Schubert Polynomials and Flow Polytopes

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Based on joint works with A. Fink, R. Liu, and K. Mészáros
Roadmap

- Schubert Polynomials
  - Zero-One Polynomials
  - Flow Polytopes
    - Right-Degree Polynomials
      - Schubert Polynomials
        - Gelfand-Tsetlin Analogues
        - Newton Polytopes
        - Saturation
        - Zero-One Polynomials
We will only be considering flow polytopes with a single source and sink. Start with a graph $G$. 

\begin{center}
\begin{tikzpicture}
  \fill (0,0) circle [radius=5pt];
  \fill (1,0) circle [radius=5pt];
  \fill (2,0) circle [radius=5pt];
  \draw (0,0) -- (2,0);
\end{tikzpicture}
\end{center}
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Fix an acyclic orientation of $G$. 
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Fix an acyclic orientation of $G$.

*In this talk edges drawn without orientations should be assumed to be oriented left to right.
Add a source $s$ and a sink $t$ connected to all the original vertices of $G$. Call the new graph $\tilde{G}$.
Add a source $s$ and a sink $t$ connected to all the original vertices of $G$. Call the new graph $\tilde{G}$.

Assign the source $s$ netflow $1$, the sink $t$ netflow $-1$, and all other vertices netflow $0$. 
A flow on $\tilde{G}$ is an assignment of nonnegative real numbers to each edge of $\tilde{G}$ so that at every vertex, outflow minus inflow equals netflow.

The flow polytope $\mathcal{F}_{\tilde{G}}$ is the convex hull in $\mathbb{R}^{E(\tilde{G})}$ of all flows on $\tilde{G}$.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 7 & 1 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{7}{12} & \frac{1}{4}
\end{pmatrix} \in \mathcal{F}_{\tilde{G}}
$$
An Example Flow Polytope

\[ \tilde{G} \]

\[ F_{\tilde{G}} \]
Theorem (Postnikov-Stanley)

If $G$ is a graph on vertices $[0, n + 1]$,

$$\text{Vol } F_G(1, 0, \ldots, 0, -1) = K_G \left(0, d_1, \ldots, d_n, -\sum_{i=1}^{n} d_i\right)$$

where $d_i = \text{indeg}_G(i) - 1$ for each vertex $i$.

$K_G(\alpha_1, \ldots, \alpha_n)$ is the Kostant partition function from representation theory. It equals the number of ways to write $\alpha$ as a sum of the positive roots $\{e_i - e_j : (i, j) \in G\}$. 
Subdividing Flow Polytopes

Flow polytopes can be subdivided combinatorially by performing a sequence of changes to the original graph.

A **reduction** on a graph $G$ is a construction of two new graphs $G_1$ and $G_2$ from a choice of two adjacent edges $(i, j), (j, k) \in G$:

![Diagram showing the reduction process](image)
Subdividing Flow Polytopes

\[ G = P_4 \]

*Not technically a picture of \( \mathcal{F}_{\tilde{G}} \), but the root polytope of \( G \).
Subdividing Flow Polytopes
Subdividing Flow Polytopes
Subdividing Flow Polytopes
More compactly, this subdivision procedure can be represented by a reduction tree.
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Subdividing Flow Polytopes

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The individual graphs appearing in a reduction tree depend on the choice of cuts used to subdivide the flow polytope.
Are there subdivision invariants?

On the one hand, we have seen the leaves of a reduction tree are dependent on choices made.

On the other hand, the simplices produced by the reduction process are always unimodular, so the number of leaves in any reduction tree is always the normalized volume of the flow polytope regardless of any choices.

**Question**

Is there any stronger invariant across all the different ways to fully subdivide a flow polytope using reductions?
Is there an invariant of different subdivisions of a flow polytope?
Subdivisions to Degree Sequences

Is there an invariant of different subdivisions of a flow polytope?
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Subdivisions to Degree Sequences
Is \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\} dependent only on the original graph?
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\textbf{Theorem (Grinberg 2017, Mészáros-S. 2017)}

\textit{Yes!}
Right-Degree Sequences

Definition

For a graph $G$, let $RD(G)$ denote the multiset of right-degree sequences of the leaves in any reduction tree of $G$.

