Convexity in Symplectic Geometry: The Atiyah-Guillemin-Sternberg Theorem

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The setting is the following:

Let (M, ω) be a $2d$-dimensional compact connected symplectic manifold, G an abelian compact Lie group, i.e., an n-dimensional torus, with a hamiltonian action on M:

$$\tau : G \times M \to M$$

The corresponding moment map is a map

$$\Phi : M \to \mathfrak{g}^*$$

which is defined uniquely up to an additive constant by the following properties:

- Φ is equivariant with respect to the action τ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$
- For $\xi \in \mathfrak{g}$, the ξ-component of the moment map is $\Phi^\xi : M \to \mathbb{R}$ given by $\Phi^\xi(p) = \langle \Phi(p), \xi \rangle$. From ξ we obtain also $\xi^\#$, the vector field on M generated by the 1-parameter subgroup $\{\exp t\xi : t \in \mathbb{R}\}$. Then Φ^ξ is a hamiltonian function for the vector field $\xi^\#$:

$$d\Phi^\xi = \iota_{\xi^\#}\omega$$

The theorem that we will prove in this talk is very simple to state, it concerns the image of the moment map in the conditions above. Recall that $\mathfrak{g}^*$ is a vector space (in particular, with G being a torus, its Lie algebra is $\mathfrak{g} = \mathbb{R}^n$, and the dual $\mathfrak{g}^*$ can be identified again with $\mathbb{R}^n$). Then:

Theorem. *(Atiyah-Guillemin-Sternberg)*
The image of Φ is a convex polytope, the convex hull of $\Phi(M^G)$.

This theorem was proved independently by Atiyah and by Guillemin and Sternberg, practically at the same time. In this talk we will follow the proof by Guillemin and Sternberg in the original article [G-S]. Atiyah's proof can be found, for example, in [McD-S].

The proof will be divided in three steps:

1. Equivariant Darboux Theorem

   In which we state but do not prove a theorem to be used in the next two steps. This theorem appears in [W] and a sketch of the proof is also given in [G-S].

2. Local Convexity

   In which we show that the image under the moment map of a neighbourhood of a fixed point $p \in M^G$ is convex.

3. Global Convexity

   In which we show that the image of the moment map is indeed convex, unsing results from Morse theory, some of which we will prove and some which we will not.

1 Equivariant Darboux Theorem

For $p \in M^G$ a fixed point, we have $\alpha_{1,p}, \alpha_{2,p}, \dots, \alpha_{d,p} \in \mathfrak{g}^*$ weights of the isotropy representation of G on the tangent space T_pM.

The equivariant Darboux theorem[1] states that there is a G-equivariant neighbourhood $U \subset M$ centered at p and coordinates $z_1, \dots, z_d$ such in which the symplectic form can be written as

$$\omega = \frac{1}{2i} \sum_{k=1}^{d} dz_k \wedge d\bar{z}_k$$

and the action τ becomes the linear action of G on $\mathbb{C}^d$ with weights $\alpha_{1,p}, \alpha_{2,p}, \dots, \alpha_{d,p}$. We remark that all other points in U^G have the same weights.

We claim that the moment map at $q \in U$ can be written in these coordinates as

$$\Phi(q) = \tilde{\Phi}(z) = \tilde{\Phi}(0) + \sum_{k=1}^{d} \alpha_{k,p} \frac{|z_k|^2}{2}$$

where $\tilde{\Phi}(0) = \Phi(p)$.

We take a moment here to convince ourselves that this claim is indeed true: The linear action of $\mathbb{S}^1$ on $(\mathbb{C}, \frac{1}{2i}\, dz \wedge d\bar{z} = r\, dr \wedge d\theta)$ is

$$\theta \cdot z = e^{i\theta} z$$

The moment map will be

$$\Phi : \mathbb{C} \to \mathrm{Lie}(\mathbb{S}^1)^* = \mathbb{R}^* \cong \mathbb{R}$$

such that

$$d\Phi = \iota_{\frac{\partial}{\partial\theta}} (r\, dr \wedge d\theta) = r\, dr = d(r^2)$$

[1] See Alan Weinstein's *Lectures in Symplectic Geometry*.

