Involutions on Grassman Manifolds

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Dedication

to Frank Ugenti and to my Father
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I would like to express my deep appreciation to my advisor, Wu-Yi Hsiang. Theorem 0 was conjectured by him, and the technique of proof is a direct generalization of a method he used to reprove the Smith theorems for projective spaces. Many of the nicest arguments of this paper contain important contributions of his. Perhaps most importantly of all, I would like to express my gratitude for his extremely valuable personal support during the preparation of this paper.

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# TABLE OF CONTENTS

I. Introduction 1

II. Cohomology of Grassmannians 5

III. Linear Models 15

IV. Binomial Coefficient Facts 23

V. The Basic Fixed Point Theorem 27

VI. Setting up of the Proof 36

VII. Inductive Hypothesis and Notation 41

VIII. Relation Coefficients 44

IX. The $v$ odd case 53

X. $v = 2$ (4) 59

XI. Inductive $S_d^{m_s} F$ Lemma 62

XII. The Core of the Proof 68

XIII. Conclusion 96

Bibliography 98

Appendix A 99

Appendix B 103

Appendix C 104

Appendix D 107

Appendix E 110

Appendix F 111

Appendix G 117
1. Introduction

In the late 1930's, P. A. Smith proved the first of his celebrated theorems about toral actions on spheres and acyclic manifolds. Enriched by the technical tool of equivariant cohomology, such theorems have since been extended to other cohomologically simple spaces. The primary purpose of this paper is to prove a similar theorem for involutions on the Grassman manifold $G_2(R^N)$ of unoriented two planes in $N$-space.

In studying involutions of $G_2(R^N)$, the first involutions that come to mind are those induced by linear involutions of $R^N$. If the linear involution is induced by

$$
\begin{pmatrix}
-1 & 0 \\
0 & 1_{N-1}
\end{pmatrix}
$$

then it is easy to see that the fixed point set will have three components of types $G_2(R^1)$, $G_2(R^{N-1})$, and $RP^{1-1} \times RP^{N-1-1}$. The following theorem is the main result of this paper:

Theorem 0. Any involution with fixed point on $G_2(R^{N+1})$ \(^1\)

(\(N\) even, \(N = 2^a + N', N' < 2^{a-1}\), and \(N \not\equiv 64 \pmod{192}\)) is $Z_2$ cohomologically modeled on a linear involution; i.e. the fixed point set has three components of $Z_2$ cohomology types $G_2(R^1)$, $G_2(R^{N+1-1})$, and $RP^{1-1} \times RP^{N-1}$ where $0 \leq 1 \leq N+1$.

\(^1\) or a space of the same $Z_2$ cohomology type (including Steenrod algebra structure.)
The foundation of this theorem lies in the equivariant cohomology theory defined by Borel. If $X$ is a $G$-space and $E_G \rightarrow B_G$ is the universal $G$-bundle, let $X_G$ be the associated bundle over $B_G$ with fibre $X$. Then the equivariant cohomology $H^*_G(X;k)$ is defined to be $H^*(X_G;k)$.

For toral groups, the basic result of Smith theory is that with the right coefficients ($k = \mathbb{Z}_2$ for $\mathbb{Z}_2$-tori) the free part of the equivariant cohomology (as an $H^*(B_G;k)$ module) determines the cohomology of the fixed point set. Thus it is quite easy to reduce the proof of theorem 0 to the determination of the equivariant cohomology of the involution.

The basic technique used to resolve this question is to study the Steenrod algebra structure of the equivariant cohomology. In particular, for any presentation of the cohomology algebra of $X_G$, the ideal of relations must be invariant under the action of the Steenrod squares. We'll show the remarkable fact that this condition suffices to force the equivariant cohomology to be isomorphic to that of one of the linear models.

Now a word about the organization of this paper. Section II reviews the cohomology of Grassman manifolds and reformulates the well known facts in terms of the classical symmetric functions $h_k$ of degree $k$. The main results are $H^*(G_2(\mathbb{R}^{N+1});\mathbb{Z}_2) = \mathbb{Z}_2[h_1,h_2]/\langle h_N,h_{N+1} \rangle$ together with various technical facts which are useful for calculations. Section III uses characteristic class theory to determine the equi-
variant cohomology of linear involutions. After a brief summary of binomial coefficient facts in section IV, section V shows how the equivariant cohomology of a linear model determines the cohomology structure of the fixed point set.

The remaining sections of this paper deal with the real work in the proof of theorem 0. Sections VI and VII set up the basic framework. It is quite easy to see that additively, $H^*_G(X;\mathbb{Z}_2) = H^*(X;\mathbb{Z}_2) \otimes H^*(BG;\mathbb{Z}_2)$. Thus it is the multiplicative structure of the equivariant cohomology which carries the geometry of the involution. Hence the equivariant cohomology may be presented as $\mathbb{Z}_2[h_1, h_2, \overline{g}/\langle f, g \rangle]$ where $f$ and $g = Sq^1f$ are the lifts of the relations $h_N$ and $h_1h_N + h_{N+1}$ defining the ordinary cohomology structure of $G_2(R^{N+1})$. Section VII concludes by setting up an inductive scheme for studying the invariance of $\langle f, g \rangle$ under the Steenrod algebra.

