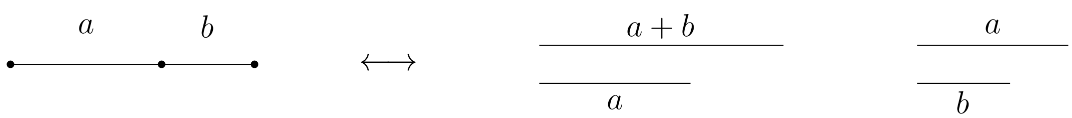
Φ: The Golden Ratio

The golden ratio is the number $\Phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618033989$. (The greek letter Φ used to represent this number is pronounced "fee".)

Where does the number Φ come from?

Suppose a line is broken into two pieces, one of length a and the other of length b (so the total length is $a+b$), and a and b are chosen in a very specific way: a and b are chosen so that the ratio of $a+b$ to a and the ratio of a to b are equal.

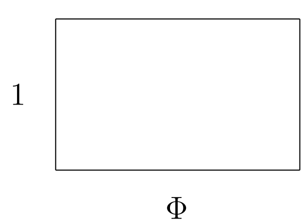


It turns out that if a and b satisfy this property so that $\frac{a+b}{a} = \frac{a}{b}$ then the ratios are equal to the number Φ! It is called the *golden* ratio because among the ancient Greeks it was thought that this ratio is the most pleasing to the eye.

Try This! You can verify that if the two ratios are equal then $\frac{a}{b} = \Phi$ yourself with a bit of careful algebra. Let $a = 1$ and use the quadratic equation to find the value of b that makes the two ratios equal. If you successfully worked out the value of b you should find $\frac{1}{b} = \Phi$.

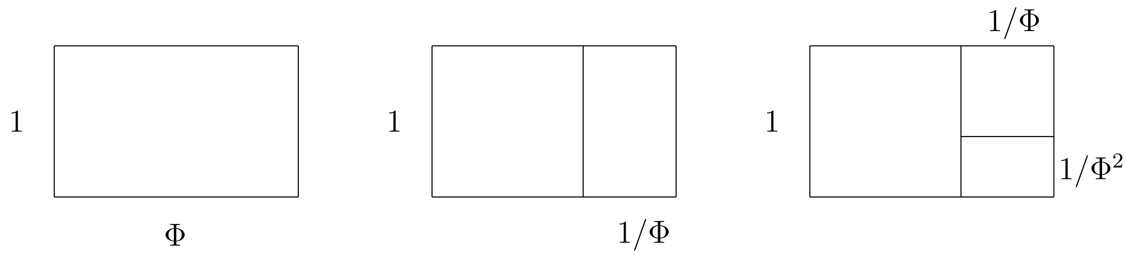
The Golden Rectangle

A rectangle is called a golden rectangle if the ratio of the sides of the rectangle is equal to Φ, like the one shown below.

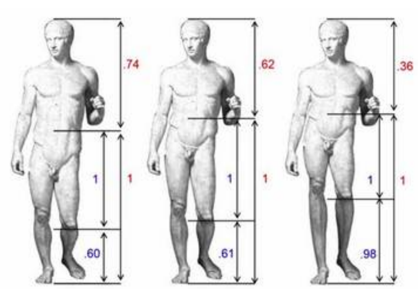


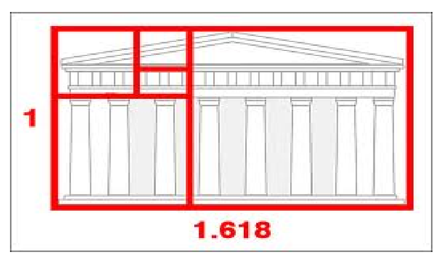
If the ratio of the sides is $\frac{1}{\Phi} = \frac{-1+\sqrt{5}}{2}$ this is also considered a golden rectangle. (Think of turning the rectangle on its side.)

