symplectic manifolds up to folded symplectomorphism, as the Darboux theorem does in the classic symplectic case:

Corollary 2.10. *(Darboux Theorem for folded symplectic manifolds) Let (M, ω) be a $2n$-dimensional folded symplectic manifold and let z be a point on the folding hypersurface Z. Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at z such that on $\mathcal{U}$ the set Z is given by $x_1 = 0$ and*

$$\omega = x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n.$$

Proof. By the classical Darboux theorem, $i^*\omega = dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n$. Now apply Proposition 2.3 with $x_1 = t$ and $\alpha = \frac{1}{2}\, dy_1$. $\square$

A folded symplectic form on a manifold M induces a line field V on the folding hypersurface Z; we will focus on the case in which this foliation is a circle fibration.

Definition 2.11. An **origami manifold** is a folded symplectic manifold (M, ω) whose nullfoliation on Z integrates to a principal $\mathbb{S}^1$-fibration, called the **nullfibration**, over a compact **base** B:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, hook] & Z \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

The form ω is called an **origami form**.

We assume that the principal $\mathbb{S}^1$-action matches the induced orientation of the nullfoliation V. Note that any folded symplectic manifold that is folded-symplectomorphic to an origami manifold must be an origami manifold as well; we say they are **origami-symplectomorphic**. If v is an oriented nonvanishing section of the nullfoliation V, we can without loss of generality scale it uniformly over each $\mathbb{S}^1$-orbit so that its integral curves all have period 2π.

As in symplectic reduction, the base B of the nullfibration is naturally symplectic. The form $i^*\omega$ descends to B, because it is invariant and horizontal. Let ω_B denote the natural reduced symplectic form on B satisfying

$$i^*\omega = \pi^*\omega_B \ .$$

The form ω_B is closed and nondegenerate.

Example 2.12. Consider the unit sphere $\mathbb{S}^{2n}$ in euclidean space $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, h)$ and let ω_0 be the restriction to $\mathbb{S}^{2n}$ of the form $dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n = r_1 dr_1 \wedge d\theta_1 + \ldots + r_n dr_n \wedge d\theta_n$. This form is folded symplectic on $\mathbb{S}^{2n}$ with folding hypersurface the equator $(2n-1)$-sphere given by intersection with the hyperplane $\{h = 0\}$. Furthermore, since

$$\iota_{\frac{\partial}{\partial\theta_1}+\ldots+\frac{\partial}{\partial\theta_n}}\omega_0 = -r_1 dr_1 - \ldots - r_n dr_n = h dh$$

vanishes on Z, the nullfoliation is the Hopf fibration: $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$, and $(\mathbb{S}^{2n}, \omega_0)$ is an origami manifold. $\diamondsuit$

2.2 From origami to symplectic: Cutting

Take an origami manifold, cut it along the folding hypersurface and consider the closures of the pieces obtained. In Example 2.12 this yields two closed hemispheres, each containing a copy of the fold. Now, collapse the $\mathbb{S}^1$-fibers on each of the copies of Z to form two copies of the base B. The pieces thus obtained are smooth manifolds, and furthermore admit a natural symplectic structure. This operation is called *symplectic cutting.*

Let (M, ω) be a symplectic manifold with a codimension two symplectic submanifold $B \overset{i}{\hookrightarrow} M$. The **radially projectivized normal bundle** of B in M is the circle bundle

$$\mathcal{N} := \mathbb{P}^+ \left(i^*TM/TB\right) = \{x \in (i^*TM)/TB, x \neq 0\}/\sim$$

where $\lambda x \sim x$ for $\lambda \in \mathbb{R}^+$.

Proposition 2.13. *Let (M^{2n}, ω) be an origami manifold with nullfibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$. Then the unions $M^+ \sqcup B$ and $M^- \sqcup B$ both admit natural symplectic structures (M_0^+, ω_0^+)*

and (M_0^-, ω_0^-), where ω_0^+ and ω_0^- coincide with ω when restricted to M^+ and M^- respectively. Furthermore, (B, ω_B) embeds as a symplectic submanifold with radially projectivized normal bundle isomorphic to $Z \to B$.

The orientation induced from the original orientation on M matches the symplectic orientation on M_0^+ and is opposite to the symplectic orientation on M_0^-.

A result very similar to this Proposition is proved in [4].

Proof. Let $\mathcal{U}$ be a tubular neighborhood of Z and $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ a Moser model diffeomorphism as in Proposition 2.3, with $\varphi^*\omega = p^*i^*\omega + d(t^2 p^*\alpha)$.

Consider $\mathcal{U}^+ = M^+ \cap \mathcal{U} = \varphi(Z \times (0, \varepsilon))$ and the diffeomorphism $\psi : Z \times (0, \varepsilon^2) \to \mathcal{U}^+$ given by $\psi(x, s) = \varphi(x, \sqrt{s})$. Then $\nu := \psi^*\omega = p^*i^*\omega + d(sp^*\alpha)$ is a symplectic form on $Z \times (0, \varepsilon^2)$ and it extends symplectically by the same formula to $Z \times (-\varepsilon^2, \varepsilon^2)$.

The nullfibration on Z induces an $\mathbb{S}^1$ action on $(Z \times (-\varepsilon^2, \varepsilon^2), \nu)$ given by $e^{i\theta} \cdot (x, s) = (e^{i\theta} \cdot x, s)$, which is hamiltonian with moment map $(x, s) \mapsto s$. We will perform symplectic cutting at the 0-level set: Consider the product space $(Z \times (-\varepsilon^2, \varepsilon^2), \nu) \times (\mathbb{C}, -idz \wedge d\bar{z})$ with the $\mathbb{S}^1$ action $e^{i\theta} \cdot (x, s, z) = (e^{i\theta} \cdot x, s, e^{-i\theta} z)$. This is a hamiltonian action with moment map $\mu(x, s, z) = s - |z|^2$. Since 0 is a regular value of μ, the set $\mu^{-1}(0)$ is a codimension one submanifold that decomposes as

$$\mu^{-1}(0) = Z \times \{0\} \times \{0\} \sqcup \{(x, s, z) : s > 0, |z|^2 = s\}.$$

Since $\mathbb{S}^1$ acts freely on each of these subsets of $\mu^{-1}(0)$, the quotient space $\mu^{-1}(0)/\mathbb{S}^1$ is a symplectic manifold and the point-orbit map is a principal $\mathbb{S}^1$ bundle. We can write $\mu^{-1}(0)/\mathbb{S}^1 \simeq B \sqcup \mathcal{U}^+$, where B embeds as a codimension two submanifold via

$$\begin{aligned} j : B &\longrightarrow \mu^{-1}(0)/\mathbb{S}^1 \\ \pi(x) &\longmapsto [x, 0, 0] \text{ for } x \in Z \end{aligned}$$

and $\mathcal{U}^+$ embeds as an open dense submanifold via

$$\begin{aligned} j^+ : \mathcal{U}^+ &\longrightarrow \mu^{-1}(0)/\mathbb{S}^1 \\ \psi(x, s) &\longmapsto [x, s, \sqrt{s}] \ . \end{aligned}$$

The symplectic form Ω_{red} on $\mu^{-1}(0)/\mathbb{S}^1$ obtained by reduction makes the above embeddings symplectic.

The normal bundle to $j(B)$ in $\mu^{-1}(0)/\mathbb{S}^1$ is the quotient over $\mathbb{S}^1$-orbits (upstairs and downstairs) of the normal bundle to $Z \times \{0\} \times \{0\}$ in $\mu^{-1}(0)$. This in turn is the product bundle $Z \times \{0\} \times \{0\} \times \mathbb{C}$, where the $\mathbb{S}^1$-action is given by $e^{i\theta} \cdot (x, 0, 0, z) = (e^{i\theta} \cdot x, 0, 0, e^{-i\theta} z)$. Performing $\mathbb{R}^+$-projectivization and taking the quotient by the $\mathbb{S}^1$-action we get the bundle $Z \to B$ with natural isomorphism:

$$\begin{tikzcd} (Z \times \{0\} \times \{0\} \times \mathbb{C}^*)/\mathbb{S}^1 \arrow[r, "\simeq"] \arrow[d] & Z \arrow[d] \\ (Z \times \{0\} \times \{0\})/\mathbb{S}^1 \arrow[r, "\simeq"] & B \end{tikzcd} \qquad \begin{tikzcd} {[x, 0, 0, re^{i\theta}]} \arrow[r, mapsto] \arrow[d, mapsto] & e^{i\theta} \cdot x \arrow[d, mapsto] \\ {[x, 0, 0]} \arrow[r, mapsto] & \pi(x)\,. \end{tikzcd}$$

Gluing the rest of M^+ along $\mathcal{U}^+$ produces a $2n$-dimensional symplectic manifold (M_0^+, ω_0^+) with a natural symplectomorphism $\overline{j^+} : M^+ \to M_0^+ \smallsetminus j(B)$ extending j^+.

For the other side, we use the map $\psi_- : Z \times (0, \varepsilon^2) \to \mathcal{U}^- := M^- \cap \mathcal{U}$, $(x, s) \mapsto \varphi(x, -\sqrt{s})$; this map is orientation reversing and we have $(\psi_-)^*\omega = \nu$. The base B embeds as a symplectic submanifold of $\mu^{-1}(0)/\mathbb{S}^1$ by the previous formula and $\mathcal{U}^-$ via the orientation-reversing symplectomorphism

$$\begin{aligned} j^- : \mathcal{U}^- &\longrightarrow \mu^{-1}(0)/\mathbb{S}^1 \\ \psi_-(x, s) &\longmapsto [x, s, -\sqrt{s}]\ . \end{aligned}$$

As in the previous case, we produce (M_0^-, ω_0^-) by gluing the rest of M^- along $\mathcal{U}^-$ and get a natural symplectomorphism $\overline{j^-} : M^- \to M_0^- \smallsetminus j(B)$ by extending j^- .