$$RD(G) = \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\}.$$
Define the **right-degree polynomial** of $G$ by

$$R_G(x) = \sum_{\alpha \in RD(G)} x^\alpha.$$ 

$RD(P_4) = \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\}$

$R_{P_4} = x_1^3 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3$
Roadmap

- Schubert Polynomials
  - Flow Polytopes
    - Right-Degree Polynomials
      - Schubert Polynomials
        - Gelfand-Tsetlin Analogues
        - Newton Polytopes
        - Saturation
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Geometrically, Schubert polynomials arise as distinguished representatives of the cohomology classes of the Schubert varieties in the flag variety of $\mathbb{C}^n$. 
Schubert Polynomials (combinatorially)

Schubert polynomials can be described using divided difference operators $\partial_i$:

$$\partial_i f = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}.$$  

If $w_0$ denotes the longest permutation $n(n-1)\cdots 21$ and $s_i$ the transposition swapping $i$ and $i+1$,

$$S_w = \begin{cases} 
  x_1^{n-1}x_2^{n-2}\cdots x_{n-1} & \text{if } w = w_0, \\
  \partial_i S_{ws_i} & \text{if } w \neq w_0, w(i) < w(i+1).
\end{cases}$$
Schubert Polynomials of $S_3$

\[
\partial_1(x_1^2 x_2) = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2
\]

\[
\partial_2(x_1 x_2) = \frac{x_1 x_2 - x_1 x_3}{x_2 - x_3} = x_1
\]

\[
\partial_1(x_1) = \frac{x_1 - x_2}{x_1 - x_2} = 1
\]

In fact,
\[
\mathcal{S}_w = \partial_{i_1} \cdots \partial_{i_k} (\mathcal{S}_{w_0})
\]

for any reduced decomposition $w^{-1} w_0 = s_{i_1} \cdots s_{i_k}$
Theorem (Lascoux-Schützenberger 1982)

Let \( w \neq \text{id} \) and \( i \) be the last descent of \( w \). Set 
\[
    j = \max\{k \mid w_k < w_i\} \quad \text{and} \quad v = w(ij).
\]
Then 
\[
    S_w = x_i S_v + \sum_{k < i} S_{v(ki)}. 
\]
Theorem (Lascoux-Schützenberger 1982)

Let \( w \neq \text{id} \) and \( i \) be the last descent of \( w \). Set 
\[ j = \max\{ k \mid w_k < w_i \} \] and \( v = w(ij) \). Then 
\[
S_w = x_i S_v + \sum_{k < i \atop v \preceq v(ki)} S_{v(ki)}.
\]

Corollary (Lascoux-Schützenberger 1982)

The coefficients of \( S_w \) are nonnegative integers.
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\[
\mathcal{S}_w = x_i \mathcal{S}_v + \sum_{k<i, v \preceq v(ki)} \mathcal{S}_{v(ki)}.
\]

Corollary (Lascoux-Schützenberger 1982)

The coefficients of \( \mathcal{S}_w \) are nonnegative integers.

What is the combinatorial meaning of the coefficients?
Compatible Sequences

Theorem (Billey-Jockusch-Stanley, 1993)

\[ \mathcal{S}_w = \sum_{\rho \in \mathcal{R}(w)} \sum_{\alpha \in C(\rho)} x_{\alpha_1} \cdots x_{\alpha_{l(w)}} \]

- $\mathcal{R}(w)$ is the set of reduced words for $w$
- For each reduced word $\rho$, $\alpha \in C(\rho)$ if
  - $\alpha$ is weakly increasing;
  - $\alpha_j \leq \rho_j$ for each $j$;
  - $\alpha_j < \alpha_{j+1}$ whenever $\rho_j < \rho_{j+1}$.

For example, $1432 = s_3s_2s_3$ is a reduced word and $(1, 1, 2) \in C(s_3s_2s_3)$. 
Compatible Sequences

Theorem (Billey-Jockusch-Stanley, 1993)

\[ S_w = \sum_{\rho \in \mathcal{R}(w)} \sum_{\alpha \in C(\rho)} x_{\alpha_1} \cdots x_{\alpha_{\ell(w)}} \]

For \( w = 1432 \):