Thus,

$$\Phi(z) = \frac{|z|^2}{2} + \text{constant}$$

Furthermore, the linear action of $\mathbb{S}^1$ on $\mathbb{C}$ with weight α is

$$\theta \cdot z = e^{i\alpha\theta} z$$

in which case it is easy to see that the moment map becomes

$$\Phi(z) = \alpha \frac{|z|^2}{2} + \text{constant}$$

Lastly, the linear action of the n-torus $\mathbb{T}^n$ on $\mathbb{C}$ with weight $\alpha \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$, $\alpha \cong (\alpha^{(1)}, \ldots, \alpha^{(n)})$ is

$$(\theta_1, \ldots, \theta_n) \cdot z = e^{i(\alpha^{(1)}\theta_1 + \ldots + \alpha^{(n)}\theta_n)} z$$

and in this case we obtain

$$d\Phi^{\frac{\partial}{\partial \theta_k}} = \alpha^{(k)} d(r^2)$$

so

$$\Phi(z) = \alpha \frac{|z|^2}{2} + \text{constant} \in \mathbb{R}^n$$

The result for $\mathbb{T}^n$ acting on $\mathbb{C}^d$ follows easily from this one.

2 Local Convexity

Consider a fixed point $p \in M^G$, a neighbourhood U and coordinates $z_1, \ldots, z_d$ with the properties given by the Equivariant Darboux Theorem.

The image of U under the moment map will be

$$\Phi(U) = \text{Im}\tilde{\Phi} = \left\{ \tilde{\Phi}(0) + \sum_{k=1}^{d} s_k \alpha_{k,p} : s_k \geq 0 \right\} = \Phi(p) + S(\alpha_{1,p}, \ldots, \alpha_{d,p})$$

where

$$S(\alpha_{1,p}, \ldots, \alpha_{d,p}) = \left\{ \sum_{k=1}^{d} s_k \alpha_{k,p} : s_k \geq 0 \right\} \subset \mathfrak{g}^*.$$

The primary aim of this step is fulfilled, as we have shown that the image under the moment map of a neighbourhood of a fixed point $p \in M^G$ is a cone with vertex $\Phi(p)$, but we will need a relative form of the result above in the following step, so we will prove it here:

Let $p \in M$ not necessarily a fixed point, and let $H \subset G$ be the stabilizer group of $p \in M^H$. We can think of the action of H on M that is simply a restriction of the action τ and apply the result above to this setting. The moment map Φ_H is obtained from Φ by composing with the linear mapping $\pi : \mathfrak{g}^* \to \mathfrak{h}^*$ induced by the inclusion of H in G:

$$\Phi_H = \pi \circ \Phi : M \to \mathfrak{g}^* \to \mathfrak{h}^*$$

Then

$$\Phi_H(U) = \Phi_H(p) + S_H(\alpha_{1,p}, \dots, \alpha_{d,p}) \subset \mathfrak{h}^*$$

with weights $\alpha_{k,p} \in \mathfrak{h}^*$, and

$$\Phi(U) = \pi^{-1}\left(\Phi_H(p) + S_H(\alpha_{1,p}, \dots, \alpha_{d,p})\right) = \Phi(p) + \pi^{-1}\left(S_H(\alpha_{1,p}, \dots, \alpha_{d,p})\right).$$

The notation will be

$$S'(\alpha_{1,p}, \dots, \alpha_{d,p}) := \pi^{-1}\left(S_H(\alpha_{1,p}, \dots, \alpha_{d,p})\right) \subset \mathfrak{g}^*.$$

3 Global Convexity

The desired result of global convexity follows easily from local convexity together with the following lemma:

Lemma. *For any $\xi \in \mathfrak{g}$, the function $\Phi^\xi : M \to \mathbb{R}$ has a unique local maximum.*

We will first see how to prove the convexity theorem, and then proceed to the proof of this lemma.