It is possible to split up a golden rectangle so that it contains a smaller golden rectangle. Take the golden rectangle shown above and draw a vertical line splitting it up into a 1 by 1 square and a rectangle. The resulting rectangle has dimensions $\Phi - 1$ by 1. Since $\Phi - 1 = 1/\Phi$, the ratio of the sides of the rectangle is $1/\Phi$, so the smaller rectangle is also golden. If the smaller rectangle is then split into a square and a rectangle, the smaller rectangle has dimensions $1/\Phi$ by $1 - 1/\Phi$. Since $1 - 1/\Phi = 1/\Phi^2$, this is a golden rectangle as well! This pattern continues; if the smallest golden rectangle is broken up into a square and a rectangle, the resulting rectangle will always be golden.

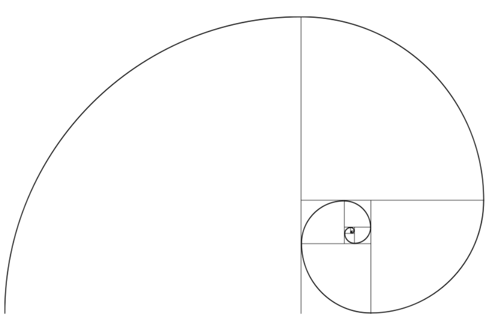


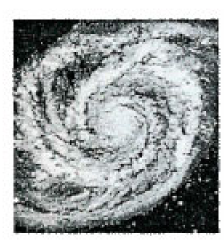
The golden ratio and golden rectangles are present in a wide array of art and architecture. The most famous example of a golden rectangle in architecture is the Parthenon of Ancient Greece.



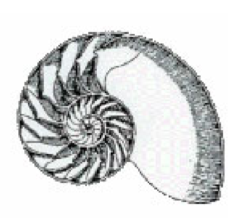


Also, if a spiral is drawn inside of a golden rectangle which has been split up into squares and smaller golden rectangles so that it crosses the corners of the smaller squares and rectangles inside, the result is the famous golden spiral, which can been see in art and nature.





Galaxy



Nautilus shell

... Φ: The Golden Ratio

Relationship to Fibonacci Numbers

The Fibonacci numbers are defined recursively, meaning the value of the n^{th} Fibonacci number depends on the value of previous Fibonacci numbers. The n^{th} Fibonacci number is denoted F_n. The values of the Fibonacci numbers are: $F_1 = 1$, $F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n = 3, 4, 5, \ldots.$$

For example, $F_3 = F_{3-1} + F_{3-2} = F_2 + F_1 = 2$, and $F_4 = F_3 + F_2 = 3$.

The **Elvis numbers** from the puzzle with the elf running up the stairs are the same as the Fibonacci numbers! To make this clear think about the very first step Elvis makes when there are n steps: If he goes up only one stair in his first step, the number of ways to climb up the rest of them is E_{n-1}, and if he skips the second stair going straight to the third in his first step, then there are E_{n-2} ways to climb the remaining stairs.

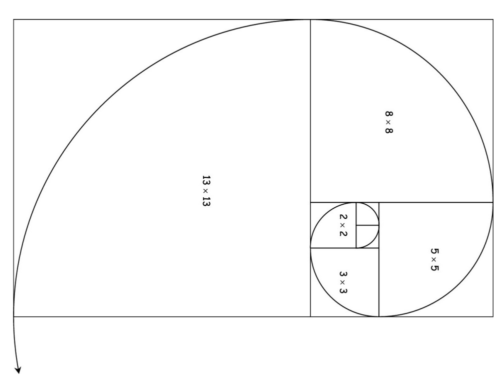
Make the connection! First, find the first 15 Fibonacci numbers. Then, write down the ratio of successive Fibonacci numbers $\frac{F_{n+1}}{F_n}$ for $n = 1, 2, \ldots, 14$.
Do you notice anything about the ratios? Are the ratios very different in value?
(Think about it before you continue.)
The interesting fact this is getting at is that as n gets larger, the ratio of successive Fibonacci numbers get closer and closer to the value Φ! Notice, the ratios are very close to Φ even when only considering the first 15 Fibonacci numbers.