<table>
<thead>
<tr>
<th>Reduced Word</th>
<th>( s_2 s_3 s_2 )</th>
<th>( s_3 s_2 s_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compatible Sequences</td>
<td>(1, 2, 2)</td>
<td>(1, 1, 2)</td>
</tr>
<tr>
<td></td>
<td>(1, 1, 3)</td>
<td>(1, 2, 3)</td>
</tr>
<tr>
<td></td>
<td>(2, 2, 3)</td>
<td></td>
</tr>
</tbody>
</table>

\[ S_w = x_1 x_2^2 + x_2^2 x_1 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3. \]
A pipe dream for $w \in S_n$ is a tiling of an $n \times n$ matrix with crosses $\rule{1cm}{1pt}$ and elbows $\rule{0.5cm}{1pt}$ such that

- All tiles in the weak south-east triangle are elbows, and
- If you write $1, 2, \ldots, n$ on the top and follow the strands (ignoring second crossings among the same strands), they come out on the left and read $w$ from top to bottom.

A pipe dream is reduced if no two strands cross twice.

![Figure: A reduced pipe dream for $w = 2143$.]
A Graphical Interpretation of Compatible Sequences

To build a reduced pipe dream from a reduced word $\rho$ and a compatible sequence $\alpha$, put crosses in positions $(\alpha_i, \rho_i - \alpha_i + 1)$

$\rho = s_3s_2s_3$, $\alpha = (2, 2, 3)$  $\rho = s_3s_2s_3$, $\alpha = (1, 2, 3)$  $\rho = s_3s_2s_3$, $\alpha = (1, 1, 2)$

$\rho = s_2s_3s_2$, $\alpha = (1, 2, 2)$  $\rho = s_3s_2s_3$, $\alpha = (1, 1, 3)$
Theorem (Fomin-Kirillov, Bergeron-Billey 1993)

\[ S_w = \sum_{P \in \text{RPD}(w)} x^P \]

where \( x^P = \prod_{i=1}^{n-1} x_i \# \text{ crosses in row } i \text{ of } P \).
Schubert Polynomial of 1432

\[ S_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 \]
All reduced pipe dreams of $w$ can be generated from a single distinguished pipe dream by a sequence of ladder moves.
Knutson and Miller also gave a combinatorial interpretation of divided difference operators on pipe dreams.

\[
\partial_2 \left( \begin{array}{c}
\end{array} \right) = \left\{ \begin{array}{c}
\end{array} \right\},
\]

\[
\text{RPD}(w) = \partial_{i_1} \cdots \partial_{i_k} (P_0)
\]

for any reduced decomposition \( w^{-1}w_0 = s_{i_1} \cdots s_{i_k} \).
To get from subdivisions of flow polytopes to Schubert polynomials, we will focus on the leaves of a particular reduction tree and connect them to pipe dreams of certain permutations.
Remember this reduction tree?
A tree is **alternating** if it has no pair of edges.

A tree is **noncrossing** if it has no pair of edges.
Noncrossing and Alternating Trees

Theorem (Mészáros 2009)

Every tree $T$ has a canonical reduction tree whose leaves are exactly the alternating noncrossing spanning trees of the directed transitive closure $\overline{T}$ of $T$. 

$T$  \hspace{5cm} \overline{T}$
Theorem (Escobar-Mészáros 2015)

For permutations of the form $w = 1w'$ where $w'$ is dominant (132-avoiding), there is a tree $T_w$ such that the reduced pipe dreams of $w$ are in bijection with the noncrossing alternating spanning trees of the directed transitive closure $\overline{T}_w$. 
How do you construct $T_w$?
How do you construct $T_w$?
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How do you construct $T_w$?
How do you construct $T_w$?
Simplices in a subdivision of $\mathcal{F}_{\widetilde{T}}$ by reductions

Leaves in any reduction tree of $T$

Leaves in the canonical reduction tree of $T$

Noncrossing alternating spanning trees of $\widetilde{T}$

If $T = T_w$ for $w = 1w'$ with $w'$ dominant

Reduced pipe dreams of $\pi$
Pipe dreams to noncrossing alternating spanning trees
Pipe Dreams to Trees
Pipe Dreams to Trees
Theorem (Escobar-Mészáros 2015)

For permutations of the form $w = 1w'$, where $w'$ is 132-avoiding, there is a tree $T_w$ such that the right-degree polynomial $R_{T_w}$ is a reparameterization of $\mathcal{S}_w$.

\[ T_{1432} \]

\[
\mathcal{S}_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_2
\]

\[
R_{T_{1432}} = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3
\]

\[
\mathcal{S}_{1432}(x_1, x_2, x_3) = x_1 x_2^2 x_3 R_{T_{1432}}(x_1^{-1}, x_2^{-1}, x_3^{-1})
\]
Right-Degree and Schubert Polynomials