Different initial choices of a Moser model φ for a tubular neighborhood $\mathcal{U}$ of Z yield symplectomorphic manifolds. $\square$

Remark 2.14. Note that the cutting procedure in the proof above produces a symplectomorphism between the tubular neighborhoods $\mu^{-1}(0)/\mathbb{S}^1$ of the embeddings of B in M_0^+ and M_0^- that from $\mathcal{U}^+$ to $\mathcal{U}^-$ is $\varphi(x, t) \mapsto \varphi(x, -t)$ and on B restricts to the identity map:

$$\begin{aligned} \gamma : \mu^{-1}(0)/\mathbb{S}^1 &\longrightarrow \mu^{-1}(0)/\mathbb{S}^1 \\ [x, s, \sqrt{s}] &\longmapsto [x, s, -\sqrt{s}]\ . \end{aligned}$$

Definition 2.15. The symplectic manifolds (M_0^+, ω_0^+) and (M_0^-, ω_0^-) obtained by cutting are called the **symplectic cut pieces** of the origami manifold (M, ω) and the embedded copies of B are called **centers**.

The symplectic cut pieces of a compact origami manifold are compact as well.

Cutting is a local operation, so it may be performed on a connected component of Z rather than on the whole fold. In particular, M can be cut by stages, one connected component of the fold at a time.

Example 2.16. Cutting the origami manifold $(\mathbb{S}^{2n}, \omega_0)$ from Example 2.12 produces $\mathbb{C}P^n$ and $\overline{\mathbb{C}P^n}$, each equipped with the same multiple of the Fubini-Study form with total volume equal to that of an original hemisphere, $n!(2\pi)^n$, and each with an embedded copy of $\mathbb{C}P^{n-1}$ as the center. $\diamondsuit$

2.3 From symplectic to origami: Blowing-up

Symplectic cutting gives us a way to split an origami manifold into symplectic components. Conversely, we would like to be able to take symplectic manifolds and use them to create an origami manifold – ideally so that once we perform symplectic cutting on that origami manifolds, we will get the original symplectic manifolds back. The obvious necessary condition is that the symplectic manifolds, M_1 and M_2, we start with must contain symplectomorphic codimension two symplectic submanifolds, B_1 and B_2, and symplectomorphic neighborhoods of those, $\mathcal{U}_1$ and $\mathcal{U}_2$. This will in fact be sufficient to create the origami manifold. The main question then is how to create a suitable $\mathbb{S}^1$-bundle over $B_1 \simeq B_2$ from the local data, so that acts as the folding hypersurface for the new origami manifold.

We choose an $\mathbb{S}^1$-action on the radially projectivized normal bundle $\mathcal{N}$ over B and let $\varepsilon > 0$.

Definition 2.17. A **blow-up model** for a neighborhood $\mathcal{U}$ of B in (M, ω) is a map

$$\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}$$

that factors as

$$\begin{tikzcd}
\mathcal{N} \times (-\varepsilon, \varepsilon) \arrow[r, "\beta"] \arrow[dr, "\beta_0"'] & \mathcal{U} \\
& \left(\mathcal{N} \times \mathbb{C}\right)/\mathbb{S}^1 \arrow[u, "\simeq"']
\end{tikzcd}$$

where $\beta_0(x,t) = [x,t]$, the $\mathbb{S}^1$-action on $\mathcal{N} \times \mathbb{C}$ is $e^{i\theta} \cdot (x,t) = (e^{i\theta} \cdot x, te^{-i\theta})$, and the vertical arrow is a bundle diffeomorphism from the image of β_0 to $\mathcal{U}$ covering the identity $B \to B$.

In practice, a blow-up model may be obtained by choosing a riemannian metric to identify $\mathcal{N}$ with the unit bundle inside the geometric normal bundle $TB^{\perp}$ and then using the exponential map: $\beta(x,t) = \exp_p(tx)$ where $p = \pi(x)$.

Lemma 2.18. *If $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ is a blow-up model for the neighborhood $\mathcal{U}$ of B in (M, ω), then the pull-back form $\beta^*\omega$ is an origami form whose nullfoliation is the circle fibration $\pi : \mathcal{N} \times \{0\} \to B$.*

Proof. The restriction $\beta|_{\mathcal{N}\times(0,\varepsilon)} : \mathcal{N} \times (0, \varepsilon) \to \mathcal{U} \smallsetminus B$ is an orientation-preserving diffeomorphism and $\beta(-x, -t) = \beta(x,t)$, so the form $\beta^*\omega$ is symplectic away from $\mathcal{N} \times \{0\}$. Since $\beta|_{\mathcal{N}\times\{0\}}$ is the bundle projection $\mathcal{N} \to B$, on $\mathcal{N} \times \{0\}$ the kernel of $\beta^*\omega$ has dimension 2 and is fibrating.

Moreover, for the vector fields ν generating the vertical bundle of $\mathcal{N} \to B$ and $\frac{\partial}{\partial t}$ tangent to $(-\varepsilon, \varepsilon)$ we have that $D\beta(\nu)$ intersects zero transversally and $D\beta(\frac{\partial}{\partial t})$ is never zero. Therefore the top power of $\beta^*\omega$ intersects zero transversally. $\square$

All blow-up models share the same germ up to diffeomorphism. More precisely, if $\beta_1 : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_1$ and $\beta_2 : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_2$ are two blow-up models for neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of B in (M, ω), then there are possibly smaller tubular neighborhoods of B, $\mathcal{V}_i \subseteq \mathcal{U}_i$ and a diffeomorphism $\gamma : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\beta_2 = \gamma \circ \beta_1$.

Proposition 2.19. *Let (M_1^{2n}, ω_1) and (M_2^{2n}, ω_2) be symplectic manifolds, and $B_i \subset M_i$ compact codimension two symplectic submanifolds. Assume that there exist tubular neighborhoods $\mathcal{U}_i$ of B_i in M_i for $i = 1, 2$ with a symplectomorphism $\gamma : \mathcal{U}_1 \to \mathcal{U}_2$ that takes $B_1 \to B_2$.*

Then there is a natural origami manifold $(\widetilde{M}, \widetilde{\omega})$ with folding hypersurface diffeomorphic to the radially projectivized normal bundle $\mathcal{N}_1$ to B_1 and nullfibration isomorphic to $\mathcal{N}_1 \to B_1$, with $\widetilde{M}^+$, $\widetilde{M}^-$ symplectomorphic to $M_1 \smallsetminus B_1$, $M_2 \smallsetminus B_2$, respectively.

Proof. Choose $\beta : \mathcal{N}_1 \times (-\varepsilon, \varepsilon) \to \mathcal{U}_1$ a blow-up model for the neighborhood $\mathcal{U}_1$. Then $\beta^*(\omega_1)$ is a folded symplectic form on $\mathcal{N}_1 \times (-\varepsilon, \varepsilon)$ with folding hypersurface $Z_1 := \mathcal{N}_1 \times \{0\}$ and nullfoliation integrating to the circle fibration $S^1 \hookrightarrow \mathcal{N}_1 \xrightarrow{\pi_1} B_1$. We define

$$\widetilde{M} = \left(M_1 \setminus B_1 \cup \overline{M_2 \setminus B_2} \cup \mathcal{N}_1 \times (-\varepsilon, \varepsilon)\right) \,/\, \sim .$$

Here $\overline{M_2 \setminus B_2}$ is simply $M_2 \setminus B_2$ with reversed orientation and we quotient by identifying via the symplectomorphisms

$$\mathcal{N}_1 \times (0, \varepsilon) \overset{\beta}{\simeq} \mathcal{U}_1 \setminus B_1 \text{ and } \mathcal{N}_1 \times (-\varepsilon, 0) \overset{\beta}{\simeq} \overline{\mathcal{U}_1 \setminus B_1} \overset{\gamma}{\simeq} \overline{\mathcal{U}_2 \setminus B_2} \ .$$

The closed 2-form defined by

$$\widetilde{\omega} := \begin{cases} \omega_1 & \text{on } M_1 \setminus B_1 \\ \omega_2 & \text{on } M_2 \setminus B_2 \\ \beta^*\omega_1 & \text{on } \mathcal{N}_1 \times (-\varepsilon, \varepsilon) \end{cases}$$

endows $\widetilde{M}$ with a structure of origami manifold with folding hypersurface Z_1, where $\widetilde{M}^+ \simeq M_1 \setminus B_1$ and $\widetilde{M}^- \simeq M_2 \setminus B_2$. □

Definition 2.20. The origami manifold $(\widetilde{M}, \widetilde{\omega})$ just constructed is called the **radial blow-up** of (M_1, ω_1) and (M_2, ω_2) through (γ, B_1).

When M_1 and M_2 are compact, the radial blow-up $\widetilde{M}$ is also compact.

Note that $(\widetilde{M}, \widetilde{\omega})$ and the radial blow-up of (M_1, ω_1) and (M_2, ω_2) through (γ^{-1}, B_2) would be origami-symplectomorphic except that they have opposite orientations.

Radial blow-up is a local operation, so it may be performed on origami manifolds (or one symplectic and one origami) at symplectomorphic symplectic submanifolds away from the already existing fold(s). For example, if we start with two origami surfaces and radially blow them up at one point (away from the folds), the resulting manifold is topologically the connected sum $M_1 \# \overline{M_2}$ with all the previous folding curves plus a new closed curve.

2.4 There and back: Cutting a radial blow-up

Radial blow-up allows us to assemble symplectic manifolds into an origami manifold. Let us see that when we cut the resulting origami manifold we recover the original symplectic manifolds:

Proposition 2.21. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and $B_i \subset M_i$ codimension-two symplectic submanifolds. Let γ be a symplectomorphism of tubular neighborhoods of B_1 and B_2 taking B_1 to B_2 and (M, ω) be the radial blow-up of (M_1, ω_1) and (M_2, ω_2) through (γ, B_1).*

Then cutting (M, ω) yields manifolds symplectomorphic to (M_1, ω_1) and (M_2, ω_2), with the symplectomorphisms carrying B to B_1 and B_2.

