RESEARCH STATEMENT

AHMAD RAFIQI

My research in mathematics is in the general fields of topology, topological dynamics and hyperbolic geometry. More specifically I've been working on problems related to surface homeomorphisms and their 'stretch-factors', as well as hyperbolic 3-manifolds that fiber over the circle.

A self-homeomorphism of a compact connected surface S has a real number associated to it called the 'dilatation' or the 'stretch factor'. For homeomorphisms that fix no collection of simple closed curves up to homotopy (*pseudo-Anosov* maps), the dilatation measures the average of how much the surface is locally being stretched by the homeomorphism. It is also the exponent of the growth rates of lengths of curves on the surface. A consequence of Thurston's classification of surface homeomorphisms is that for pseudo-Anosov maps, this growth rate neither depends on the representative homeomorphism in the given mapping class, nor the curve you choose to iterate, nor the metric of constant curvature you place on the surface to measure the lengths of the curves. It is thus a topological invariant of the mapping class. Finally, the mapping class group of the surface acts properly discontinuously by precomposition of marking on the Teichmüller space $\mathcal{T}(S)$ of marked complex structures on S. The quotient is the Moduli space $\mathcal{M}(S)$ of complex structures on S, and logarithms of dilatations of pseudo-Anosov maps are the lengths of closed geodesics in Moduli space under the Tecihmüller metric.

Past research

Constructing pseudo-Anosov maps with given dilatations:
Fried [6] proved that if λ is the dilatation associated to the homeomorphism of a compact surface then both λ and λ^{-1} are roots of monic polynomials over the integers (so λ is an algebraic unit) and that all its Galois conjugates (excepts perhaps λ^{-1}) lie in the open annulus $\{z \in \mathbb{C} : 1/\lambda < |z| < \lambda\}$. Numbers satisfying these properties are called *biPerron* units. In his paper, Fried conjectured that all biPerron units are dilatations of surface homeomorphisms and not just *vice-versa* as he has proved. However this conjecture has turned out be rather difficult and there are many related works. Thurston [17] gave a partial answer to the $\mathrm{Out}(F_n)$-version of this question (see [17] Theorem 1.8).

There are many constructions of pseudo-Anosov maps, see [1], [9], [10], [11], [12], [13], [15], [16] for instance. Thurston [17] gave a few examples of constructions of pseudo-Anosov maps for given bi-Perron algebraic units. He constructed post-critically finite piecewise-linear maps of the unit interval with $\pm\lambda$ as the slopes, and surfaces were constructed by computing the ω-limit set of extensions of the interval maps to the plane that stretched horizontally and shrunk vertically by λ. These examples motivated my previous work ([2], with H. Baik and C. Wu) to construct pseudo-Anosov maps with prescribed bi-Perron units as dilatations. We considered those biPerron units that were the leading eigenvalues of matrices with entries in $\{0, 1\}$ (with some additional properties, see [2]). For these we were able to construct closed orientable surfaces (with quadratic differentials coming from the shape of the matrix) and define homeomorphisms on them such that the dilatation was the fixed biPerron unit. Even though this construction gave us many new examples of pseudo-Anosov maps, it does not produce all pseudo-Anosov maps. Whether such a construction works for all biPerron units is unknown.

Typical properties of biPerron units and dilatations:
Hamenstädt [8] showed that if we fix the genus of an orientable surface S_g, and a radius R, the proportion of dilatations smaller than R on S_g that are *totally real* (that is, with only real Galois

2010 *Mathematics Subject Classification.* 37B40 37E30 37F30 54H20 .
Date: October 16, 2017.

conjugates) approaches 1 as $R \to \infty$. This came as a surprise since there seemed no reason to believe that biPerron units should be totally real, and thus suggested that Fried's conjecture may be false. It motivated my study of the asymptotic behavior of biPerron units. For its formulation, let $g \geq 2$ be fixed, and R be any positive real number. Say $\mathcal{B}_g(R)$ is the set of bi-Perron units no larger than R whose minimal polynomial has degree at most $2g$, and say $\mathcal{D}_g(R)$ is the set of dilatations no larger than R of pseudo-Anosov maps with orientable invariant foliations on S_g, (an orientable invariant foliation is guaranteed to exist if one takes a branched double cover of the surface and lifts the map, see [3]).

Eskin-Mirzakhani-Rafi [5] and Hamenstädt [7] independently showed that the number of periodic orbits of length less than $\log(R)$ for the Teichmüller flow on the moduli space of area one abelian differentials on S_g grows like $R^{4g-3}/\log(R)$ as $R \to \infty$. This is an upper bound for the asymptotics of $|\mathcal{D}_g(R)|$ since a periodic orbit of length less than $\log(R)$ corresponds to a dilatation smaller than R and multiple periodic orbits may have the same length. In [3] (with H. Baik and C. Wu) we showed that $|\mathcal{B}_g(R)|$ grows like $R^{g(g+1)/2}$ as $R \to \infty$. Since dilatations $\mathcal{D}_g(R)$ form a subset of biPerron units $\mathcal{B}_g(R)$, this together with [5] and [7] allowed us to conclude that for $g \geq 6$ the proportion of dilatations $\mathcal{D}_g(R)$ inside biPerron units $\mathcal{B}_g(R)$ approached 0 as $R \to \infty$, (Theorem 1 [3]). Note that Fried's conjecture is equivalent to $\mathcal{B}_g(R)$ being contained in $\mathcal{D}_n(R)$ for some large enough n. So although our result does not disprove Fried's conjecture, it strongly suggests that it may be incorrect.