Sections VIII through XII carry out the details of this program. Because of a large number of similar though unfortunately distinct arguments, the general policy followed is to carefully go through typical arguments and to summarize the other cases in appendices. Sections XI and XII however contain the generic general case argument in complete detail.

It should be noted that most of the arguments for theorem 0 work as well for actions of $\mathbb{Z}_2$-tori ($\mathbb{Z}_2^r$) under the sole dimension restriction that $N$ be even. Also, there is a clearly analogous approach in the $N$ odd case. These arguments as well as applications to the action of arbitrary
Lie groups will be covered in a subsequent paper.
II. Cohomology of Grassmannians

It is well known that $H^*(BO(m);Z_2) \cong H^*(BZ^m_2;Z_2) = Z_2[t_1, \ldots, t_m]$ may be thought of as a polynomial algebra generated by the first $m$ elementary symmetric functions $\{\sigma_i\}$ of the variables $t_i$. Also, as the inclusion $G_m(R^N) \rightarrow G_m(R^\infty) = BO(m)$ induces an injection in homology, we know $H^*(G_m(R^N);Z_2) = Z_2[\sigma_1, \ldots, \sigma_m]/I$ where $I$ is an ideal in the ring of symmetric functions.

The purpose of this section will be to reformulate this picture in terms of the so-called classical symmetric polynomials $h_k$ defined by the following $k$ by $k$ determinant:

$$
\begin{vmatrix}
\sigma_1 & 1 & 0 & \cdots & 0 \\
\sigma_2 & \sigma_1 & 1 & \cdots & 0 \\
\vdots & \sigma_2 & \sigma_1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \sigma_m & \sigma_{m-1} \\
0 & \cdots & \cdots & 0 & \vdots \\
0 & \cdots & \cdots & 0 & 0 \\
0 & \cdots & \sigma_1 & 0 & \sigma_2 \\
0 & 0 & 0 & 0 & \sigma_2 \sigma_1 \\
\end{vmatrix}
$$

(1) \quad h_k =

We shall also discuss those algebraic properties of these symmetric functions which are relevant to doing calculations on $H^*(G_2(R^N);Z_2)$.

The following theorem (essentially known to Wen Tsun Wu) explains the importance of the $h_i$:
Theorem II.1

(1) $H^*(G_m(R^N); Z_2) \cong Z_2[h_1, \ldots, h_m]/I$ where $I$ is generated by $\{h_j : N-m+1 \leq j \leq N\}$.

(2) A module basis for $H^*(G_m(R^N); Z_2)$ is given by $\{\prod_{j=1}^{m} h_{i_j} : 0 \leq i_j \leq N-m\}$.

We shall use the usual multi-index notation $h_\lambda$ for the element $\prod_{j=1}^{m} h_{i_j}$ where $\lambda = (i_1, \ldots, i_m)$ and $|\lambda| = \sum_{j=1}^{m} i_j$.

Before proving theorem II.1, we shall develop some of the algebraic properties of $h_k \in Z[t_1, \ldots, t_m]$. Expansion of the defining determinant (1) in minors of the first column immediately gives the basic recursion formula:

**Lemma II.2** \[ h_N = \sum_{j=1}^{m} (-1)^{j-1} \sigma_j h_{N-j} \]

Lemma II.2 shows us $Z[h_1, \ldots, h_m] = Z[\sigma_1, \ldots, \sigma_m]$.

It is worthwhile to convert this lemma into generating function language. Setting $H = \sum_{k=0}^{\infty} h_k$, the above recursion formula translates into $(\sum_{j=0}^{m} (-1)^j \sigma_j)H = 1$. Thus we obtain:

**Lemma II.3** The generating function of the $h_k$ is given by

\[ H = \prod_{j=1}^{m} \frac{1}{(1-t_j)} \]

The formula $1/(1-t) = \sum_{j=0}^{\infty} t^j$ immediately shows us that the coefficient of any degree $k$ monomial $t^\lambda$ in $h_k$ is 1. So for $m = 2$, $h_k = (t_1^{k+1} - t_2^{k+1})/(t_1 - t_2)$.

The following proposition is the real heart of the
proof of theorem 11.1:

**Proposition 11.4** Let $\mathbf{h}_I$ be a multi-index of length $m$. Then:

1. The collection of all $\mathbf{h}_I$ form a module basis for the symmetric polynomials in $\mathbb{Z}[t_1, \ldots, t_m]$.
2. There exist integers $B^J_I$ so that

$$
\mathbf{h}_k \mathbf{h}_I = \sum_J B^J_I h^J_I |I| + k - |J|
$$

where the summation is over all multi-indices $J$ of length $m-1$.