Schubert Polynomials

$1 w'$ Case

Right-Degree Polynomials

Similar properties?
Roadmap

Flow Polytopes

Right-Degree Polynomials

Schubert Polynomials

Gelfand-Tsetlin Analogues

Newton Polytopes

Saturation

Zero-One Polynomials
Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, a **semistandard Young tableaux** (SSYT) of shape $\lambda$ is a filling of the Young diagram of $\lambda$ with numbers from $[n]$ such that:

- The entries weakly increase left-to-right in each row,
- The entries strictly increase top-to-bottom in each column.

The **weight** of a SSYT $T$ is the vector $wt(T)$ whose $i$th component counts the number of $i$’s in $T$.

SSYT$(2, 1, 0)$:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
\end{array}
\]
Schur Polynomials

Definition

Given a partition $\lambda \in \mathbb{Z}^n_{\geq 0}$, the **Schur polynomial** $s_\lambda$ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x_1^{\text{wt}(T)_1} \cdots x_n^{\text{wt}(T)_n}$$

**SSYT(2, 1, 0):**

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 1 & 3 & 3 \\
2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

$s_{210}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$
Definition
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$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x_1^{\text{wt}(T)_1} \cdots x_n^{\text{wt}(T)_n}$$

\text{SSYT}(2, 1, 0) :

\begin{align*}
1 & 1 \\
2 & 2 \\
1 & 3 \\
2 & 3 \\
1 & 2 \\
3 & 3 \\
1 & 3 \\
2 & 2 \\
2 & 3
\end{align*}

$s_{210}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3 + x_2 x_3^2$
For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the \textbf{Gelfand-Tsetlin polytope} $GT(\lambda)$ is the set of all triangular arrays

\[
\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\
x_{22} & x_{23} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{nn}
\end{array}
\]

such that

\[
x_{i-1,j-1} \geq x_{i-1,j} \geq x_{ij}
\]

Integral points in $GT(\lambda)$ biject to semistandard Young tableaux of shape $\lambda$. 
Gelfand-Tsetlin Patterns and SSYT

2 1 0
2 1 2
2 1

2 1 0
2 1 1
2 1

2 1 0
2 0 2
2 1

1 1
1 2
1 3
1 1

1 1

2 1 0
2 0 1
2 1 0
2 1 0

1 2
1 3
2 2
2 3

3
3
3
3
Gelfand-Tsetlin Patterns and SSYT

\[
\begin{array}{cccc}
2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\
2 & 1 & 2 & 1 & 1 & 1 & 0 & 2 & 0 & 1 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 2 \\
2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{cc}
1 & 3 \\
2 & 2 \\
\end{array}
\]

\[
\begin{array}{cc}
1 & 1 \\
3 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\
2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 2 \\
3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 3 & 3 \\
3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{cc}
2 & 3 \\
2 & 3 \\
\end{array}
\]

\[
\begin{array}{cc}
2 & 3 \\
3 & 3 \\
\end{array}
\]

### Gelfand-Tsetlin Patterns and SSYT

- **Pattern 1:**
  - Row 1: 2, 1, 0
  - Row 2: 2, 1
  - Row 3: 2

- **Pattern 2:**
  - Row 1: 2, 1, 0
  - Row 2: 2, 1
  - Row 3: 1

- **Pattern 3:**
  - Row 1: 2, 1, 0
  - Row 2: 2, 0
  - Row 3: 1

- **Pattern 4:**
  - Row 1: 2, 1, 0
  - Row 2: 1, 0
  - Row 3: 1

- **Pattern 5:**
  - Row 1: 2, 1
  - Row 2: 2

- **Pattern 6:**
  - Row 1: 1, 2
  - Row 2: 2

- **Pattern 7:**
  - Row 1: 1, 3
  - Row 2: 2

- **Pattern 8:**
  - Row 1: 1
  - Row 2: 3

- **Pattern 9:**
  - Row 1: 1, 2, 3
  - Row 2: 3

- **Pattern 10:**
  - Row 1: 2, 2
  - Row 2: 3

- **Pattern 11:**
  - Row 1: 2, 3
  - Row 2: 3
Gelfand-Tsetlin Patterns and SSYT

\[
\begin{array}{cccc}
2 & 1 & 0 \\
2 & 1 \\
2
\end{array} & \begin{array}{cccc}
2 & 1 & 0 \\
2 & 1 \\
1
\end{array} & \begin{array}{cccc}
2 & 1 & 0 \\
1 & 1 \\
1
\end{array} & \begin{array}{cccc}
2 & 1 & 0 \\
2 & 0 \\
2
\end{array}
\]