Proof. We will construct a symplectomorphism ρ_1 between the cut space (M_0^+, ω_0^+) of Definition 2.15 and the original manifold (M_1, ω_1).

Let $\mathcal{N}$ be the radially projectivized normal bundle to B_1 in M_1 and $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_1$ a blow-up model. The cut space M_0^+ is obtained by gluing the reduced space

$$\mu^{-1}(0)/\mathbb{S}^1 = \left\{(x, s, z) \in Z \times [0, \varepsilon^2) \times \mathbb{C} : \ s = |z|^2\right\} /\mathbb{S}^1$$

with the manifold $M_1 \smallsetminus B_1$ via $[x, t^2, t] \sim \beta(x, t)$ for $t > 0$ over $\mathcal{U}_1 \smallsetminus B_1$. The gluing diffeomorphism uses the maps

$$\begin{array}{rcl} \mathcal{N} \times (0, \varepsilon) & \longrightarrow & \mu^{-1}(0)/\mathbb{S}^1 \\ (x, t) & \longmapsto & [x, t^2, t] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{N} \times (0, \varepsilon) & \longrightarrow & \mathcal{U}_1 \smallsetminus B_1 \\ (x, t) & \longmapsto & \beta(x, t) \end{array}$$

and is in fact a symplectomorphism. The symplectic form ω_0^+ on M_0^+ is equal to the reduced symplectic form on $\mu^{-1}(0)/\mathbb{S}^1$ and equal to ω_1 on $M_1 \smallsetminus B_1$.

Let us define the map $\rho_1 : M_1 \to M_0^+$ that is the identity on $M_1 \smallsetminus B_1$ and on $\mathcal{U}_1$ is the composed diffeomorphism

$$\begin{array}{rccccc} \delta_1 : & \mathcal{U}_1 & \longrightarrow & (\mathcal{N} \times \mathbb{C})/\mathbb{S}^1 & \longrightarrow & \mu^{-1}(0)/\mathbb{S}^1 \\ & & & [x, z] & \longmapsto & [x, |z|^2, z], \end{array}$$

where the first arrow is the inverse of the bundle isomorphism given by the blow-up model. To show that ρ_1 is well-defined we must check that $u_1 \in \mathcal{U}_1 \smallsetminus B_1$ is equivalent to its image

$\delta_1(u_1) \in \mu^{-1}(0)/\mathbb{S}^1 \smallsetminus B$. Indeed, u_1 must correspond to $[x, z] \in (\mathcal{N} \times \mathbb{C})/\mathbb{S}^1$ with $z \neq 0$. We write z as $z = te^{i\theta}$ with $t > 0$. Since $[x, z] = [e^{i\theta}x, t]$, we have $u_1 = \beta(e^{i\theta}x, t)$ and $\delta_1(u_1) = [e^{i\theta}x, |t|^2, t]$. These two are equivalent under $\beta(x, t) \sim [x, t^2, t]$, so ρ_1 is well-defined.

Furthermore, M_1 and M_0^+ are symplectic manifolds equipped with a diffeomorphism that is a symplectomorphism on the common dense subset $M_1 \smallsetminus B_1$, so M_1 and M_0^+ must be globally symplectomorphic.

We will now turn to (M_2, ω_2) and (M_0^-, ω_0^-). The cut space M_0^- is obtained gluing the same reduced space $\mu^{-1}(0)/\mathbb{S}^1$ with the manifold $M_2 \smallsetminus B_2$ via $[x, t^2, t] \sim \gamma(\beta(x, t))$ for $t < 0$ over $\mathcal{U}_2 \smallsetminus B_2$, more precisely through the diffeomorphisms

$$\begin{array}{rcl} \mathcal{N} \times (-\varepsilon, 0) & \longrightarrow & \overline{\mu^{-1}(0)/\mathbb{S}^1} \\ (x, t) & \longmapsto & [x, t^2, t] \end{array}$$

and

$$\begin{array}{rcccl} \mathcal{N} \times (-\varepsilon, 0) & \longrightarrow & \overline{\mathcal{U}_1 \smallsetminus B_1} & \xrightarrow{\gamma} & \overline{\mathcal{U}_2 \smallsetminus B_2} \\ (x, t) & \longmapsto & \beta(x, t) & \longmapsto & \gamma(\beta(x, t)) . \end{array}$$

The symplectic form ω_0^- on M_0^- is equal to the reduced symplectic form on $\mu^{-1}(0)/\mathbb{S}^1$ and equal to ω_2 on $M_2 \smallsetminus B_2$.

Let us define the map $\rho_2 : M_2 \to M_0^-$ that is the identity on $M_2 \smallsetminus B_2$ and on $\mathcal{U}_2$ is the composed diffeomorphism

$$\begin{array}{rccccccc} \delta_2 : & \mathcal{U}_2 & \xrightarrow{\gamma^{-1}} & \mathcal{U}_1 & \longrightarrow & (\mathcal{N} \times \mathbb{C})/\mathbb{S}^1 & \longrightarrow & \mu^{-1}(0)/\mathbb{S}^1 \\ & & & & & [x, z] & \longmapsto & [x, |z|^2, z], \end{array}$$

where the second arrow is the inverse of the bundle isomorphism given by the blow-up model. To show that ρ_2 is well-defined we must check that $u_2 = \gamma(u_1) \in \mathcal{U}_2 \smallsetminus B_2$ is equivalent to its image $\delta_2(u_2) \in \mu^{-1}(0)/\mathbb{S}^1 \smallsetminus B$. Indeed u_1 must correspond to $[x, z] \in (\mathcal{N} \times \mathbb{C})/\mathbb{S}^1$ with $z \neq 0$. We write z as $z = -te^{i\theta}$ with $t < 0$. Since $[x, z] = [-e^{i\theta}x, t]$, we have $u_2 = \gamma\left(\beta(-e^{i\theta}x, t)\right)$ and $\delta_2(u_2) = [-e^{i\theta}x, |t|^2, t]$. These two are equivalent under $\gamma(\beta(x, t)) \sim [x, t^2, t]$, so ρ_1 is well-defined.

As before, we conclude that M_2 and M_0^- must be globally symplectomorphic. $\square$

2.5 And vice-versa: Radially blowing-up cut pieces

It was to be expected that, using reasonable definitions of cutting and radial blowing-up, cutting a radial blow-up would yield the original symplectic manifolds. However, asking if radially blowing-up cut pieces yields the original origami manifold is ultimately the same as asking if an origami manifold is completely determined by its symplectic cut pieces (plus a symplectomorphism γ as in Remark 2.14, which is also obtained when cutting). The answer is yes, and this is a remarkable property of origami manifolds that will be fundamental in the results of the next chapter.

Proposition 2.22. *Let (M,ω) be an origami manifold with nullfibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$, with (M_1,ω_1) and (M_2,ω_2) its symplectic cut pieces, B_1 and B_2 the respective natural symplectic embedded images of B, and $\gamma_1 : \mathcal{U}_1 \to \mathcal{U}_2$ the natural symplectomorphism of tubular neighborhoods of B_1 and B_2 as in Remark 2.14. Let $(\widetilde{M},\widetilde{\omega})$ be the radial blow-up of (M_1,ω_1) and (M_2,ω_2) through (γ_1, B_1).*

Then (M,ω) and $(\widetilde{M},\widetilde{\omega})$ are origami-symplectomorphic.

Proof. Let $\mathcal{U}$ be a tubular neighborhood of Z and $\varphi : Z \times (-\varepsilon,\varepsilon) \to \mathcal{U}$ a Moser model diffeomorphism as in Proposition 2.3, with $\varphi^*\omega = p^*i^*\omega + d(t^2p^*\alpha)$. Let $\mathcal{N}$ be the radially projectivized normal bundle to B_1 in M_1. By Proposition 2.13, the natural embedding of B in M_1 with image B_1 lifts to a bundle isomorphism from $\mathcal{N} \to B_1$ to $Z \to B$. Under this isomorphism, we pick the following natural blow-up model for the neighborhood $\mu^{-1}(0)/\mathbb{S}^1$ of B_1 in (M_1,ω_1):

$$\begin{array}{rccc} \beta : & Z \times (-\varepsilon,\varepsilon) & \longrightarrow & \mu^{-1}(0)/\mathbb{S}^1 \\ & (x,t) & \longmapsto & [x,t^2,t] \ . \end{array}$$

By construction, (see proof of Proposition 2.13) the reduced form ω_1 on $\mu^{-1}(0)/\mathbb{S}^1$ is such that $\beta^*\omega_1 = \varphi^*\omega$, so in this case the origami manifold $(\widetilde{M},\widetilde{\omega})$ has

$$\widetilde{M} = \left(M_1 \smallsetminus B_1 \cup \overline{M_2 \smallsetminus B_2} \cup Z \times (-\varepsilon,\varepsilon)\right) \ / \ \sim$$

where we quotient via

$$Z \times (0,\varepsilon) \overset{\beta}{\simeq} \mu^{-1}(0)/\mathbb{S}^1 \subset \mathcal{U}_1 \smallsetminus B_1$$

and

$$Z \times (-\varepsilon,0) \overset{\beta}{\simeq} \overline{\mu^{-1}(0)/\mathbb{S}^1} \overset{\gamma}{\simeq} \overline{\mu^{-1}(0)/\mathbb{S}^1} \subset \overline{\mathcal{U}_2 \smallsetminus B_2}$$

and we have

$$\widetilde{\omega} := \begin{cases} \omega_1 & \text{on } M_1 \smallsetminus B_1 \\ \omega_2 & \text{on } M_2 \smallsetminus B_2 \\ \beta^*\omega_1 & \text{on } Z \times (-\varepsilon, \varepsilon) \ . \end{cases}$$

