

Topology, Binary Numbers & Puzzle Games

TOPOLOGY

Topology is a branch of mathematics that is concerned with invariant (unchanging) properties of different objects. It is known as rubber geometry because topologists like to think about what properties shapes still have after they are stretched out and moved around. A topologist looks at shapes differently than you and I do in day to day life. For example, I'm sure we would all say a triangle and circle are two different shapes. However, a topologists would these shapes are the same.

Why? Imagine a triangle made of silly putty is sitting on top of a piece of paper, which will represent a plane. We can easily smooth it into a circle and it still divides the paper into two regions, one inside the shape and the other outside the shape, so it is in this sense that the shapes are the same. This is also an example of an invariant property: even if the triangle is stretched and pushed around so it no longer has sharp angles, or has four angles instead of only three, it still has one hole and thus divides our paper into two regions. However, we are not allowed to break the silly putty or pinch it together. So, no matter how we stretch or bend our putty triangle the resulting shape will always have the property that it has one hole, and separates the paper into two regions. Furthermore, this also tells us that a triangle, circle and rectangle are all the same, but are topologically distinct from a figure eight because it has two holes and divides the paper into three regions, no matter how we bend and stretch it.

Inside or Outside?

One of the first theorems a topologist will learn is the **Jordan Curve Theorem**:

Any closed and simple (doesn't cross itself) curve in the plane, such as a triangle or circle, divides the plane into two regions, one inside the curve and one outside the curve.

Seems obvious, right? The truth is though, it is not obvious how to prove it using rigorous mathematics. (Which is why it was worthy of the label "theorem" when Jordan figured out how to do so!)

Topology and Sprouts.

How many distinct first moves are there in the two dot game of sprouts?

Topologically, there are only two distinct opening moves. If the two dots are connected by a line, the plane still only has one region. If instead Player 1 decides to draw a

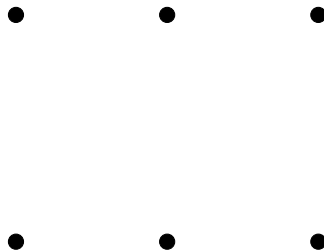
loop connecting a dot to itself this creates a closed curve and thus splits the plane into an inside and outside region. It does not matter which of the two dots the loop emanates from, nor does it matter if the second dot is inside or outside the loop.

The equivalent positions in the three dot game shown on the puzzle sheet are of course topologically equivalent as well, and each is equivalent to the figure eight.

Isolating Points.

If a line is drawn connecting two dots and this line creates a closed curve either by itself (i.e. it's a loop) or by connecting to an existing line creating a closed curve, the Jordan curve theorem guarantees one of the existing regions of the plane is separated into two regions. You no doubt noticed this when you were playing. Since lines cannot be drawn through existing lines, the idea of inside and outside is significant as it allows one to isolate dots, thus restricting the possibilities one's opponent has on her next turn.

The Utilities Puzzle. The Jordan curve theorem is related to another well known puzzle: Three houses each need to be hooked up to the water, gas and electricity plants. The catch is none of the pipelines can intersect. Can you figure out a way to connect each home to each of the three utilities?



The Jordan curve theorem can be used to show that there is no way to solve this problem! Try to think about how considering the idea of inside and outside can be used to prove there is no way to draw the nine necessary lines without intersection.

You can check the Archimedes Lab webpage to see if you were on the right track:

http://www.archimedes-lab.org/How_to_Solve/Water_gas.html.

BINARY NUMBERS

In the binary number system, every number can be represented using only the two digits 0 and 1. (Remember the prefix bi- means two.) One can think of this in terms of false, or off, and true, or on. This is actually how your computer thinks; every circuit in the computer gets assigned either the digit 0 if it is off, or 1 if it is on. So, if you have studied computer science you are probably already familiar with the binary number system, but it will be discussed here for those of us who are not. Before we describe how to represent numbers in the binary system, let's review our usual number system.

Base-10 Number System. In grade school we all learn how to count and represent numbers in a base-10 system. It is called base-10 because each number can be represented using 10 digits: 0, 1, 2, . . . , 9. If we dissect exactly what is going on with the base-10 system, it will be much easier to understand how to represent numbers in the binary (or base-2) system. Let's examine the number 1234. We say that 4 is in the ones position, 3 is in the tens position, 2 is in the hundreds position, and 1 is in the thousands position. Where does this terminology come from? Notice that a natural way to think of this number is as follows.

$$1234 = \underline{1}000 + \underline{2}00 + \underline{3}0 + \underline{4}$$

We have broken 1234 into a sum of numbers, which are each a multiple of a power of 10. That is

$$\begin{aligned} 1234 &= \underline{1} \times 1000 + \underline{2} \times 100 + \underline{3} \times 10 + \underline{4} \times 1 \\ &= 1 \times 10^4 + 2 \times 10^3 + 3 \times 10^2 + 4 \times 10^0, \end{aligned}$$

where the exponential function 10^k means 10 is multiplied k times ($10 \times 10 \times \cdots \times 10$), and by convention $10^0 = 1$. So, for example, we say that 2 is the hundreds digit because it is the digit which is multiplied by $100 = 10^2$.