Proof: Since the algebra of symmetric polynomials is a polynomial algebra on the elementary symmetric functions, we can easily count the number of elements of given degree and note it to be the same as the number of $m$-fold products $\mathbf{h}_I$ of given degree. Since any symmetric polynomial may obviously be expressed as a polynomial in $e_1, \ldots, e_m$ and hence $h_1, \ldots, h_m$, it is clear that repeated application of formula (2) will express any symmetric element as an integral combination of the $h_I$. Since the number of $h_I$ of given degree is equal to the dimension of the symmetric polynomials of that degree, it is clear from (2) that the $h_I$ form a module basis.

Thus we need only prove multiplication formula (2). The proof is by induction on $m$. The case $m = 1$ is trivial. Assume the proposition to be true for $m - 1$.

Define $Y_k \in \mathbb{Z}[t_1, \ldots, t_{m-1}]$ to be the function $h_k$ of $m-1$ variables $t_1, \ldots, t_{m-1}$. Then $h_k = \sum_{j=0}^k t_m^j Y_j$. Also set
\( X_k = \sum_{j=0}^{k} Y_j \). Now the \( \{X_j\} \) (J a multi-index of length m-1) form a (non-homogeneous) module basis for the symmetric functions in \( Z[t_1, \ldots, t_{m-1}] \). By inductive hypothesis, there exist integers \( B^J_i \) so that \( X_i - \sum J B^J_i X_j = 0 \). Then \( h_1 - \sum J B^J_i h_J \) vanishes when \( t_i = 1 \) \((1 \leq i \leq m)\) and hence is in the ideal generated by \( \prod_{i=1}^{m} (1-t_i) \).

So we see that \( h_1 = \sum J B^J_i h_J + Q(t_1, \ldots, t_m) \prod_{i=1}^{m} (1-t_i) \) where \( \deg Q \leq |J| - m - 1 \). Hence

\[
    h_1 \sum_{k=0}^{\infty} h_k = h_1 / (\prod_{i=1}^{m} (1-t_i))
\]

\[
    = (\sum J B^J_i h_J) \sum_{k=0}^{\infty} h_k + Q(t_1, \ldots, t_m)
\]

Equating terms of degree \(|J| + k\), we get

\[
    h_1 h_k = \sum J B^J_i h_J h^{|J|+k-|J|}
\]

Note that the above determination of the "structure constants" \( B^J_i \) is constructive. The structure constants for products of \( m+1 \) homogeneous elements \( \{h\} \) in \( Z[\sigma_1, \ldots, \sigma_m] \) are identical to those for products of \( m \) inhomogeneous elements \( \{X_i\} \) in \( Z[\sigma_1, \ldots, \sigma_{m-1}] \).

Proof of 11.1: First observe that \( h_{N-m+1}, \ldots, h_N \) are relations in \( H^*(G_m(R^N);Z_2) \). The easiest way to see this is to consider the Whitney duality formula applied to the universal bundle \( \gamma_m \) and its complement \( \gamma_{N-m} \) over \( G_m(R^N) \). Then \( w_i(\gamma_m) = \sigma_i \) and let \( \tau_j = w_j(\gamma_{N-m}) \). \( w(\gamma_m \oplus \gamma_{N-m}) = 1 \) yields

\[
    \sum_{j=0}^{m} \sigma_j \tau_{r-j} = 0 \text{ for all } r.
\]
Thus it is clear that the $\tau_j$ satisfy the recursion formula

$$\tau_r = \sum_{j=1}^{m} \sigma_j \tau_{r-j}$$

and so in fact $\tau_r = h_r$. Since $\gamma_{N-m}$ is $N-m$ dimensional, $h_r = 0$ for $r \geq N-m+1$.

By II.4 and the fact that $H^*(BO(m);Z_2) \rightarrow H^*(G_m(R^N);Z_2)$ is surjective, we immediately conclude that the set $\{h_i\}$ with $0 \leq i_j \leq N-m$ span $H^*(G_m(R^N);Z_2)$. Since the cardinality of this set is the same as the number of Schubert cycles, we see that $\{h_i\}$ form a module basis for $H^*(G_m(R^N);Z_2)$. The multiplication formula in II.4 together with the relations $h_r = 0$ for $r > N-m$ clearly suffice to describe the multiplicative structure, and so the ideal of relations in $H^*(G_m(R^N);Z_2)$ is generated by $\{h_{N-m+1}, \ldots, h_N\}$.

Note that with trivial changes, the above proof suffices to compute the integral cohomology ring of the complex or quaternionic Grassman manifolds. In these cases as well as the real case, the classes $h^*_k$ may be identified with the duals of the "special Schubert cycles" $(k,0,\ldots,0)$ in Chern's notation.

Because the $h^*_k$ may be thought of as Stiefel-Whitney classes, Wu's formula immediately gives the Steenrod algebra structure of $H^*(G_m(R^N);Z_2)$. Explicitly, for $r < k$:

$$S^{r}h^*_{k} = \sum_{u=0}^{r} \binom{k-r+u-1}{u} h^*_{r-u} h^*_k + u$$

It is worthwhile to specialize some of the above considerations to the case of two planes. So for the remainder
of this section, let \( m = 2 \). Theorem 11.1 and proposition 11.4 may be restated as:

**Corollary 11.5**

1. \( H^*(G_2(R^N); Z_2) = Z_2[h_1, h_2]/\langle h_{N-1}, h_N \rangle \)

2. The collection \( \{ h_a h_b : 0 \leq a, b \leq N-2 \} \) form a module basis for this ring.

3. \( h_a h_b h_c = \sum_{j=0}^{a-1} h_j h_{a+b+c-j} + \sum_{j=0}^{a} h_b j h_{a+c-j} \)

(The multiplication formula in (3) is also valid over \( Z \).)