\[
\begin{array}{cc}
1 & 1 \\
2
\end{array} & \begin{array}{cc}
1 & 2 \\
2
\end{array} & \begin{array}{cc}
1 & 3 \\
2
\end{array} & \begin{array}{cc}
1 & 1 \\
3
\end{array}
\]

\[
\begin{array}{cccc}
2 & 1 & 0 \\
2 & 0 \\
1
\end{array} & \begin{array}{cccc}
2 & 1 & 0 \\
1 & 0 \\
1
\end{array} & \begin{array}{cccc}
2 & 1 & 0 \\
2 & 0 \\
0
\end{array} & \begin{array}{cccc}
2 & 1 & 0 \\
1 & 0 \\
0
\end{array}
\]

\[
\begin{array}{cc}
1 & 2 \\
3
\end{array} & \begin{array}{cc}
1 & 3 \\
3
\end{array} & \begin{array}{cc}
2 & 2 \\
3
\end{array} & \begin{array}{cc}
2 & 3 \\
3
\end{array}
\]
Schur Polynomials

**Theorem**

For a partition \( \lambda \in \mathbb{Z}^n_{\geq 0} \), the **Schur polynomial** \( s_\lambda \) is given by

\[
s_\lambda = \sum_{A \in GT(\lambda) \cap \mathbb{Z}^n \choose \binom{n}{2}} x_1^{wt(A)_1} x_2^{wt(A)_2} \cdots x_n^{wt(A)_n}
\]

where \( wt(A)_i = (x_{ii} + \cdots + x_{in}) - (x_{i+1,i+1} + \cdots + x_{i+1,n}) \)

\[
GT(2, 1, 0) = \begin{bmatrix} 2 & 1 & 0 \\ x_{11} & x_{12} \\ x_{22} \end{bmatrix} \quad \begin{bmatrix} 2 \geq x_{11} \geq 1 \\ 1 \geq x_{12} \geq 0 \\ x_{11} \geq x_{22} \geq x_{12} \end{bmatrix}
\]

\[
\begin{array}{cccccccccc}
2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\
2 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\[
x_1x_2^2 \quad x_1^2x_2 \quad x_2^2x_3 \quad x_1x_2x_3 \quad x_1x_2x_3 \quad x_2x_3^2 \quad x_1x_3^2
\]

\[
s_{210} = x_1x_2^2 + x_1^2x_2 + x_2^2x_3 + 2x_1x_2x_3 + x_1^2x_3 + x_2x_3^2 + x_1x_3^2
\]
Theorem (Mészáros-S. 2017)

For any simple graph $G$ on $[n + 1]$, the multiset $RD(G)$ of right-degree sequences of $G$ is exactly the collection of first columns of the integer arrays

\[
\begin{array}{cccc}
   a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
   a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
   a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\
   \vdots & \vdots & \vdots & \ldots & \vdots \\
   a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}
\end{array}
\]

- $a_{ij} \geq 0$ for all $(i, j)$
- $a_{ij} \leq a_{i+1,j}$ if $(i, j) \in E(G)$
- $a_{ij} = a_{i+1,j}$ if $(i, j) \notin E(G)$
- $a_{ii} = \#E(G|[i,n+1]) - \sum_{j=i+1}^{n} a_{ij}$
A Polytope Sitting Above $R_G$

$T_{1432}$

$a_{11}$
$a_{12}$  $a_{22}$
$a_{13}$  $a_{23}$  $a_{33}$

\[0 \leq a_{11} = 3 - a_{12} - a_{13}\]
\[0 \leq a_{12} \leq a_{22} = 2 - a_{23}\]
\[0 \leq a_{13} = a_{23} \leq a_{33} = 1\]

\[
\begin{array}{cccccccc}
3 & 2 & 2 & 1 & 1 \\
0 & 2 & 1 & 2 & 0 & 1 & 2 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

$R_{T_{1432}} = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3$
**Theorem (Mészáros-S. 2017)**