The natural symplectomorphisms (from the proof of Proposition 2.13) $\overline{j^+} : M^+ \to M_1 \smallsetminus B_1$ and $\overline{j^-} : M^- \to M_2 \smallsetminus B_2$ extending $\varphi(x,t) \mapsto \left[x, t^2, t\right]$ make the following diagrams commute:

$$\begin{tikzcd} M^+ \supset \mathcal{U}^+ \arrow[rr, "j^+"] & & \mu^{-1}(0)/\mathbb{S}^1 \subset M_1 \smallsetminus B_1 \\ & Z \times (0,\varepsilon) \arrow[ul, "\varphi"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

$$\begin{tikzcd} M^- \supset \mathcal{U}^- \arrow[rr, "j^-"] & & \mu^{-1}(0)/\mathbb{S}^1 \subset \overline{M_2 \smallsetminus B_2} \\ & Z \times (-\varepsilon,0) \arrow[ul, "\varphi"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Therefore, the map $M \to \widetilde{M}$ defined by $\overline{j^+}$, $\overline{j^-}$ and φ^{-1} is a well-defined diffeomorphism pulling back $\widetilde{\omega}$ to ω. $\square$

Chapter 3

Group actions on origami manifolds

3.1 Moment images of origami manifolds

Consider a smooth action of a Lie group G on an origami manifold (M, ω)

$$\psi : G \to \mathrm{Diff}(M).$$

Suppose G acts by origami-symplectomorphisms, i.e., $\psi(g)^*\omega = \omega$ for each $g \in G$. Note that such a G-action preserves the folding hypersurface and its nullfibration. As in the symplectic case, we say that ψ is a **hamiltonian action** if there exists a map $\mu : M \to \mathfrak{g}^*$, equivariant with respect to the given action on M and the coadjoint action on $\mathfrak{g}^*$, such that for each $X \in \mathfrak{g}$ we have

$$d\mu^X = \iota_{X^\#}\omega,$$

where $\mu^X : M \to \mathbb{R}$, given by $\mu^X(p) = \langle \mu(p), X \rangle$, is the component of μ along X, and $X^\#$ is the vector field on M generated by the one-parameter subgroup $\{\exp tX : t \in \mathbb{R}\} \subset G$. The map μ is called the **moment map**.

Guillemin-Sternberg [7] and Atiyah [1] proved that the image of the moment map of a compact connected symplectic manifold with a torus action is a convex polytope. We are going to see that for origami manifolds the moment images are superpositions of convex polytopes, one for each connected component of $M \setminus Z$.

Definition 3.1. Let Δ_1 and Δ_2 be polytopes in $\mathbb{R}^n$ and F_1, F_2 faces of Δ_1 and Δ_2, respectively. We say that Δ_1 **agrees** with Δ_2 near F_1 and F_2 if $F_1 = F_2$ and there is an open subset $\mathcal{U}$ of $\mathbb{R}^n$ containing F_1 such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$.

Theorem 3.2. *Let (M, ω, G, μ) be a compact connected origami manifold with nullfibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$ and a hamiltonian action of an m-dimensional torus G with moment map $\mu : M \to \mathfrak{g}^*$. Then:*

(a) The image $\mu(M)$ of the moment map is the union of a finite number of convex polytopes Δ_i, $i = 1, \ldots, N$, each of which is the image of the moment map restricted to the closure of a connected component of $M \smallsetminus Z$;

(b) Over each connected component Z' of Z, $\mu(Z')$ is a facet of each of the two polytopes corresponding to the neighboring components of $M \smallsetminus Z$ (and furthermore the two polytopes agree near that facet) if and only if the nullfibration on Z' is given by a circle subgroup of G.

Such images $\mu(M)$ are called **origami polytopes**.

Proof. (a) Since the G-action preserves ω, it also preserves each connected component of the folding hypersurface Z and its nullfoliation V. Choose an oriented nonvanishing section V, average it so that it is G-invariant and scale it uniformly over each orbit so that its integral curves all have period 2π. This produces a vector field v which generates an action of $\mathbb{S}^1$ on Z that commutes with the G-action. This $\mathbb{S}^1$-action also preserves the moment map μ: For any $X \in \mathfrak{g}$ with corresponding vector field $X^{\#}$ on M, we have over Z

$$\mathcal{L}_v \mu^X = \iota_v d\mu^X = \iota_v \iota_{X^{\#}} \omega = \omega(X^{\#}, v) = 0 \,.$$

Using this v, the cutting construction from Section 2.2 has a hamiltonian version. Let (M_i, ω_i), $i = 1, \ldots, N$, be the compact connected components of the symplectic cut pieces and B_i be the union of the components of the centers B which naturally embed in M_i. Each $M_i \smallsetminus B_i$ is symplectomorphic to a connected component $\mathcal{W}_i \subset M \smallsetminus Z$ and M_i is the closure of $M_i \smallsetminus B_i$. Each (M_i, ω_i) inherits a hamiltonian action of G with moment map μ_i that matches $\mu|_{\mathcal{W}_i}$ over $M_i \smallsetminus B_i$ and is the well-defined $\mathbb{S}^1$-quotient of $\mu|_Z$ over B_i.

By the Atiyah-Guillemin-Sternberg convexity theorem [1, 7], each $\mu_i(M_i)$ is a convex polytope Δ_i. Since $\mu(M)$ is the union of the $\mu_i(M_i)$, we conclude that

$$\mu(M) = \bigcup_{i=1}^{N} \Delta_i.$$

(b) Let Z' be a connected component of Z with nullfibration $Z' \to B'$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the two neighboring components of $M \smallsetminus Z$ on each side of Z', $(M_1, \omega_1, G, \mu_1)$ and $(M_2, \omega_2, G, \mu_2)$ the corresponding cut spaces with moment polytopes Δ_1 and Δ_2.

Let $\mathcal{U}$ be a G-invariant tubular neighborhood of Z' with a G-equivariant diffeomorphism $\varphi : Z' \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ such that

$$\varphi^*\omega = p^*i^*\omega + d\left(t^2 p^*\alpha\right),$$

where G acts trivially on $(-\varepsilon, \varepsilon)$, $p : Z' \times (-\varepsilon, \varepsilon) \to Z'$ is the projection onto the first factor, $t \in (-\varepsilon, \varepsilon)$ and α is a G-invariant $\mathbb{S}^1$-connection on Z' as in Remark 2.9.

Without loss of generality, $Z' \times (0, \varepsilon)$ and $Z' \times (-\varepsilon, 0)$ correspond via φ to the two sides $\mathcal{U}_1 =: \mathcal{U} \cap \mathcal{W}_1$ and $\mathcal{U}_2 =: \mathcal{U} \cap \mathcal{W}_2$, respectively. The involution $\tau : \mathcal{U} \to \mathcal{U}$ translating $t \mapsto -t$ in $Z' \times (-\varepsilon, \varepsilon)$ is a G-equivariant (orientation-reversing) diffeomorphism preserving Z', switching $\mathcal{U}_1$ and $\mathcal{U}_2$ but preserving ω. Hence the moment map satisfies $\mu \circ \tau = \mu$ and $\mu(\mathcal{U}_1) = \mu(\mathcal{U}_2)$.

When the nullfibration is given by a subgroup of G, we cut the G-space $\mathcal{U}$ at the level Z'. The image $\mu(Z')$ is the intersection of $\mu(\mathcal{U})$ with a hyperplane and thus a facet of both Δ_1 and Δ_2. Each $\mathcal{U}_i \cup B'$ is equivariantly symplectomorphic to a neighborhood $\mathcal{V}_i$ of B' in $(M_i, \omega_i, G, \mu_i)$ with $\mu_i(\mathcal{V}_i) = \mu(\mathcal{U}_i) \cup \mu(Z')$, $i = 1, 2$. Since $\mu_1(\mathcal{V}_1) = \mu_2(\mathcal{V}_2)$, we conclude that Δ_1 and Δ_2 agree near the facet $\mu(Z')$.

For a general nullfibration, we cut the $G \times \mathbb{S}^1$-space $\mathcal{U}$ with moment map (μ, t^2) at Z', the $\mathbb{S}^1$-level $t^2 = 0$. The image of Z' by the $G \times \mathbb{S}^1$-moment map is the intersection of the image of the full $\mathcal{U}$ with a hyperplane. Let $\pi : \mathfrak{g}^* \times \mathbb{R} \to \mathfrak{g}^*$ be the projection onto the first factor. We conclude that the image $\mu(Z')$ is a facet of a polytope $\widetilde{\Delta}$ in $\mathfrak{g}^* \times \mathbb{R}$, so it can be of codimension zero or one; see Example 3.4.

If $\pi|_{\widetilde{\Delta}} : \widetilde{\Delta} \to \Delta_1$ is one-to-one, then facets of $\widetilde{\Delta}$ map to facets of Δ_1 and $\widetilde{\Delta}$ is contained in a hyperplane surjecting onto $\mathfrak{g}^*$. The normal to that hyperplane corresponds to a circle subgroup of $\mathbb{S}^1 \times G$ acting trivially on $\mathcal{U}$ and surjecting onto the $\mathbb{S}^1$-factor. This allows us to express the $\mathbb{S}^1$-action in terms of a subgroup of G.

If $\pi|_{\widetilde{\Delta}} : \widetilde{\Delta} \to \Delta_1$ is not one-to-one, it cannot map the facet $\widetilde{F}_{Z'}$ of $\widetilde{\Delta}$ corresponding to Z' to a facet of Δ_1: Otherwise, $\widetilde{F}_{Z'}$ would contain nontrivial *vertical* vectors $(0, x) \in \mathfrak{g}^* \times \mathbb{R}$, which would forbid cutting. Hence, the normal to $\widetilde{F}_{Z'}$ in $\widetilde{\Delta}$ must be transverse to $\mathfrak{g}^*$, and the corresponding nullfibration circle subgroup is not a subgroup of G.