Let $x \in \mathfrak{g}^*$ be a point in the boundary of the image of the moment map, $p \in M$ be a pre-image of x, $H \subset G$ the stabilizer of p and $\alpha_{1,p}, \dots, \alpha_{d,p} \in \mathfrak{h}^*$ the corresponding weights. Then

$$\Phi(U) = x + S'(\alpha_{1,p}, \dots, \alpha_{d,p}).$$

Let S_k be a boundary component of $S'(\alpha_{1,p}, \dots, \alpha_{d,p})$. Since S_k is at least codimension 1, we can choose $\xi \in \mathfrak{g}$ such that $l_\xi \equiv 0$ on S_k and $l_\xi < 0$ on the interior of $S'(\alpha_{1,p}, \dots, \alpha_{d,p})$ (here $l_\xi = \langle \xi, \cdot \rangle$). Then, if $l_\xi(x) = a$, we have for all $q \in U$

$$\Phi^\xi(q) = (l_\xi \circ \Phi)(q) \leq a$$

which implies that a is a local maximum of Φ^ξ. By the lemma above, it is in fact an absolute maximum, so $\Phi^\xi(M) \leq a$.

Applying this argument to all faces S_k of $S'(\alpha_{1,p}, \dots, \alpha_{d,p})$ we conclude that $\Phi(M)$ sits inside the cone

$$\Phi(M) \subset x + S'(\alpha_{1,p}, \dots, \alpha_{d,p}).$$

So we have proved that $\Phi(M)$ behaves like a convex set relative to its boundary, which implies that is it a convex set, which finishes the proof of the theorem.

Now, we will prove the lemma using Morse theory.

Definition. *A smooth function $f : M \to \mathbb{R}$ is Morse-Bott if each connected component of the critical set of f, C_f, is a submanifold of M and if at each critical point $p \in C_f$ the Hessian Hess f_p is nondegenerate in the directions normal to C_f at p.*

We define the index of Hess f_p to be

$$(i_-, i_+) = (\# \{\textit{negative eigenvalues}\}, \# \{\textit{positive eigenvalues}\})$$

If f is Morse-Bott then the index of Hess f_p is constant along each connected component C of C_f and it is called the index of the critical set C.

Equipping M with a Riemann metric, f defines a gradient vector field ∇f on M. The flow generated by this vector field is $\varphi_t : M \to M$, $t \in \mathbb{R}$, and this allows us to define for each component C_i of C_f the stable manifold

$$W_i = \{p \in M : \varphi_t(p) \to C_i \text{ as } t \to +\infty\}.$$

An important result in Morse theory is the following:

Theorem. *If f is Morse-Bott then each W_i is a fibre bundle with fibre a i_--cell over C_i, so*

$$\dim(W_i) = i_- + \dim(C_i),$$

and M is given as a disjoint union of these W_i,

$$M = \bigcup_i W_i$$

Corollary. *If $f : M \to \mathbb{R}$ is Morse-Bott and the index i_- of all critical manifolds of f is even, then f attains a unique local maximum.*

Proof. Let $C_1, \ldots, C_k$ be the critical manifolds of local maxima, $f \equiv a_i$ on C_i, and let $C_{k+1}, \ldots, C_N$ bet the remaining critical manifolds. We make two remarks:

One is that by definition of W_i, the stable manifolds corresponding to local maxima must be $2d$-dimensional, and so $W_i, \ldots, W_k$ are open subsets of M.

The other is that by the nondegeneracy condition on the Hessian, the codimension of W_i is exactly i_-. Now, stable manifolds not corresponding to local maxima must have $i_- > 0$ and hence, by hypothesis, $i_- \geq 2$. So the manifolds $W_{k+1}, \ldots, W_N$ have codimension ≥ 2.

But a manifold of codimension ≥ 2 cannot disconnect M, so $M - \bigcup_{i>k} W_i$ is connected but also it is $\bigcup_{i \leq k}$, a disjoint union of k open sets, so we must have $k = 1$, a unique local maximum. ☐

Finally, we are left only with showing that all components of the moment map, Φ^ξ, are in the conditions of the corollary above.