For any graph $G$ on $[n + 1]$, there exists a polytope $P(G)$ and a function $\text{wt}: P(G) \to \mathbb{R}^n$ such that

$$R_G = \sum_{A \in P(G) \cap \mathbb{Z}^{\binom{n}{2}}} x_1^{\text{wt}(A)_1} x_2^{\text{wt}(A)_2} \cdots x_n^{\text{wt}(A)_n}. $$

**Conjecture (Liu-Mészáros-S. 2017)**

For any permutation $w$, there is a polytope $P_w$ and a function $\text{wt}$ such that

$$G_w = \sum_{A \in P_w \cap \mathbb{Z}^{\binom{n}{2}}} x_1^{\text{wt}(A)_1} x_2^{\text{wt}(A)_2} \cdots x_n^{\text{wt}(A)_n}. $$
Theorem (Liu-Mészáros-S. 2019)

For any permutation $w$ avoiding 3142 and 4132, there is a natural polytope $P_w$ and a function $w^t$ such that

$$
\mathcal{S}_w = \sum_{A \in P_w \cap \mathbb{Z}^{n \choose 2}} \chi^{w^t(A)}. 
$$

For $w \in S_n$ avoiding 3142 and 4132, we give a recipe to get partitions $\lambda^{(i)}$ for $i \in [n]$ such that

$$
P_w = GT(\lambda^{(1)}) + GT(\lambda^{(2)}) + \cdots + GT(\lambda^{(n)}).
$$
For $w = 256413$: 

[Diagram of a Rothe diagram for $w = 256413$]
For $w = 256413$: 
For $w = 256413$: 

![Rothe Diagrams](image-url)
For $w = 256413$ the Rothe diagram $D(w)$ is
For $w = 256413$ the Rothe diagram $D(w)$ is

The partitions associated to $w$ are $\lambda^{(3)} = (1, 1, 0)$, and $\lambda^{(4)} = (2, 2, 2, 1)$.

$$P_w = GT(2, 2, 2, 1) + GT(1, 1, 0).$$
Roadmap

- Flow Polytopes
- Right-Degree Polynomials
- Schubert Polynomials
- Gelfand-Tsetlin Analogues
- Newton Polytopes
- Saturation
- Zero-One Polynomials
So what are these polytopes literally projecting to?
Any polynomial $f = \sum_{z \in \mathbb{Z}^n} a_z x^z \in \mathbb{C}[x_1, \ldots, x_n]$ has an associated integer polytope called its Newton polytope:

$$\text{Newton}(f) = \text{Conv}(z : a_z \neq 0)$$

$$\text{Newton}(1 + x + y + x^2 + xy^2 + x^2y^2) = \text{Conv}((0, 1), (1, 1), (2, 1))$$

$$\text{Newton}(1 + y + xy + xy^2 + x^2 + x^2y^2) = \text{Conv}((0, 0), (1, 0), (2, 0))$$
What kind of polytopes are the Newton polytopes of Schubert polynomials?

\[
\mathcal{S}_{1243} = x_1 + x_2 + x_3
\]

\[
\mathcal{S}_{13524} = x_2 x_3^2 + x_1 x_3^2 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + 2x_1 x_2 x_3
\]
\[ \mathcal{S}_{21543} = x_1^3 x_2 + x_1^3 x_3 + x_1^3 x_4 + x_1^2 x_2^2 + x_1^2 x_3^2 + 2 x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1 x_2^2 x_3 x_4 + x_1 x_2 x_3^2 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 \]
The standard **permutedahedron** in $\mathbb{R}^n$ is the convex hull of all rearrangements of the vector $(1, 2, \ldots, n)$.

**Definition (Postnikov 2005, Edmonds 1970)**

A **generalized permutedahedron** is any polytope obtained by deforming the standard permutedahedron by moving the vertices in any way so that all edge directions are preserved.
Theorem (Mészáros-S. 2017)

For any graph $G$, $\text{Newton}(R_G)$ is a generalized permutahedron.

$T_{1432}$

Newton($R_{T_{1432}}$) =

$$
(1, 1, 1)
\quad (2, 0, 1)
\quad (1, 2, 0)
\quad (2, 1, 0)
\quad (3, 0, 0)
$$
An Answer For $R_G$

**Theorem** (Mészáros-S. 2017)

For any graph $G$, $\text{Newton}(R_G)$ is a generalized permutahedron.