The Fibonacci Spiral

Since the ratios of successive Fibonacci numbers are a close approximation to the golden ratio they can be use to create a series of squares inside of a rectangle, similar to breaking up a golden rectangle into squares and rectangles, and the spiral drawn through the corners of the squares is very close to the golden spiral.

Make your own!

1. On a piece of graph paper, trace a square and the square above it.
2. Trace a 2 by 2 square whose right edge is the left edge of the two 1 by 1 squares.
3. Trace a 3 by 3 square whose top edge shares the bottom edge of the 2 by 2 and first 1 by 1 square.



4. Trace a 5 by 5 square whose left edge shares the right of the 3 by 3 and 1 by 1 squares.
5. Trace an 8 by 8 square whose bottom edge shares the top edge of the 5 by 5, second 1 by one and 2 by 2 squares.

Do you see the pattern? Each block will fit nicely along the edge of the two previous blocks since $F_n = F_{n-1} + F_{n-2}$. It is necessary to keep rotating around as you add blocks so the shared edge is the bottom, right, top and then left edge of the new square. Once the squares are drawn, start spiralling out from the first square you drew. It should look something like the above picture when you're done. (You can continue on up to whichever Fibonacci number you like once you get the hang of it!)

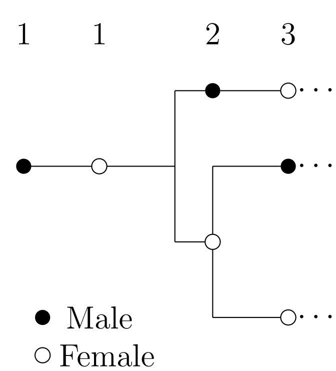
The Golden Ratio and the Fibonacci Numbers in Nature

The golden ratio and Fibonacci numbers can be found in many places in nature. For example, leaves want to be arranged so that a leaf is not blocked by the leaves above it, this way each leaf has the same access to sunlight. In many plants, leaves spiral around a stem according to the golden ratio or Fibonacci numbers.

Fibonacci himself was interested in how quickly rabbits breed, and under his simplified breeding model the number of rabbits present after each breeding season were what are now known as the Fibonacci numbers. This model was not very accurate since it assumed the rabbits never die, however the Fibonacci numbers do describe the number of parents, grandparents, great grandparents, etc of honey bees very well.

Honey Bee Family Tree There is one queen in a honey bee colony. The unfertilized eggs of the queen bee result in male worker bees and the fertilized eggs of the queen bee become female worker bees, whom are usually sterile. A female becomes a reproductive queen only if chosen to be fed royal jelly. In any case, this means that males have on parent (the queen) and female bees have two parents (the queen and a male).

Make a family tree which traces back through the ancestry of a male worker bee and note the number of ancestors in each generation. Do these numbers look familiar?



Links to More Information

These links will also be posted on the Puzzle Sessions webpage:
http://www.math.cornell.edu/Community/community.html

Ratios: http://cs.gmu.edu/cne/modules/dau/algebra/fractions/frac5_frm.html

Quadratic Equations: http://www.enchantedlearning.com/math/algebra/quadratic

Patterns and the Golden Ratio:

- http://www.mathsisfun.com/numbers/golden-ratio.html (Specifically, check out the Calculating It and Most Irrational sections.)
- http://www.miqel.com/fractals_math_patterns/visual-math-phi-golden.html
- **Short Video:** Donald Duck in Mathemagic Land
  http://www.youtube.com/watch?v=YVODhFLe0mw

The Golden Ratio and Fibonacci Numbers in Nature:

- http://www.mathsisfun.com/numbers/nature-golden-ratio-fibonacci.html
- http://www.miqel.com/fractals_math_patterns/visual-math-phi-golden.html
- **Fibonacci's Rabbits:**
  http://scramble.hubpages.com/hub/Fibonacci-Numbers-Golden-Ratio-In-Nature