Proof: Everything is obvious except the multiplication formula (3). Applying the method of proposition 11.4, we need only study the elements \( X_k \in Z[t] \) defined by \( X_k = \sum_{j=0}^{k} t^j \).

Claim: \( X_i X_k + \sum_{j=0}^{i-1} X_j - \sum_{j=0}^{i} X_{k+j} = 0 \)

Proof of claim: Multiplication by \( 1 - t \) gives

\[
\sum_{j=0}^{i} t^j (1-t^{k+1}) + \sum_{j=0}^{i-1} (1-t^j+1) = \sum_{j=0}^{i} (1-t^{k+j+1}) = 0
\]

Formula (3) now follows immediately from the argument of proposition 11.4.

In writing out expressions like formula (3) in 11.5, all terms \( h_x h_y \) are generally of the same total degree \( d = x + y \). Accordingly, when the total degree \( d \) is clear from context, we shall abbreviate the expression \( h_x h_{d-x} \) as \( \overline{h}_x \).

It was noted before that for \( m = 2 \), \( h_k = (t_1^{k+1} - t_2^{k+1})/(t_1 - t_2) \). Thus most questions in \( H^*(G_2(R^N); Z) \) may be
readily reformulated in terms of the symmetric sums $s_k = t_1^k + t_2^k$. The double products \{$s_a s_b : a, b \geq 1$\} span the ideal generated by $h_1^2 = s_1^2 = s_2$ in $\mathbb{Z}_2[\{h_1, h_2\}]$. We shall also use the notation $\overline{s_a} = s_a s_{d-a}$. The $s_k$ have a somewhat simpler multiplication law than the $h_k$.

**Lemma 11.6**

1. \[ s_a s_b s_c = \overline{s_a} + \overline{s_b} + \overline{s_c} \]
2. \[ s_a s_b s_c s_d = \overline{s_a} + \overline{s_b} + \overline{s_c} + \overline{s_{a+b}} + \overline{s_{a+c}} + \overline{s_{b+c}} + \overline{s_{a+b+c}} \]

**Proof:** \[ s_{a+1} s_{b+1} = h_1^2 h_a h_b = \overline{h_1} + \overline{h_a} + \overline{h_{a+2}}. \] Then

\[ s_{a+1} s_{b+1} s_{c+1} = h_1^3 h_a h_b h_c = h_3 h_a h_b h_c \]

\[ = \overline{h_1} + \overline{h_a} + \overline{h_{a+2}} + \overline{h_b} + \overline{h_{b+2}} + \overline{h_c} + \overline{h_{c+2}} \]

\[ = \overline{s_{a+1}} + \overline{s_{b+1}} + \overline{s_{c+1}} \]

(2) follows trivially from (1).

The major advantage of the $s_k$ in computation is that the Cartan formula gives the optimal expression when applied to $Sq^r s_a s_b$. (Note $Sq^r s_a = \binom{a}{r} s_{a+r}$.) This is not the case when one attempts to calculate $Sq^r h_a h_b$.

The formula $h_1^2 h_a h_b = s_{a+1} s_{b+1}$ readily gives the following recursive formula:

\[(*) \quad Sq^k h_a h_b = h_1^2 Sq^{k-2} h_a h_b + \sum_{j=0}^{k} \binom{a+1}{k-j} \binom{b+1}{k-j} \overline{h_{a+j}} \]
The cases $k \leq 4$ of this formula will be of special interest to us.

**Lemma 11.7**

1. \[ Sq^1 h_a h_b = (a+1)\overline{h}_{a+1} + (b+1)\overline{h}_{b+1} \]

2. \[ Sq^2 h_a h_b = \binom{a-1}{2} \overline{h}_{a+2} + (a+1)(b+1)\overline{h}_{a+1} + \binom{b-1}{2} \overline{h}_a + \overline{h}_1 \]

3. \[ Sq^4 h_a h_b = \binom{a-1}{4} \overline{h}_{a+4} + \binom{a-1}{3} (b+1)\overline{h}_{a+3} + \]
\[ \left[ \binom{a-1}{2} \binom{b-1}{2} + 1 \right] \overline{h}_{a+2} + \]
\[ (a+1) \binom{b-1}{3} \overline{h}_{a+1} + \binom{b-1}{4} \overline{h}_a + \overline{h}_3 + \]
\[ \binom{a+b+2}{2} \overline{h}_1 \]

The ring $H^*(G_2(R^N); Z_2)$ satisfies certain filtration properties that result in great simplifications in the study of ideal stability conditions $Sq^k f \in \langle f, g \rangle$. First define:

\[ H^r = \text{span}\{h_x h_y : x + y = r\} \]
\[ S^b_a = \text{span}\{h_x h_y : a \leq x \leq b\} \]

Then it follows from Wu's formula and corollary 11.5 that:

**Corollary 11.8**

1. \[ H^r \cdot S^b_a \subseteq S^{r-1}_0 + S^{b+r}_a \]

2. \[ Sq^r S^b_a \subseteq S^{r-1}_0 + S^{b+r}_a \]
Although we shall not use these results elsewhere in this paper, it is worthwhile to state the following generalizations of 11.5 and 11.7.