$\square$

Example 3.3. Consider $(\mathbb{S}^{2n}, \omega_0, \mathbb{T}^n, \mu)$, where $(\mathbb{S}^{2n}, \omega_0)$ is a sphere as in Example 2.12 with $\mathbb{T}^n$ acting by

$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_1, \dots, z_n, h) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n, h)$$

and moment map defined by

$$\mu(z_1, \dots, z_n, h) = \left(\frac{|z_1|^2}{2}, \dots, \frac{|z_n|^2}{2} \right)$$

whose image is the n-simplex

$$\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum_i x_i \leq \frac{1}{2} \right\}.$$

The image $\mu(Z)$ of the folding hypersurface is the $(n-1)$-dimensional affine simplex that is the facet opposite from the orthogonal corner. Figure 3-1 gives the moment images of $\mathbb{S}^4$ and $\mathbb{S}^6$.

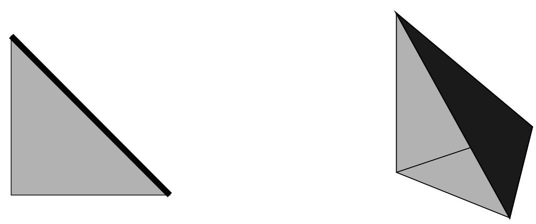

Figure 3-1: Origami polytopes for $\mathbb{S}^4$ and $\mathbb{S}^6$

The nullfoliation is the Hopf fibration given by the diagonal circle subgroup of $\mathbb{T}^n$. The moment image n-simplex is the union of two identical n-simplices, each of which is the moment polytope of one of the copies of $\mathbb{CP}^n$ obtained by cutting; see Example 2.16. $\diamondsuit$

Example 3.4. Consider $(\mathbb{S}^2 \times \mathbb{S}^2, \omega_s \oplus \omega_f, \mathbb{S}^1, \mu)$, where $(\mathbb{S}^2, \omega_s)$ is a standard symplectic sphere, $(\mathbb{S}^2, \omega_f)$ is a folded symplectic sphere with folding hypersurface given by a parallel, and $\mathbb{S}^1$ acts as the diagonal of the standard rotation action of $\mathbb{S}^1 \times \mathbb{S}^1$ on the product manifold. Then the moment map image is a line segment and the image of the folding hypersurface is a nontrivial subsegment. Indeed, the image of μ is a 45^o projection of the image of the moment map for the full $\mathbb{S}^1 \times \mathbb{S}^1$ action, the latter being a rectangle in which the folding hypersurface surjects to one of the sides.

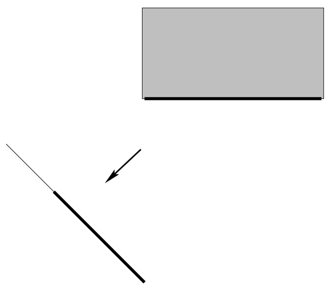

Figure 3-2: One-dimensional origami polytope where the image of the fold is not a facet

By considering the first or second factors of the $\mathbb{S}^1 \times \mathbb{S}^1$ action alone, we get the two extreme cases in which the image of the folding hypersurface is either the full line segment or simply one of the boundary points.

The analogous six-dimensional examples $(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, \omega_s \oplus \omega_s \oplus \omega_f, \mathbb{T}^2, \mu)$ produce moment images which are rational projections of a cube, with the folding hypersurface mapped to rhombi.

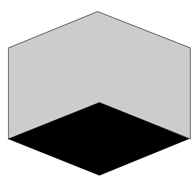

Figure 3-3: Two-dimensional origami polytope where the image of the fold is not a facet

$\diamondsuit$

3.2 Toric origami manifolds

In the symplectic world, a closed connected symplectic $2n$-dimensional manifold equipped with an effective hamiltonian action of an n-dimensional torus and with a corresponding moment map is called a *toric symplectic manifold* or *Delzant space*. Delzant's theorem [5] says that the moment polytope determines the Delzant space up to an equivariant symplectomorphism intertwining the moment maps. A polytope which occurs as the moment image of a Delzant space is a **Delzant polytope**. This is a polytope in $\mathbb{R}^n$ such that n edges of the form $p + tu_i$, $t \geq 0$ meet at each vertex p, with $u_i \in \mathbb{Z}^n$, and for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Definition 3.5. A **toric origami manifold** (M, ω, G, μ) is a compact connected origami manifold (M, ω) equipped with an effective hamiltonian action of a torus G with $\dim G = \frac{1}{2} \dim M$ and with a choice of a corresponding moment map μ.

We will see that in the toric case the condition of part (b) of Theorem 3.2 always holds:

Corollary 3.6. *When (M, ω, G, μ) is a toric origami manifold the moment map image of each connected component Z' of Z is a facet of the two polytopes corresponding to the neighboring components of $M \smallsetminus Z$ and these polytopes agree near the facet $\mu(Z')$.*

Proof. On a toric origami manifold, principal orbits, i.e. those with trivial isotropy, form a dense open subset of M [2, p.179]. Any connected component Z' of Z has a G-invariant tubular neighborhood modeled on $Z' \times (-\varepsilon, \varepsilon)$ with a $G \times \mathbb{S}^1$ hamiltonian action having moment map (μ, t^2). As the orbits are isotropic submanifolds, the principal orbits of the $G \times \mathbb{S}^1$-action must still have dimension $\dim G$, so their stabilizer must be a one-dimensional compact connected subgroup surjecting onto $\mathbb{S}^1$. Thus, over those connected components of Z the nullfibration is given by a subgroup of G. $\square$

The moment image of an origami manifold is a superposition of polytopes with certain compatibility conditions. These polytopes are the moment images of the closures of connected components of $M \smallsetminus Z$, and are also the moment polytopes of the connected components M_i of the symplectic cut pieces.

As seen in the proof of Theorem 3.2, each M_i inherits a hamiltonian G-action, thus making each $(M_i, \omega_i, G, \mu_i)$ a Delzant space. In the next section we will see that all (compatible) superpositions of Delzant polytopes occur as moment images of toric origami manifolds, and furthermore classify them up to equivariant origami-symplectomorphism intertwining moment maps.

Example 3.7. Let $(M_1, \omega_1, \mathbb{T}^2, \mu_1)$ and $(M_2, \omega_2, \mathbb{T}^2, \mu_2)$ be symplectic toric manifolds with moment polytopes that agree near a facet. For example, consider the Hirzebruch surfaces with Delzant polytopes as in Figure 3-4, but translated so the vertical edges coincide.

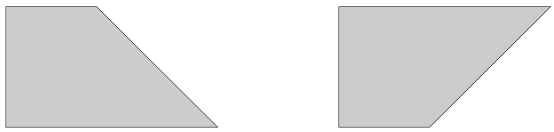

Figure 3-4: Delzant polytopes that agree near a facet: two Hirzebruch surfaces

Let $(B, \omega_B, \mathbb{T}^2, \mu_B)$ be a symplectic $\mathbb{S}^2$ with a hamiltonian (noneffective) $\mathbb{T}^2$-action and hamiltonian embeddings j_i into $(M_i, \omega_i, \mathbb{T}^2, \mu_i)$ as preimages of the vertical edges.

In order to proceed we need the following lemma:

Lemma 3.8. *Let $G = \mathbb{T}^n$ be an n-dimensional torus and $(M_i^{2n}, \omega_i, \mu_i)$, $i = 1, 2$, two symplectic toric manifolds. If the moment polytopes $\Delta_i := \mu_i(M_i)$ agree near facets $F_1 \subset \mu_1(M_1)$ and $F_2 \subset \mu_2(M_2)$, then there are G-invariant neighborhoods $\mathcal{U}_i$ of $B_i = \mu_i^{-1}(F_i)$, $i = 1, 2$, with a G-equivariant symplectomorphism $\gamma : \mathcal{U}_1 \to \mathcal{U}_2$ extending a symplectomorphism $B_1 \to B_2$ and such that $\gamma^* \mu_2 = \mu_1$.*

Proof. Let $\mathcal{U}$ be an open set containing $F_1 = F_2$ such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$. Perform symplectic cutting [9] on M_1 and M_2 by slicing Δ_i along a hyperplane parallel to F_i such that the resulting moment polytope $\widetilde{\Delta}_i$ containing F_i is in the open set $\mathcal{U}$. Suppose the hyperplane is close enough to F_i to guarantee that $\widetilde{\Delta}_i$ is still a Delzant polytope. Then $\widetilde{\Delta}_1 = \widetilde{\Delta}_2$. By Delzant's theorem, the corresponding cut spaces $\widetilde{M}_1$ and $\widetilde{M}_2$ are G-equivariantly symplectomorphic, with the symplectomorphism pulling back one moment map to the other.

Since symplectic cutting is a local operation, restricting the previous symplectomorphism gives us a G-equivariant symplectomorphism between G-equivariant neighborhoods $\mathcal{U}_i$ of B_i in M_i pulling back one moment map to the other. $\square$

By this Lemma, there exists a $\mathbb{T}^2$-equivariant symplectomorphism $\gamma : \mathcal{U}_1 \to \mathcal{U}_2$ between invariant tubular neighborhoods $\mathcal{U}_i$ of $j_i(B)$ extending a symplectomorphism $j_1(B) \to j_2(B)$

such that $\gamma^*\mu_2 = \mu_1$. The corresponding radial blow-up has the origami polytope depicted in Figure 3-5.