Current research

One way to obtain more dilatations from previous ones is the Teichmüller polynomial. For compact, orientable 3-manifolds Thurston defined a seminorm on the second homology group (or by duality on the first cohomology group) which is a norm if every embedded surface representing a nonzero element of homology has negative Euler characteristic. The unit ball for the Thurston norm is a rational polytope in this case and Thurston [18] showed that for a top-dimensional face F of the unit ball, if an integral point in the open cone $\mathbb{R}_+ \cdot F$ is represented by a fibration of the 3-manifold over the circle, then all integral points in that cone are represented by fibrations. In this case, F is called a *fibered face*. For each rational point in the fibered face, one has an embedded surface for a fiber and a homeomorphism of it as the monodromy of the fibration.

McMullen [14] defined a polynomial invariant for fibered faces called the *Teichmüller polynomial*. This multivariate polynomial encodes in it all the dilatations of the monodromies of the fibrations within the fibered face. My current research is about trying to find flat structures on the different fibers within a fibered face as well as to further attempt to resolve Fried's conjecture with help from the abundance of examples coming from the Teichmüller polynomial.

Future directions

In [2] we gave a sufficient condition for the leading eigenvalue of a matrix to be a pseudo-Anosov dilatation on a closed surface. One direction would be to investigate whether surfaces of finite type could be obtained for a bigger class of biPerron units. However, if we do not require the resulting surface to be of finite-type, we can significantly weaken the conditions, and obtain pseudo-Anosov like maps. More precisely, we still have two invariant and transverse, measured foliations on the surface - except perhaps with infinitely many singularities - one expanded and the other contracted by the dilatation. If the singularities of the foliation accumulate at a finite number of points we get a generalized pseudo-Anosov map (in the sense of [4]). In general we obtain a special kind of an end-periodic map. The mapping torus of a generalized pseudo-Anosov map is a 3-manifold with a depth-1 taut foliation, and can be approximated by fibered 3-manifolds the dilatations of the monodromies of which approach the fixed biPerron unit we start with. Another direction of possible research is to investigate whether the Teichmüller polynomial can be generalized to manifolds where the Thurston "norm" is only a seminorm. Experiments with examples suggest that such a definition should be possible.

References

[1] Arnoux, P., Yoccoz, J.(1980). *Construction de difféomorphisms pseudo-Anosov*, C. R. Acad. Sci. Paris 292, 75–78.

[2] Baik, H., Rafiqi, A. and Wu, C. (2016). *Constructing pseudo-Anosov maps with given dilatations*, Geom. Dedicata, Volume 180, Issue 1, 39–48.

[3] Baik, H., Rafiqi, A., & Wu, C. (2016), *Is a typical bi-Perron algebraic unit a pseudo-Anosov dilatation?* Accepted: Ergodic Theory and Dynamical Systems.

[4] de Carvalho, A. and Hall, T. (2004). *Unimodal generalized pseudo-Anosov maps.* Geom. Topol., 8, 1127–1188.

[5] Eskin, A., Mirzakhani, M., Rafi, K. *Counting closed geodesics in strata*, arXiv:1206.5574.

[6] Fried, D. (1985). *Growth rate of surface homeomorphisms and flow equivalence.* Ergod. Theory Dyn. Syst., 5(04), 539–563.

[7] Hamenstädt, U. (2013). *Bowen's construction for the Teichmüller flow*, J. Mod. Dynamics 7, 489–526.

[8] Hamenstädt, U. *Typical properties of periodic Teichmüller geodesics*, arXiv:1409.5978.

[9] Hironaka, E. (2010). *Small dilatation mapping classes coming from the simplest hyperbolic braid*, Algebr. Geom. Topol., 10(4):2041–2060.

[10] Lanneau, E., Thiffeault, J. (2011). *On the minimum dilatation of braids on punctured discs*, Geom. Dedicata, 152:165–182.

[11] Lanneau, E., Thiffeault, J. (2011). *On the minimum dilatation of pseudoAnosov homeromorphisms on surfaces of small genus*, Ann. Inst. Fourier (Grenoble), 61(1):105–144.

[12] Leininger, C. (2004) *On groups generated by two positive multi-twists: Teichmuller curves and Lehmer's number*, Geom. Topol., 8:1301–1359.

[13] Long, D. (1985). *Constructing pseudo-Anosov maps*, In Knot theory and manifolds (Vancouver, B.C., 1983), volume 1144 of Lecture Notes in Math., 108–114. Springer, Berlin.

[14] McMullen, C. T. *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations.* Annales scientifiques de l'Ecole normale suprieure. Vol. 33. No. 4. 2000.

[15] Penner, R. (1988). *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer.Math. Soc., 310(1):179–197.

[16] Shin, H., Strenner, B. (2015). *Pseudo-Anosov mapping classes not arising from Penner's construction*, Geom. Topol. 19, 3645–3656.

[17] Thurston, W. (2014). *Entropy in dimension one.* arXiv:1402.2008.

[18] Thurston, W. P. *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. 59 (1986), no. 339, ivi and 99130

Department of Mathematics, Cornell University, 310 Malott Hall, Ithaca, NY 14853-4201 USA

E-mail address: ar776@cornell.edu