**Lemma 11.9**

1. \( S_q^m h_a h_b = \sum_{u=1}^{m-1} p^u(a, b, m) h_u + \sum_{u=0}^{m} q^u(a, b, m) h_{a+u} \)

2. \( q^u(a, b, m) = q^u(a, b, m-2) + q^{u-2}(a, b, m-2) + \)
\[
\binom{a+1}{u} \binom{b+1}{m-u}
\]

3. \( p^u(a, b, m) = p^u(a, b, m-2) + p^{u-2}(a, b, m-2) \) for \( u > 1 \)

4. \( p^1(a, b, m) = \sum_{u=3}^{m-3} p^u(a, b, m-2) + \sum_{u=0}^{m-2} q^u(a, b, m-2) \)

5. \( p^{2r}(a, b, m) = 0 \) for all \( r \)

6. \( q^u(a, b, m) = \sum_{k=0}^{[m/4]} \binom{a-1-2k}{u-2k} \binom{b-1-2k}{m-u+2k} \)

7. \( p^u(a, b, m) \) only depends on \( u, m \), and the total degree \( a + b \).

**Proof (outline):** (1) - (5) follow readily from induction applied to recursion formula (*). (6) follows from repeated expansion of \( \binom{a+1}{u} \binom{b+1}{m-u} \) using the formula \( \binom{x}{u} = \binom{x-2}{u} + \binom{x-2}{u-2} \). (7) then follows from (4) and (6) using the binomial theorem \( \sum_{u=0}^{v} \binom{x}{u} \binom{y}{u-v} = \binom{x+y}{v} \).
Lemma 11.10 \[ h_a h_b h_c h_d = \]
\[
\left[ \sum_{u=0}^{[(a-2)/2]} \overline{h}_{2u+1} + \overline{h}_{a+b-2u-1} + a \overline{h}_b + (a+1) \sum_{u=a}^{b-1} \overline{h}_u \right] + \\
\left[ \sum_{u=0}^{[(a-1)/2]} \overline{h}_{c+2u} + \overline{h}_{a+b+c-2u} + (a+1) \sum_{u=a}^{b} \overline{h}_{c+u} \right]
\]

Proof (outline): Apply corollary 11.5 and interchange the order of summation.
III. Linear Models

When a $\mathbb{Z}_2$ torus $T$ acts linearly on $\mathbb{R}^N$, it induces a direct sum decomposition of $\mathbb{R}^N$ into weight spaces $\mathbb{R}^{n_i}_{\theta_i}$. Each $\theta_i$ is a homomorphism from $T$ to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and the action of $T$ on $\mathbb{R}^{n_i}_{\theta_i}$ is given by identifying the nontrivial element of $\mathbb{Z}_2$ with the action of $-1$ on $\mathbb{R}^{n_i}_{\theta_i}$.

The $\theta_i$ are called the weights of the representation. By transgression in the universal bundle $E_T \to B_T$, they may be thought of as elements $\bar{\theta}_i$ of $H^1(B_T; \mathbb{Z}_2)$. In this section, we'll determine the equivariant cohomology of the induced action on $G_m(\mathbb{R}^N)$, and show it to be simply expressible in terms of the elementary symmetric functions of the $\theta_i$ (counting multiplicities.)

First, let's observe that the fixed point set of a linear $T$ action on $G_m(\mathbb{R}^N)$ is readily described in terms of the weight space decomposition.

Lemma III.1. Suppose a $T$ action on $G_m(\mathbb{R}^N)$ is induced by a linear action of $T$ with weights $\{\theta_i\}_{i=1}^S$ and weight spaces $\mathbb{R}^{n_i}_{\theta_i}$. Then the set of components of the fixed point set will be $\{x_{i=1}^S G_{r_i}(\mathbb{R}^{n_i}) : r_i \leq n_i \text{ and } \sum r_i = m\}$.