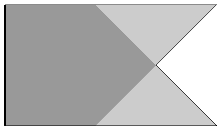

Figure 3-5: Origami polytope for the radial blow-up of two Hirzebruch surfaces

Different shades of grey distinguish regions where each point represents two orbits (darker) or one orbit (lighter), as results from the superposition of two Hirzebruch polytopes. $\diamondsuit$

3.3 Classification of toric origami manifolds

A toric origami manifold is determined by its symplectic cut pieces (plus a symplectomorphsim γ as in Remark 2.14) whose connected components are Delzant spaces. These, in turn, are determined by their moment polytopes, so the toric origami manifold is determined by its moment data, collected in the form of an *origami template.*

Definition 3.9. An n-dimensional **origami template** is a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a (nonempty) finite collection of oriented n-dimensional Delzant polytopes and $\mathcal{F}$ is a collection of pairs of facets of polytopes in $\mathcal{P}$ satisfying the following properties:

(a) For each pair $\{F_1, F_2\} \in \mathcal{F}$, the corresponding polytopes $\Delta_1 \ni F_1$ and $\Delta_2 \ni F_2$ agree near those facets and have opposite orientations;

(b) If a facet F occurs in a pair in $\mathcal{F}$, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;

(c) The topological space constructed from the disjoint union $\sqcup\Delta_i$, $\Delta_i \in \mathcal{P}$ by identifying facet pairs in $\mathcal{F}$ is connected.

Theorem 3.10. *Toric origami manifolds are classified by origami templates up to equivariant origami-symplectomorphisms preserving the moment maps. More specifically, there* *is a bijective correspondence*

$$\begin{array}{rcl} \{2n\text{-}dim\text{'}l\ toric\ origami\ manifolds\} & \longrightarrow & \{n\text{-}dim\text{'}l\ origami\ templates\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M). \end{array}$$

Proof. Let M_1 be the disjoint union of all the Delzant spaces associated with the positively oriented polytopes in $\mathcal{P}$, and similarly M_2 for the negatively oriented ones. The origami manifold corresponding to the template is the radial blow-up of M_1 and M_2 along the inverse images of the pairs of facets occurring in $\mathcal{F}$, using the symplectomorphism γ from Lemma 3.8.

The uniqueness part follows from an equivariant version of Proposition 2.22. □

Example 3.11. The template of the manifold constructed in Example 3.7 is $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ contains the two Hirzebruch polytopes and $\mathcal{F}$ consists of the pair of vertical edges. ♢

Example 3.12. Unlike ordinary toric manifolds, toric origami manifolds may come from non-simply connected templates. Let M be the manifold $\mathbb{S}^2 \times \mathbb{S}^2$ (with different areas) blown up at two points. The associated polytope Δ is a rectangle with two corners removed. We can construct an origami template $(\mathcal{P}, \mathcal{F})$ where $\mathcal{P}$ consists of four copies of Δ arranged in a square and $\mathcal{F}$ of the four pairs of edges coming from the blowups, as illustrated in Figure 3-6.

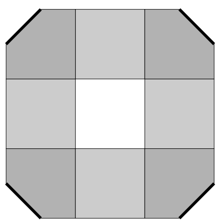

Figure 3-6: Non-simply connected origami polytope

Note that the associated origami manifold is also not simply connected.

We can form higher-dimensional analogues of this example, which fail to be k-connected for $k \geq 2$. In the case $k = 2$, for instance, let Δ' be the polytope associated to $M \times \mathbb{S}^2$, and construct an origami template $(\mathcal{P}', \mathcal{F}')$ just as before, to obtain the three-dimensional polytope of Figure 3-7.

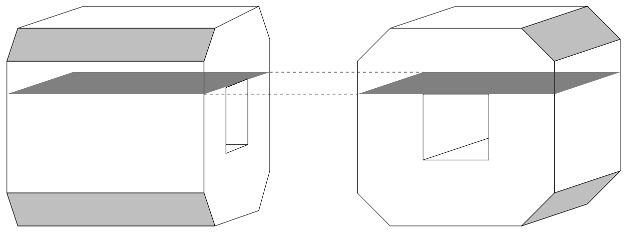

Figure 3-7: Non-2-connected origami polytope

We now superimpose these two solids along the dark shaded facets (the bottom facets of the top copies of Δ'), giving us a ninth pair of facets and the desired non-2-connected template. $\diamondsuit$

Example 3.13. We can classify all two-dimensional toric origami manifolds via their one-dimensional templates. These are disjoint unions of n segments connected at vertices with zero angle, each segment gives a component of $M \setminus Z$. In these pictures the segment length, which accounts for symplectic area of the corresponding component, is ignored. Instead of drawing segments superimposed, we open up angles slightly to show the number of components. All internal vertices are marked with bullets for folds and each one gives a component of Z. An endpoint on the picture corresponds to a fixed point on the manifold. There are two families:

- Manifolds diffeomorphic to a sphere $\mathbb{S}^2$ with two fixed points (the north and south pole) and the fold Z consisting of $n \geq 0$ disjoint circles:

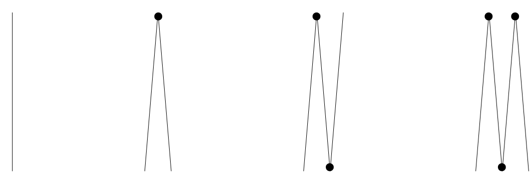

Figure 3-8: Toric origami 2-spheres

- Manifolds diffeomorphic to a torus $\mathbb{T}^2$ with no fixed points and the fold Z consisting of an even number $2n \geq 2$ of disjoint circles:

$\diamondsuit$

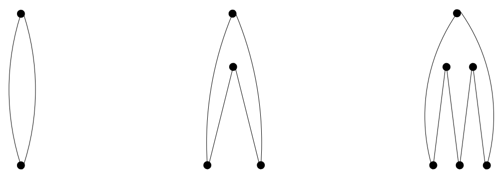

Figure 3-9: Toric origami 2-tori

3.4 Take a walk on the non-origami side

Moment images of origami manifolds, and especially those of toric origami manifolds, are very rigid. This is due to the origami hypothesis, and in fact (non-origami) folded symplectic manifolds have moment images that can be practically arbitrary.

Take a toric symplectic $2n$-manifold $(M', \omega', \mathbb{T}^n, \mu)$ and use a regular closed hypersurface inside the moment image to scoop out the $\mathbb{T}^n$-invariant open subset of the manifold M' that corresponds to the region inside the hypersurface. Let f be a defining function for the hypersurface such that f is positive on the interior. Now, consider the manifold

$$M = \left\{(p, x) \in M' \times \mathbb{R} \, : \, x^2 = f(p)\right\} .$$

This manifold is naturally a folded symplectic manifold that inherits an effective hamiltonian $\mathbb{T}^n$-action and a moment map from the original M'. Its moment map image is exactly the interior of the hypersurface (each point on the interior of the image represents two orbits, one for positive and one for negative x, and points on the fold are mapped to the boundary). For instance, we can use this method to build a 4-dimensional folded symplectic manifold with moment image as in Figure 3-10[1].

The nullfoliation on the fold Z is not fibrating: At points where the slope of the curve is irrational, the corresponding leaf is not compact, so this manifold is not origami.

As illustrated by Figure 3-10, it is not possible to classify toric folded symplectic manifolds by combinatorial moment data as was the case for toric origami manifolds. Such a classification must be more intricate for the general folded case, and in [8] C. Lee gives a partial result that sheds some light on the type of classification that might be possible:

A toric folded symplectic manifold $(M, \omega, \mathbb{T}, \mu)$ is a compact connected folded symplectic manifold (M^{2n}, ω) endowed with an effective hamiltonian action of a half-dimensional torus

[1]Image from [6], reprinted and altered with permission of the authors

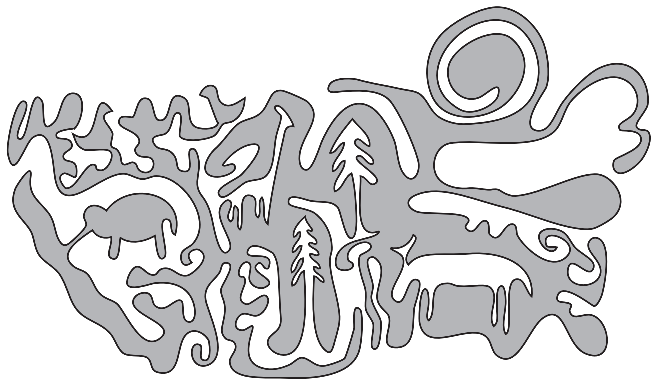

Figure 3-10: Moment image of a folded (non-origami) symplectic manifold

$\mathbb{T}^n$ and a corresponding moment map μ. The *orbital moment map* is the map on the orbit space $M/\mathbb{T}$ induced by the moment map. Two toric folded symplectic 4-manifolds $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are folded-symplectomorphic if $H^2(M/\mathbb{T}, \mathbb{Z}) = 0$ and there exists a diffeomorphism between orbit spaces preserving orbital moment maps.

When $(M, \omega, \mathbb{T}, \mu)$ is a toric origami manifold, $M/\mathbb{T}$ can be realized as the topological space obtained by gluing the polytopes of its origami template along the common facets (see point (c) in Definition 3.9). This space has the same homotopy type as the graph obtained by replacing each polytope by a point and each glued double facet by an edge between the points corresponding to the polytopes that the facet belongs to. Therefore, $H^2(M/\mathbb{T}, \mathbb{Z}) = 0$. The existence of a diffeomorphism between orbit spaces implies that $(M', \omega', \mathbb{T}, \mu')$ is an origami manifold as well, and that its origami template is the same as that of $(M, \omega, \mathbb{T}, \mu)$, which makes them origami-symplectomorphic by Theorem 3.3.

Chapter 4

Going nonorientable

4.1 Nonorientable origami manifolds

In Section 3.3 we saw that given an origami template, we can construct the corresponding origami manifold by blowing up the Delzant spaces that correspond to the polytopes along the paired facets. Consider Figure 4-1, and use this method to create a manifold: Start with three copies of $\mathbb{S}^2 \times \mathbb{S}^2$ blown up at two points and radially blow up the three marked pairs of facets.