Theorem. *For any $\xi \in \mathfrak{t}$, the function Φ^ξ is Morse-Bott and all its critical manifolds have even i_- index.*

Proof. Let $\xi \in \mathfrak{g}$ and $p \in C_{\Phi\xi}$. Then p is a fixed point for the action of the 1-parameter subgroup $\{\exp -t\xi : t \in \mathbb{R}\}$ and we apply the equivariant Darboux theorem for the action of this subgroup on M. Let $\alpha_{1,p}(\xi), \ldots, \alpha_{d,p}(\xi)$ be the weights of the isotropy representation of $\{\exp -t\xi : t \in \mathbb{R}\}$ on T_pM[2].

The moment map for this action is $\Phi_{\{\exp -t\xi : t\in\mathbb{R}\}} = \pi \circ \Phi$, π induced by the inclusion of $\{\exp -t\xi : t \in \mathbb{R}\}$ in G, as in the end of section 2. But this is exactly the ξ-component of the moment map for the G-action, Φ^ξ.

The equivariant Darboux theorem then tells us that in local coordinates,

$$\Phi^\xi(q) = \Phi^\xi(p) + \sum_{k=1}^{d} \alpha_{k,p}(\xi)\, |z_k|^2$$

We can assume that for some $0 \le j \le d$, $\alpha_{j+1,p}(\xi) = \ldots = \alpha_{d,p}(\xi) = 0$. Now it's easy to see from the formula above that

$$C_{\Phi^\xi} \cap U \cong \left\{(0, \ldots, 0, z_{j+1}, \ldots, z_d) \in \mathbb{C}^d\right\}$$

so C_{Φ^ξ} is a $2(d-j)$-dimensional submanifold of M.

Furthermore, the Hessian of Φ^ξ at p is a diagonal matrix

$$\begin{pmatrix} \alpha_{1,p}(\xi) & & & & & & & \\ & \alpha_{1,p}(\xi) & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_{j,p}(\xi) & & & & \\ & & & & \alpha_{j,p}(\xi) & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

so i_- is twice the number of negative $\alpha_{k,p}$'s and therefore an even number. □

4 Two examples

The circle $\mathbb{S}^1$ acts on the sphere $(\mathbb{S}^2, d\theta \wedge dh)$ by rotation, and the moment map is simply the height function, $\Phi = h$:

$$d\Phi = \iota_{\frac{\partial}{\partial\theta}}(d\theta \wedge dh) = dh$$

The image of the moment map is the interval $[-1, 1]$, which is the convex hull of the images of the two fixed points:

$$\Phi(\text{North pole}) = 1$$
$$\Phi(\text{South pole}) = -1$$

[2]The choice of notation here is not innocent. In fact, these correspond to the evaluation at ξ of the weights of the G-action $\alpha_{1,p}, \ldots, \alpha_{d,p}$.

Another example is the $\mathbb{T}^n$ action on $\mathbb{C}P^n$ given by

$$(\theta_1, \ldots, \theta_n) \cdot [z_0; \ldots; z_n] = \left[z_0; e^{i\theta_1} z_1; \ldots; e^{i\theta_n} z_n\right]$$

The moment map for this action is

$$\Phi([z_0; \ldots; z_n]) = \left(\frac{|z_1|^2}{|z_0|^2 + \ldots + |z_n|^2}, \ldots, \frac{|z_n|^2}{|z_0|^2 + \ldots + |z_n|^2}\right)$$

The fixed points are $[1; 0; \ldots; 0], [0; 1; \ldots; 0], \ldots, [0; \ldots; 0; 1]$ and they map to

$$\begin{array}{c} \Phi([1; 0; \ldots; 0]) = (0, 0, \ldots, 0) \\ \Phi([0; 1; \ldots; 0]) = (1, 0, \ldots, 0) \\ \Phi([0; \ldots; 0; 1]) = (0, \ldots, 0, 1) \end{array}$$

The convex hull of the images of these points is exactly the moment polytope $\mathrm{Im}(\Phi)$, the simplex

$$\{x \in \mathbb{R}^n : x_1 + \ldots + x_n \leq 1 \text{ and } x_i \geq 0 \text{ for all } i \ \}$$

References

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