Proof: It clearly suffices to prove the following:

(*) If $p \in G_m(\mathbb{R}^N)$ is a plane fixed by $T$, and $v$ is a vector in $p$, then $\mathbb{1}_{\theta_i}(v) \in p$ where $\mathbb{1}_{\theta_i}$ is the projection onto the weight space $\mathbb{R}^{n_i}_{\theta_i}$. 
Statement (*) is readily proven by induction on $s$, the number of distinct weights. The case $s = 1$ is trivial.
Given the truth of (*) for $s$ weights, write an arbitrary vector $v$ in some fixed plane $p$ as $v_1 + v_2$ where $v_2 \in R_{s+1}^n$ and $v_1 \in \bigoplus_{i=1}^s R_{\theta_i}^{n_i}$.
For all $j \leq s$, there exist $g_j \in T$ so that, $\theta_{s+1}(g_j) = -1$ and $\theta_j(g_j) = 1$. Then $v + g_j v \in p \cap \bigoplus_{j=1}^s R_{\theta_i}^{n_i}$, and by inductive hypothesis $\eta_j(v) \in p$ for all $j \leq s$. Since $\eta_{s+1}(v) = 1 - \sum_{j=1}^s \eta_j(v) \in p$ and the induction is complete.

The key to calculating the equivariant cohomology of a linear action on $G^m(R^N)$ is to note that the bundles $\gamma_m$ and $\gamma_{N-m}$ are equivariant bundles. Then the Borel construction may be applied to them to give vector bundles over $(G^m(R^N))_T$.
Now cohomology information about $(G^m(R^N))_T$ will be recorded in the characteristic class information of $(\gamma_m)_T$ and $(\gamma_{N-m})_T$. To process this information, we shall need the following easy lemmas:

**Lemma III.2** Given a representation $\theta : T \to \text{O}(1)$ and a space $X$, let $\xi$ be the trivial line bundle $X \times R$ over $X$ with (non-trivial) $T$ action given by $g(x,s) = (x, \theta(g)s)$. Then $w_1(\xi_T) = \eta \circ \overline{\theta}$ where $\eta : X_T \to B_T$ and $\overline{\theta}$ is the transgression of $\theta$.

**Proof:** The $T$-equivariant map $X \times R \to R$ induces a bundle map

$$\xi_T \to R_T$$

$$\downarrow$$

$$X_T \to B_T$$
The weight $\theta$ induces a bundle map

$$ R_T \rightarrow \gamma_1 $$

$$ \downarrow \quad \downarrow $$

$$ B_T \rightarrow B_{Z_2} = \mathbb{R}P^\infty $$

where $\gamma_1$ is the canonical bundle over $\mathbb{R}P^\infty$.

Combining these two bundle maps with the functoriality of Stiefel-Whitney classes proves the lemma.

**Lemma III.3** The determinant of the $n$ by $n$ matrix

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
& \sigma_1 & 1 & 0 & \cdots \\
& & \sigma_1 & 1 & 0 & \cdots \\
& & & \ddots & \ddots & \ddots \\
0 & & & & \sigma_m \\
& \cdots & \cdots & \cdots & \cdots & \sigma_m \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

row $i$ $\rightarrow$

\[
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
\end{pmatrix}
\]

is $h_{n-i}$.

**Proof:** This is obvious from the definition $\det(a_{jk}) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_k a_{k\sigma(k)}$. (For a nonzero term, $\sigma(k) = k+1$ for all $k < i$ and $\sigma(i) = 1$.)
We're now ready to compute the equivariant cohomology of a linear action on $G_m(R^N)$. First note that since $\gamma_m$ and $\gamma_{N-m}$ are equivariant and $H^\ast(G_m(R^N))$ is generated by their Stiefel Whitney classes, we can immediately conclude that the fiber of the bundle

$$G_m(R^N) \rightarrow G_m(R^N)_T$$

$$\rightarrow B_T$$

is totally nonhomologous to zero. Also, as any linear automorphism of $R^N$ (N odd) is homotopic (through automorphisms) to $\pm 1$, such a transformation will induce a trivial action on $H^\ast(G_m(R^N))$. Any linear transformation of $G_m(R^N)$ extends to $G_m(R^{N+1})$ and thus will also act trivially on $H^\ast(G_m(R^N)) \subseteq H^\ast(G_m(R^{N+1}))$ when N is even. Thus the action of the fundamental group of the base on the cohomology of the fibre in the above fibration will be trivial. So we need not worry about local coefficient systems.

Hence additively, for any linear involution $H^\ast(G_m(R^N))_T = H^\ast(G_m(R^N);Z_2) \otimes H^\ast(B_T;Z_2)$. All the information about the action is really contained in the multiplicative structure.

**Theorem III.4** In the notation above,

$$H^\ast(G_m(R^N);Z_2) = H^\ast(B_T;Z_2)[h_1, \ldots, h_m]/\langle f^{N-m+1}, \ldots, f^N \rangle$$

where

$$f^{N-k} = \sum_{j=0}^N h_j h_{N-k-j}$$

and $\rho_j$ is the j'th elementary symmetric function of the weights $\bar{\sigma}_j$ (counting multiplicities.)
Proof: Let $\sigma_i = w_i((\gamma_m)_T) (0 \leq i \leq m)$ and $\tau_i = w_i((\gamma_{N-m})_T)$. Now define elements $h_i \in H^*(G_m(R^N))$ by the same determinant used to define $h_i \in H^*(G_m(R))$; i.e.

\[
\begin{bmatrix}
\sigma_1 & 1 & 0 \\
\sigma_2 & \sigma_1 & 1 \\
\vdots & \sigma_2 & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\sigma_m & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \sigma_m & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]