**Question** (Mészáros-S. 2017)

Is $\text{Newton}(\mathcal{G}_w)$ a generalized permutahedron for any $w \in S_n$?
Conjecture (Monical-Tokcan-Yong 2017)

For any $w \in S_n$, $\text{Newton}(S_w)$ is a generalized permutahedron.
Conjecture (Monical-Tokcan-Yong 2017)

For any $w \in S_n$, $\text{Newton}(S_w)$ is a generalized permutahedron.

- $\{\text{Schubert polynomials } S_w\} \supseteq \{\text{Schur polynomials } s_\lambda\}$
Schubert Newton Polytopes

Conjecture (Monical-Tokcan-Yong 2017)

For any \( w \in S_n \), \( \text{Newton}(\mathcal{G}_w) \) is a generalized permutahedron.

- \( \{ \text{Schubert polynomials } \mathcal{G}_w \} \supseteq \{ \text{Schur polynomials } s_\lambda \} \)
- \( \text{Newton}(s_\lambda) = \text{Conv(} \text{all permutations of } \lambda \text{)} = \text{permutahedron} \)
Conjecture (Monical-Tokcan-Yong 2017)

For any \( w \in S_n \), \( \text{Newton}(S_w) \) is a generalized permutahedron.

\[
\begin{align*}
\{\text{Schubert polynomials } S_w\} & \supseteq \{\text{Schur polynomials } s_\lambda\} \\
\text{Newton}(s_\lambda) &= \text{Conv}(\text{all permutations of } \lambda) = \text{permutahedron} \\
\text{Newton}(S_w) \text{ should be a generalized permutahedron}
\end{align*}
\]
Schubert Newton Polytopes

Conjecture (Monical-Tokcan-Yong 2017)

For any $w \in S_n$, $\text{Newton}(\mathcal{G}_w)$ is a generalized permutahedron.

- $\{\text{Schubert polynomials } \mathcal{G}_w\} \supseteq \{\text{Schur polynomials } s_\lambda\}$
- $\text{Newton}(s_\lambda) = \text{Conv}(\text{all permutations of } \lambda) = \text{permutahedron}$
- $\text{Newton}(\mathcal{G}_w)$ should be a generalized permutahedron

Theorem (Fink-Mészáros-S. 2017)

The Newton polytopes of the Schubert polynomials are generalized permutahedra.
What does $RD(G)$ look like? Specifically, how do the points in $RD(G)$ sit inside the Newton polytope of $R_G$?

Newton($R_T_{1432}$) =

(1, 1, 1) (2, 1, 0) (3, 0, 0)

(1, 2, 0)

Theorem (Mészáros-S. 2017)

Newton($R_G$) is a generalized permutahedron whose integral points are exactly $RD(G)$. 
Definition (Monical-Tokcan-Yong 2017)

A polynomial $f$ is said to have **saturated Newton polytope** (SNP) if every integer point in the Newton polytope corresponds to a monomial with nonzero coefficient in $f$.

\[ S_{13524} = x_2 x_3^2 + x_1 x_3^2 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + 2 x_1 x_2 x_3 \]
Theorem (Monical-Tokcan-Yong 2017)

The following all have SNP:

- Schur polynomials
- Skew-Schur polynomials
- Stanley symmetric function
- \((q, t)\) evaluations of symmetric Macdonald polynomials
Theorem (Monical-Tokcan-Yong 2017)

The following all have SNP:

- Schur polynomials
- Skew-Schur polynomials
- Stanley symmetric function
- $(q, t)$ evaluations of symmetric Macdonald polynomials

Conjecture (Monical-Tokcan-Yong 2017)

The following all have SNP:

- Schubert polynomials
- Key polynomials
- Double Schubert polynomials
- Grothendieck polynomials
Theorem

The following all have SNP:

- Schubert polynomials (Fink-Mészáros-S. 2017)
- Key polynomials (Fink-Mészáros-S. 2017)
- $1w'$ Grothendieck polynomials (Mészáros-S. 2017)
- Symmetric Grothendieck polynomials (Escobar-Yong 2017)
A heuristic explanation for SNP?