Binary Numbers. Since the binary system is base-2, numbers are represented using 2 digits (0 and 1) and powers of 2. Let's start by figuring out how to represent powers of 2 in binary. (It may be helpful to refer back to the base-10 example as you work through the examples.)

$$1 = 2^0 = \underline{1} \times 2^0, \text{ so in binary } 1 = 1.$$

$$2 = 2^1 = \underline{1} \times 2^1 + \underline{0} \times 2^0, \text{ so in binary } 2 = 10.$$

$$4 = 2^2 = \underline{1} \times 2^2 + \underline{0} \times 2^1 + \underline{0} \times 2^0, \text{ so } 4 = 100.$$

Now, let's write a few more numbers in binary. Remember, each power of 2 is multiplied by either a 1 or 0, which indicates whether or not that particular power shows up when the number is written as a sum of powers of 2. (i.e. it is true or false that this power of 2 appears in the sum.)

Since we must write numbers as a sum of powers of 2, this table will come in handy:

k	0	1	2	3	4	5	6	7	8	9	10
2^k	1	2	4	8	16	32	64	128	256	512	1024

- $5 = 4 + 1 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 101$.
- $10 = 8 + 2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 1010$
- To write 77 in binary, notice that 64 is the greatest power of 2 which is less than or equal to 77. We can start the process of writing 77 as a sum of powers of 2 by writing $77 = 64 + 13$. Next, we look for a greatest power of 2 which is less than or equal to 13, which is 8, and then continue the process: $77 = 64 + 8 + 5 = 64 + 8 + 4 + 1$. Thus, the binary representation of 77 is 1001101.
- $122 = 64 + 32 + 16 + 8 + 2 = 1111010$.

You try. Write each of the following as a binary number.

- a) 6 b) 23 c) 964 d) 1234

Analyzing Nim. When playing Nim, you may have noticed that if the configuration after your turn was such that there were pennies in only two of the rows and that these rows have an equal number of pennies, you were in a winning position. Why is this? If there are an equal number of pennies, all you have to do is match what your opponent does on the next turn so that there are still an equal number of pennies in each row on your opponent's next turn. This ensures that when your opponent takes the last penny in a row there will be one last penny for you to take in the other row. It is therefore a good strategy (no matter how many pennies the game starts with) to try to reach this configuration.

Connection to Binary Numbers. If there are an equal number of pennies in two rows and the binary representation of the number of pennies in each row are summed, the result will be a number with even digits in every column. (Note the sum will not be the actual binary representation of the sum.) This is the case because there can only be a 0 or 1 in each column; if the column has two zeros the result will be a zero, and if it has ones, the result will be 2. For example, if there are six pennies

in two rows, we have

$$\begin{array}{r} 6 : 110 \\ 6 : 110 \\ \hline 220 \end{array}$$

How is this useful in **creating a strategy**? For now let's still consider a situation with pennies in only two rows. Suppose there are 7 and 5 pennies. Then the binary sum looks like

$$\begin{array}{r} 7 : 111 \\ 5 : 101 \\ \hline 212 \end{array}$$

which has an odd digit. Suppose we choose our move so that the binary sum of the remaining number of pennies in the two rows will have all even digits. If we take away two pennies from the top row at our turn, the 1 in the middle column becomes a 0 and the sum changes to 202, which has all even digits. This is good for us because it also means there are 5 pennies in each row on our opponents turn so we are sure to win!

This generalizes to situations when there are pennies in more than two rows: if the binary sum on our turn contains odd numbers and we can make a move such that the remaining number of pennies has only even digits, we are in a winning position. For example, the binary sum at the start of the game with a 2-3-4 configuration is

$$\begin{array}{r} 2 : 010 \\ 3 : 011 \\ 4 : 100 \\ \hline 121. \end{array}$$

This has two odd digits. Notice we must take pennies from the bottom row in order to get an even number, namely 0, in the left most column of the sum. If the remaining number of pennies in the bottom row is odd, there will be a 1 in the right most column of the binary representation and the sum of that column will be even. However, if we take only 1 away so that there are 3 remaining, there will be an additional 1 in the middle column of the binary representation so the sum of that column would become odd. The goal is to to have all even digits, so the best move would be to remove 3

pennies from the bottom row:

2 : 0 1 0

3 : 0 1 1

1 : 0 0 1

0 2 2.

Now, no matter what move the opponent makes the binary representations will differ in at least one column, which means there will be at least one odd number. On our turn, we can remove pennies so that the sum has all even digits. If we continue in this manner we are guaranteed to win!

Recall the winning configurations you thought of while playing the game. Write them out in binary, as done above, and verify that the winning configurations have all even digits. Try playing with more pennies and see if you can find a winning strategy using binary numbers.

LINKS TO MORE INFORMATION

Topology:

- Good basic introduction: (you can click on this at the Puzzle Night website)
<http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Estes/unit/dayten/topology1.html>
- Mbius strip (this is fun):
http://www.themathwebsite.com/Classics/Classic_Mbius_Strip.php

Jordan Curve Theorem and Utilities:

- http://www.archimedes-lab.org/How_to_Solve/Water_gas.html
- http://www.cut-the-knot.org/do_you_know/3Utilities.shtml

Binary Number System: <http://www.unm.edu/~tbeach/terms/binary.html>

Analysis of Nim:

- <http://www.cs4fn.org/binary/nim/nimsecret.php>
- <http://library.thinkquest.org/06aug/01132/Gametreepage.htm>