$h_i = $

Then these $h_i$ will satisfy the same multiplication formula as the $h_i \in H^*(G_m(R^N))$ (proposition II.4.2). The set of $m$-fold products $h_i (i = (i_1, \ldots, i_m), 0 \leq i_j \leq m)$ are a basis for $H^*_T(G_m(R^N))$ as an $H^*(B_T)$ module. So to determine the multiplicative structure of $H^*_T(G_m(R^N))$, it suffices to expand an arbitrary $m$-fold product $h_i$ in terms of $m$-fold products with $i_j \leq N-m$. If we can show that $f^{N-m+1}, \ldots, f^N$ are relations in $H^*_T$, then by the basic recurrence relations II.2, it will be clear that $h_A = \sum_{j=1}^{N} \rho_j h_{A-j}$ for all $A > N-m$. This will then fully describe the ring structure of $H^*_T(G_m(R^N))$.

To show that $f^{N-m+1}, \ldots, f^N$ are relations, consider the Whitney duality formula applied to $(\gamma_m)_T \oplus (\gamma_{N-m})_T = (G_m(R^N) \times R^N)_T$. By lemma III.2, $w_k((G_m(R^N) \times R^N)_T) = \rho_k$. Thus:


\[
\begin{align*}
\sigma_1 + \tau_1 &= \rho_1 \\
\sigma_2 + \sigma_1 \tau_1 + \tau_2 &= \rho_2 \\
\vdots \\
\sigma_m \tau_{N-m} &= \rho_N
\end{align*}
\]

Viewing the first \(N-m+k\) equations in this system as a homogeneous linear system (in the variables \(\tau_0=1, \tau_1, \ldots, \tau_{N-m}\) and \(\tau_{N-m+1} = \ldots = \tau_{N-m+k-1} = 0\)) with nontrivial solution, we conclude that the determinant of the \(N-m+k\) by \(N-m+k\) matrix

\[
\begin{vmatrix}
\sigma_1 + \rho_1 & 1 & 0 \\
\sigma_2 + \rho_2 & \sigma_1 & \cdot \\
\sigma_3 + \rho_3 & \sigma_2 & \cdot \\
\vdots & \vdots & \vdots \\
\sigma_m + \rho_m & \sigma_{m-1} & \cdot \\
\rho_{m+1} & \sigma_m & \cdot \\
\vdots & \vdots & \cdot \\
\rho_{N-m+k} & \cdot & \cdot
\end{vmatrix}
\]

is zero. Applying lemma 11.3, we see immediately that this is zero.
The nature of these relations $f^A = \sum_{j=0}^{N} \rho_j h_{A-j}$ ($A > N-m$) can be further clarified by considering the equivariant inclusion $i : G_m(R^N) \to G_m(R^{N'})$ for $N' > N$. Then modulo trivial bundles, $i^* \gamma_{N'-m} = \gamma_{N-m}$ and the relation $f^A = 0$ is just the statement that $i^* w_A(\gamma_{N'-m}) = 0$. (In general, $f^k = w_k(\gamma_{N-m})$.)

Since the relations $f^A$ may be thought of as characteristic classes, Wu's formula applies and we immediately obtain:

\textbf{Lemma 111.5} If $r < A$

$$\text{Sq}^r f^A = \sum_{u=0}^{r} f^{r-u} f^{A-u} \binom{A-r+u-1}{u}$$

In the case of a linear involution of $G_2(R^{N+1})$ ($N$ even), it is worthwhile to calculate Lemma 111.5 somewhat more explicitly. Since $\text{Sq}^1 h_N = h_1 h_N + h_{N+1}$, the relation $f^{N+1}$ is determined by $f^N$. Thus $\text{sq}^1 f^N = (h_1 + \rho_1) f^N + f^{N+1}$ is an alternative (used elsewhere in this paper) for the relation of degree $N+1$.

The expansion formula $h_{N+r} = (h_r + h_1 h_{r-1}) h_N + h_{r-1} h_{N+1}$ immediately tells us $f^{N+r} = (h_r + h_1 h_{r-1}) f^N + h_{r-1} f^{N+1}$. Using this, Lemma 111.5 may be reformulated as follows:

\textbf{Lemma 111.6} For a linear involution of $BO(2)$ with at most $N+1$ nonzero weights (counting multiplicities),

$$\text{Sq}^m f^N = b^m f^N + c^m \text{Sq}^1 f^N$$

where
\[ c^m = \sum_{j=0}^{m-1} \rho_j \sum_{u=1}^{m-j} \binom{N-m+u-1}{u} h_{u-1} h_{m-u-j} \]

\[ b^m = \rho_1 c^m + \sum_{j=0}^{m} \rho_j \sum_{u=0}^{m-j} \binom{N-m+u-1}{u} h_u h_{m-u-j} \]