GT(2, 1, 0)

Newton(s_{210})
Asking when a Schubert polynomial is zero-one is the same as asking when it equals the integer point transform of its Newton polytope \( \text{Newton}(\mathcal{S}_w) \).

\[
\mathcal{S}_{13524} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3
\]
Theorem (Fink-Mészáros-S. 2019)

The Schubert polynomial $S_w$ is zero-one if and only if $w$ avoids the patterns 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, 315642, and 315642.

\[ S_{13524} = x_2 x_3^2 + x_1 x_3^2 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + 2x_1 x_2 x_3 \]
The Schubert polynomial $\mathcal{S}_w$ is zero-one if and only if $w$ avoids the patterns 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642.

$$\mathcal{S}_{13524} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3$$
A Characterization of Zero-One Schubert Polynomials

Theorem (Fink-Mészáros-S. 2019)

For a permutation \( w \), the following are equivalent:

1. \( \mathcal{S}_w \) is zero-one,
2. \( w \) is multiplicity-free,
3. \( D(w) \) does not contain any instance of configuration A, B, or B',
4. \( w \) avoids 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642.

The proof proceeds by showing (1) \( \Rightarrow \) (4), (4) \( \Rightarrow \) (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1).
Theorem (Fink-Mészáros-S. 2019)

For a permutation $w$, the following are equivalent:

1. $\mathcal{G}_w$ is zero-one,
2. $w$ is multiplicity-free,
3. $D(w)$ does not contain any instance of configuration A, B, or $B'$,
4. $w$ avoids 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642.

The proof proceeds by showing (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).
Theorem (Fink-Mészáros-S. 2019)

Fix $w \in S_n$ and let $\sigma \in S_{n-1}$ be the pattern obtained by deleting $w_k$ from $w$. Then

$$G_w(x_1, \ldots, x_n) - M(x_1, \ldots, x_n)G_\sigma(x_1, \ldots, \widehat{x_k}, \ldots, x_n)$$

has nonnegative coefficients, where

$$M(x_1, \ldots, x_n) = \left( \prod_{(k,i) \in D(w)} x_k \right) \left( \prod_{(i,w_k) \in D(w)} x_i \right).$$
If $w = 2463571$ and $k = 5$, then $\sigma = 245361$.

\[
\mathcal{S}_w(x_1, \ldots, x_7) - x_3 x_5 \mathcal{S}_\sigma(x_1, x_2, x_3, x_4, x_6, x_7) \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_7].
\]
Corollary (Fink-Mészáros-S. 2019)

Fix \( w \in S_n \) and let \( \sigma \in S_m \) be any pattern contained in \( w \). If \( k \) is a coefficient of a monomial in \( \mathcal{S}_\sigma \), then \( \mathcal{S}_w \) contains a monomial with coefficient at least \( k \).
Corollary (Fink-Mészáros-S. 2019)

Fix $w \in S_n$ and let $\sigma \in S_m$ be any pattern contained in $w$. If $k$ is a coefficient of a monomial in $\mathcal{S}_\sigma$, then $\mathcal{S}_w$ contains a monomial with coefficient at least $k$.

Schubert polynomials $\mathcal{S}_w$ with coefficients less than or equal to $k$ are closed under pattern containment.
Corollary (Fink-Mészáros-S. 2019)

Fix $w \in S_n$ and let $\sigma \in S_m$ be any pattern contained in $w$. If $k$ is a coefficient of a monomial in $S_\sigma$, then $S_w$ contains a monomial with coefficient at least $k$.

Schubert polynomials $S_w$ with coefficients less than or equal to $k$ are closed under pattern containment.

That is, Schubert polynomials $S_w$ with coefficients less than or equal to $k$ are characterized as avoiding a list of patterns $\text{Patt}_k$. 

A Corollary

Corollary (Fink-Mészáros-S. 2019)

Fix \( w \in S_n \) and let \( \sigma \in S_m \) be any pattern contained in \( w \). If \( k \) is a coefficient of a monomial in \( S_\sigma \), then \( S_w \) contains a monomial with coefficient at least \( k \).

Schubert polynomials \( S_w \) with coefficients less than or equal to \( k \) are closed under pattern containment.

That is, Schubert polynomials \( S_w \) with coefficients less than or equal to \( k \) are characterized as avoiding a list of patterns \( \text{Patt}_k \).

Question

For which values of \( k \) is \( \text{Patt}_k \) a finite set?
Thanks For Listening!

Schubert Polynomials

Right-Degree Polynomials

Schubert Polynomials

Gelfand-Tsetlin Analogues
Newton Polytopes
Saturation
Zero-One Polynomials

Thank you for listening!