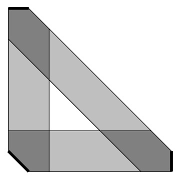

Figure 4-1: Nonorientable origami template

This is not a template in the sense of Definition 3.9 because the condition regarding orientation is not satisfied, but since radial blowing up is a local operation, it is possible to construct the corresponding manifold. While failing to be globally orientable, this manifold is not essentially different from the origami manifolds studied in Chapters 1 and 2. More precisely, a small enough tubular neighbourhood of the folding hypersurface, like the collar given by Proposition 2.3, is an open (oriented) origami manifold. In this case, we say that we have a **coorientable fold**.

This suggests that we should allow for nonorientable origami manifolds. Indeed, Definition 2.1 of folded symplectic and 2.11 of origami forms make sense without reference to orientability. We therefore extend the earlier definitions to include nonorientable origami manifolds that are, in the above sense, essentially different from those previously considered:

Example 4.1. Consider the projective real space $\mathbb{RP}^{2n} = \mathbb{S}^{2n}/\mathbb{Z}_2$, with the 2-form ω_0 induced by the restriction to $\mathbb{S}^{2n}$ of the $\mathbb{Z}_2$-invariant form $dx_1 \wedge dx_2 + \ldots + dx_{2n-1} \wedge dx_{2n}$ in $\mathbb{R}^{2n+1}$ [4]. The folding hypersurface is $\mathbb{RP}^{2n-1} \simeq \{[x_1, \ldots, x_{2n}, 0]\}$ and the nullfibration is the $\mathbb{Z}_2$-quotient of the Hopf fibration: $\mathbb{S}^1 \hookrightarrow \mathbb{RP}^{2n-1} \twoheadrightarrow \mathbb{CP}^{n-1}$. This origami $\mathbb{RP}^{2n}$ is the $\mathbb{Z}_2$-quotient of the oriented origami $2n$-sphere from Example 2.12. Note that the collar neighbourhood of the fold is a bundle of Möbius bands. $\diamondsuit$

When the collar neighbourhood of the fold is nonorientable, we say we have a **non-coorientable fold**.

4.2 Cutting and radially blowing-up in the general case

The orientable double cover $(\overline{M}, \overline{\omega})$ (with a choice of orientation) of a nonorientable origami manifold (M, ω) is an oriented origami manifold, so we can perform cutting on M by working on $(\overline{M}, \overline{\omega})$: The double cover involution yields a symplectomorphism from one symplectic cut piece to the other, so we regard these pieces as a trivial double cover of one of them, and call their $\mathbb{Z}_2$-quotient the *symplectic cut space* of (M, ω). In general,

Definition 4.2. The **symplectic cut space** of an origami manifold (M, ω) is the natural $\mathbb{Z}_2$-quotient of the symplectic cut pieces of its orientable double cover.

Example 4.3. In the case where $M \setminus Z$ is connected, the symplectic cut space is also connected: Cutting the origami manifold $(\mathbb{RP}^{2n}, \omega_0)$ from Example 4.1 produces a single copy of $\mathbb{CP}^n$. $\diamondsuit$

In order to generalize the blowing-up construction to the general case, we replace the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) in Proposition 2.19 by their union (M, ω), and likewise the symplectic submanifolds B_1 and B_2 by their union B. Let (M, ω) be a symplectic manifold with a codimension-two symplectic submanifold B. It is useful to introduce notion of *model involution.*

Definition 4.4. A **model involution** of a tubular neighborhood $\mathcal{U}$ of B is a symplectic involution $\gamma : \mathcal{U} \to \mathcal{U}$ preserving B such that on the connected components $\mathcal{U}_i$ of $\mathcal{U}$, where

$\gamma(\mathcal{U}_i) = \mathcal{U}_i$, we have $\gamma|_{\mathcal{U}_i} = \mathrm{id}_{\mathcal{U}_i}$.

Recall that $\mathcal{N}$ is the radially projectivized normal bundle to B in M. Then a model involution γ induces a bundle map $\Gamma : \mathcal{N} \to \mathcal{N}$ covering $\gamma|_B$ by the formula

$$\Gamma([v]) = [d\gamma_p(v)] \text{ for } v \in T_pM, p \in B \ .$$

This is well-defined because $\gamma(B) = B$.

When, as in the orientable case, B is the disjoint union of B_1 and B_2 and $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$, if $\gamma(B_1) = B_2$ then

$$\gamma_1 := \gamma|_{\mathcal{U}_1} : \mathcal{U}_1 \to \mathcal{U}_2 \text{ and } \gamma|_{\mathcal{U}_2} = \gamma_1^{-1} : \mathcal{U}_2 \to \mathcal{U}_1 \ .$$

In that case, $B/\gamma \simeq B_1$ and $\mathcal{N}/-\Gamma \simeq \mathcal{N}_1$ is the radially projectivized normal bundle to B_1.

Proposition 4.5. *Let (M, ω) be a symplectic manifold, B a compact codimension-two symplectic submanifold and $\mathcal{N}$ its radially projectivized normal bundle. Let $\gamma : \mathcal{U} \to \mathcal{U}$ be a model involution of a tubular neighborhood $\mathcal{U}$ of B and $\Gamma : \mathcal{N} \to \mathcal{N}$ the induced bundle map.*

Then there is a natural origami manifold $(\widetilde{M}, \widetilde{\omega})$ with folding hypersurface diffeomorphic to $\mathcal{N}/-\Gamma$ and nullfibration isomorphic to $\mathcal{N}/-\Gamma \to B/\gamma$.

Proof. Choose $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ a blow-up model for the neighborhood $\mathcal{U}$ such that $\gamma \circ \beta = \beta \circ \Gamma$. This is always possible: For components $\mathcal{U}_i$ of $\mathcal{U}$ where $\gamma(\mathcal{U}_i) = \mathcal{U}_i$ this condition is trivial; for disjoint neighborhood components $\mathcal{U}_i$ and $\mathcal{U}_j$ such that $\gamma(\mathcal{U}_i) = \mathcal{U}_j$ this condition amounts to choosing any blow-up model on one of these components and transporting it to the other by γ.

Then $\beta^*\omega$ is a folded symplectic form on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ with folding hypersurface $\mathcal{N} \times \{0\}$ and nullfoliation integrating to the circle fibration $\mathbb{S}^1 \hookrightarrow \mathcal{N} \xrightarrow{\pi} B$. We define

$$\widetilde{M} = \left(M \setminus B \bigcup \mathcal{N} \times (-\varepsilon, \varepsilon) \right) / \sim$$

where we quotient by

$$(x, t) \sim \beta(x, t) \text{ for } t > 0 \text{ and } (x, t) \sim (-\Gamma(x), -t) \, .$$

The forms ω on $M \smallsetminus B$ and $\beta^*\omega$ on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ induce an origami form $\widetilde{\omega}$ on $\widetilde{M}$ with folding hypersurface $\mathcal{N}/-\Gamma$. Indeed, β is a symplectomorphism for $t > 0$, and $(-\Gamma, -\mathrm{id})$ on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ is a symplectomorphism away from $t = 0$ (since β and γ are) and at points where $t = 0$ it is a local diffeomorphism. $\square$

The origami manifold $(\widetilde{M}, \widetilde{\omega})$ just constructed is called the **radial blow-up** of (M, ω) through (γ, B). When M is compact, $\widetilde{M}$ is as well.

Example 4.6. Let M be a 2-sphere, B the union of two (distinct) points of M, and γ defined by a symplectomorphism from a Darboux neighborhood of one point to a Darboux neighborhood of the other. Then the radial blow-up $\widetilde{M}$ is a Klein bottle and $\widetilde{\omega}$ a form which folds along a circle. $\diamondsuit$

Example 4.7. Let M be a 2-sphere, B one point on it, and γ the identity map on a neighborhood of that point. Then the radial blow-up $\widetilde{M}$ is $\mathbb{RP}^2$ and $\widetilde{\omega}$ is a form that folds along a circle. $\diamondsuit$

The quotient $\mathcal{N} \times (-\varepsilon, \varepsilon)/(-\Gamma, -\mathrm{id})$ provides a collar neighborhood of the fold in $(\widetilde{M}, \widetilde{\omega})$.

When B splits into two disjoint components interchanged by γ, this collar is orientable so the fold is coorientable. Example 4.6 illustrates a case where, even though the fold is coorientable, the radial blow-up $(\widetilde{M}, \widetilde{\omega})$ is not orientable.

When γ is the identity map, as in Example 4.7, the collar is nonorientable and the fold is not coorientable. In the latter case, the collar is a bundle of Möbius bands $\mathbb{S}^1 \times (-\varepsilon, \varepsilon)/(-\mathrm{id}, -\mathrm{id})$ over B.

In general, γ will be the identity over some connected components of B and will interchange other components, so some components of the fold will be coorientable and others will not.

While cutting of general origami manifolds is performed by working on the double cover, blow-up is performed directly, so in order to prove results similar to Propositions 2.21 and 2.22 we need the fact that the blow-up of the double cover is the double cover of the blow-up:

Lemma 4.8. *Let (M, ω) be the blow-up of the symplectic manifold (M_s, ω_s) through (γ, B). We write $B = B_0 \sqcup B_1 \sqcup B_2$ and the domain of γ as $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2$ where γ is the identity map on $\mathcal{U}_0$ and exchanges $\mathcal{U}_1$ and $\mathcal{U}_2$.*

Let $(\overline{M_s}, \overline{\omega_s})$ be the trivial double cover of (M_s, ω_s) with $\overline{B} = B^\uparrow \sqcup B^\downarrow$, $\overline{\mathcal{U}} = \mathcal{U}^\uparrow \sqcup \mathcal{U}^\downarrow$ the *double covers of B and $\mathcal{U}$; let $\overline{\gamma} : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ be the lift of γ satisfying $\overline{\gamma}(\mathcal{U}_0^{\uparrow}) = \mathcal{U}_0^{\downarrow}$, $\overline{\gamma}(\mathcal{U}_1^{\uparrow}) = \mathcal{U}_2^{\downarrow}$ and $\overline{\gamma}(\mathcal{U}_2^{\uparrow}) = \mathcal{U}_1^{\downarrow}$; and let $(\overline{M}, \overline{\omega})$ be the blow-up of $(\overline{M_s}, \overline{\omega_s})$ through $(\overline{\gamma}, \overline{B})$.*

Then $(\overline{M}, \overline{\omega})$ is an orientable double cover of (M, ω).

Proof. Being the double cover of an oriented manifold, we write

$$\overline{M_s} = M_s^{\uparrow} \sqcup M_s^{\downarrow}$$

with each component diffeomorphic to M_s. By construction of $\overline{\gamma}$, the blow-up can be performed as in Proposition 2.19 and yields an orientable origami manifold $(\overline{M}, \overline{\omega})$ with

$$\overline{M}^{+} \simeq M_s^{\uparrow} \smallsetminus B^{\uparrow} \text{ and } \overline{M}^{-} \simeq M_s^{\downarrow} \smallsetminus B^{\downarrow}$$

and a fold $\mathcal{N}^{\uparrow}$ fibering over $B^{\uparrow}$. There is a natural two-to-one smooth projection $\overline{M} \to M$ taking $M_s^{\uparrow} \smallsetminus B^{\uparrow}$ and $M_s^{\downarrow} \smallsetminus B^{\downarrow}$ each diffeomorphically to $M \smallsetminus Z$ where Z is the fold of (M, ω), and that takes the fold $\mathcal{N}^{\uparrow} \simeq \mathcal{N}$ of $(\overline{M}, \overline{\omega})$ to $Z \simeq \mathcal{N}/-\Gamma$, where $\Gamma : \mathcal{N} \to \mathcal{N}$ is the bundle map induced by γ (the map $-\Gamma$ has no fixed points). □

Corollary 4.9. *Let (M, ω) be the radial blow-up of the symplectic manifold (M_s, ω_s) through (γ, B). Then the cutting of (M, ω) yields a manifold symplectomorphic to (M_s, ω_s) where the symplectomorphism carries the base to B.*

Proof. Let (M_{cut}, ω_{cut}) be the symplectic cut space of (M, ω). Let $(\overline{M_s}, \overline{\omega_s})$ and $(\overline{M_{cut}}, \overline{\omega_{cut}})$ be the trivial double covers of (M_s, ω_s) and (M_{cut}, ω_{cut}). By Lemma 4.8, the radial blow-up $(\overline{M}, \overline{\omega})$ of $(\overline{M_s}, \overline{\omega_s})$ through $(\overline{\gamma}, \overline{B})$ is an orientable double cover of (M, ω). By Definition 4.2, $(\overline{M_{cut}}, \overline{\omega_{cut}})$ is the symplectic cut space of $(\overline{M}, \overline{\omega})$. By Proposition 2.22, $(\overline{M_s}, \overline{\omega_s})$ and $(\overline{M_{cut}}, \overline{\omega_{cut}})$ are symplectomorphic relative to the centers. It follows that (M_s, ω_s) and (M_{cut}, ω_{cut}) are symplectomorphic relative to the centers. □

Corollary 4.10. *Let (M, ω) be an origami manifold with nullfibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$.*

Let (M_{cut}, ω_{cut}) be its symplectic cut space, B_{cut} the natural symplectic embedded image of B in M_{cut} and $\gamma : \mathcal{U} \to \mathcal{U}$ a natural symplectomorphism of a tubular neighborhood $\mathcal{U}$ of B_{cut}.

Then (M, ω) is origami-symplectomorphic to $(\widetilde{M}, \widetilde{\omega})$, the radial blow-up of (M_{cut}, ω_{cut}) through (γ, B_{cut}).

Proof. We prove this at the level of orientable double covers. By Proposition 2.21, the orientable double cover of (M, ω) is origami-symplectomorphic to the blow-up of its cut space. By definition, the cut space of the double cover of (M, ω) is the double cover of (M_{cut}, ω_{cut}) and, by Lemma 4.8, the blow-up of this latter double cover is the double cover of $(\widetilde{M}, \widetilde{\omega})$. $\square$

4.3 Convexity and classification results in the general case

The definitions of hamiltonian action, moment map and toric action hold also for nonorientable origami manifolds, so we will extend the results of Chapter 3 to all origami manifolds.

Example 4.11. Recall that the moment polytope of a toric $\mathbb{CP}^2$ is a triangle as shown on the right in Figure 4-2. The moment image of a toric sphere $\mathbb{S}^4$ is shown on the left with a darker shading, two copies of this triangle glued along one edge. In the center we have the moment image of a toric $\mathbb{RP}^4$, a single copy of the triangle with a single folded edge.

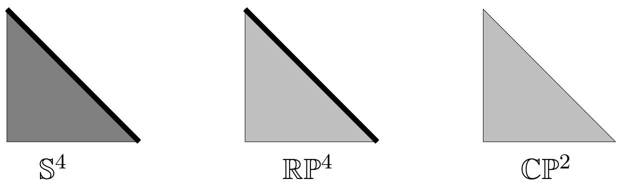

Figure 4-2: Three origami templates: $\mathbb{S}^4$, $\mathbb{RP}^4$ and $\mathbb{CP}^2$

This exhibits $\mathbb{S}^4$ as a double cover of $\mathbb{RP}^4$ at the level of templates. $\diamondsuit$

Whenever the connected component of the fold is coorientable, the situtation is similar to that of the original oriented case, in the sense that the neighboring polytopes must agree near the folding edge. In the non-coorientable case, the folding edge only belongs to one polytope, so there is no such condition to satisfy. Theorem 3.2 then becomes:

Theorem 4.12. *Let (M, ω, G, μ) be a compact connected origami manifold with nullfibration $\mathbb{S}^1 \hookrightarrow Z \xrightarrow{\pi} B$ and a hamiltonian action of an m-dimensional torus G with moment map $\mu : M \to \mathfrak{g}^*$. Then:*

(a) The image $\mu(M)$ of the moment map is the union of a finite number of convex polytopes Δ_i, $i = 1, \ldots, N$, each of which is the image of the moment map restricted to the closure of a connected component of $M \setminus Z$;

(b) *Over each connected component Z' of Z, $\mu(Z')$ is a facet of each of the one or two polytopes corresponding to the neighboring components of $M \smallsetminus Z$ (and furthermore the polytopes, if two, agree near that facet) if and only if the nullfibration on Z' is given by a subgroup of G.*

To prove this, we lift the hamiltonian torus action to the orientable double cover of M. The lifted moment map is the composition of the two-to-one projection with the original double map. The result then follows from Theorem 3.2.

Working again with orientable double covers, we get as in Corollary 3.6 that the condition of part (b) of the Theorem above holds for toric origami manifolds. We can now proceed to a classification of toric origami manifolds analogous to that of Theorem 3.10. We begin by re-defining templates (compare with Definition 3.9):

Definition 4.13. An n-dimensional **origami template** is a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a (nonempty) finite collection of n-dimensional Delzant polytopes and $\mathcal{F}$ is a collection of facets and pairs of facets of polytopes in $\mathcal{P}$ satisfying the following properties:

(a) For each pair $\{F_1, F_2\} \in \mathcal{F}$, the corresponding polytopes $\Delta_1 \ni F_1$ and $\Delta_2 \ni F_2$ agree near those facets;

(b) If a facet F occurs in a set in $\mathcal{F}$, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;

(c) The topological space constructed from the disjoint union $\sqcup \Delta_i$, $\Delta_i \in \mathcal{P}$ by identifying facet pairs in $\mathcal{F}$ is connected.

With the updated definitions of origami manifolds and origami templates, that include both the orientable and the nonorientable case, we again have a classification theorem:

Theorem 4.14. *Toric origami manifolds are classified by origami templates up to equivariant origami-symplectomorphisms preserving the moment maps. More specifically, there is a bijective correspondence*

$$\begin{array}{rcl} \{2n\text{-}\mathit{diml\ toric\ origami\ manifolds}\} & \longrightarrow & \{n\text{-}\mathit{diml\ origami\ templates}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M). \end{array}$$

The proof is similar to that of Theorem 3.10: To build the toric orgami manifold from the template, blow up the Delzant spaces associated with the polytopes in $\mathcal{P}$ along the inverse images of the facets in $\mathcal{F}$. The model involution γ is built from the identity map for single facets $\{F\} \in \mathcal{F}$ and the symplectomorphism in Lemma 3.8 for pairs $\{F_1, F_2\} \in \mathcal{F}$.

Example 4.15. We can add the following two families of (nonorientable) origami manifolds to the listing of toric origami surfaces of Example 3.13:

- Manifolds diffeomorphic to a projective plane $\mathbb{RP}^2$ with one fixed point and the fold Z consisting of one non-coorientable circle and $n \geq 0$ coorientable disjoint circles:

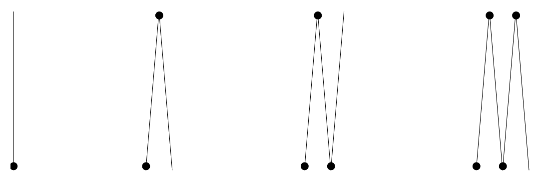

Figure 4-3: Toric origami projective planes

- Manifolds diffeomorphic to a Klein bottle with no fixed points and the fold Z consisting of two non-coorientable circles and $n \geq 0$ coorientable disjoint circles:

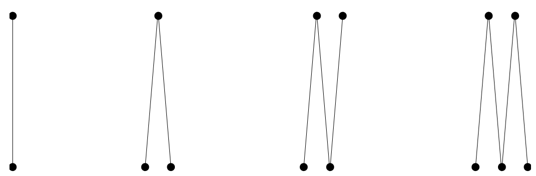

Figure 4-4: Toric origami Klein bottles

In summary, toric origami surfaces are diffeomorphically either spheres, tori, projective planes or Klein bottles. $\diamondsuit$

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