**Proof:** By lemma 111.5

\[ S^{m_0}_N = \left[ \sum_{u=0}^{m} \binom{N-m+u-1}{u} h_u f^{m-u} \right] f^N + \left[ \sum_{u=0}^{m} \binom{N-m+u-1}{u} h_{u-1} f^{m-u} \right] \left[ f^{N+1} + h_1 f^N \right] \]

So

\[ c^m = \sum_{u=1}^{m} \binom{N-m+u-1}{u} \sum_{j=0}^{m-u} \rho_j h_{u-1} h_{m-u} \]

and

\[ b^m = \rho_1 c^m + \sum_{u=0}^{m} \binom{N-m+u-1}{u} \sum_{j=0}^{m-u} \rho_j h_u h_{m-u} \]

But

\[ \sum_{u=0}^{m} \sum_{j=0}^{m-u} \phi_{ju} = \sum_{j=0}^{m} \sum_{u=0}^{m-j} \phi_{ju} \]

Note that for the linear involution

\[
\begin{pmatrix}
-1 & 0 \\
0 & I_{N-1}
\end{pmatrix}
\]

we have \( \rho_j = \binom{1}{j} t^j \) where \( t \) is the generator of \( H^1(B_T; Z_2) \).
IV. Binomial Coefficient Facts

The basic method for calculating binomial coefficients modulo 2 is to use the relationship

\[
\binom{b}{a} = \prod_{i=0}^{\infty} \binom{b_i}{a_i},
\]

where

\[b = \sum b_i \cdot 2^i, \quad a = \sum a_i \cdot 2^i, \quad \text{and} \quad a_i, b_i \in \{0,1\}.
\]

It follows from this that

\[
\binom{1}{k} = \prod_{i=0}^{\infty} \binom{1}{k_i \cdot 2^i} = \prod_{\text{all } k_i = 1} \binom{1}{2^i}
\]

Also the i’th bit of 1 is given by \(\binom{1}{2^i}\). Thus we have:

\[
\binom{1}{2^i}
\]

**Lemma IV.1** The sequence \(a_k = \binom{1}{k}\) is completely characterized by the set of recursion relations:

1. \(a_0 = 1\)

2. \(v = 2^i (2^i+1) \implies a_v = a_{2^i} a_{v-2^i}\)

Moreover \(1 = \sum a_{2^i} \cdot 2^i\).

If we generalize (2) to include \(a_v = 0\) for \(v > N+1\), then we can conclude \(1 \leq N+1\).

Recall that for the linear involution given by 1 -1's along the diagonal, one of the relations in \(H^*_T(G_2(R^{N+1}); Z_2)\) was \(f = \sum_{j=0}^{N} \binom{1}{j} t^j h_{N-j}\). Lemma IV.1 will be our tool for
recovering the coefficients \( \binom{1}{j} \) of \( t^j h_{N-j} \).

Certain combinatorial properties of binomial coefficients will be useful for us. Firstly, if \( j < 2^k \) and \( 1 \equiv 1' \pmod{2^k} \), then \( \binom{1}{j} = \binom{1'}{j} \). When no confusion will result, we shall generally extend this relation to negative values of \( 1 \); e.g. \( \binom{-1}{j} = \binom{2^k-1}{j} \).

The ordinary binomial coefficients satisfy the Vandermonde convolution formula over \( \mathbb{Z} \):

\[
\binom{n}{m} = \sum_{v=0}^{v=n} \binom{v}{j} \binom{n-v}{m-j}
\]

(This is obvious from the identification of binomial coefficients with the counting of combinations.) When \( v \) is a power of 2, this formula becomes extremely simple over \( \mathbb{Z}_2 \).

**Corollary IV.2**

\[
\binom{n}{m} = \binom{n-2^k}{m} + \binom{n-2^k}{m-2^k}
\]

**Proof:** If \( v = 2^k \) in the Vandermonde convolution formula, then \( \binom{v}{j} = 0 \) unless \( j = 0 \) or \( 2^k \).

The following lemma will not be used later in this paper. However it is a rather pretty result, and is of great technical value as a "reserve" for many of the arguments used later on.

**Lemma IV.3** For fixed \( w \) (with \( 2^w \leq w \leq 2^{n+1} \)), integers
$0 \leq x \leq w$ for which \( \binom{x}{w-x} \) = 1 modulo 2 correspond bijectively to sequences (including the empty sequence corresponding to \( x = w \)) \( 0 \leq a_0 < b_0 < \ldots < a_k < b_k < \ldots < a_j < b_j \leq n \) with \( w_{a_k} = 0 \) and \( w_{b_k} = 1 \). Explicitly, the binary expansions of \( x \) and \( w-x \) are given by:

1. \( b_{k-1} < 1 < a_k \) implies \( x_1 = w_1 \) and \( (w-x)_1 = 0 \).
2. \( 1 = a_k \) implies \( x_1 = 1 \) and \( (w-x)_1 = 1 \).
3. \( a_k < 1 < b_k \) implies \( x_1 = 1 \) and \( (w-x)_1 = w_1 \).
4. \( 1 = b_k \) implies \( x_1 = 0 \) and \( (w-x)_1 = 0 \).

In order to prove this lemma, we need to review addition in base 2.

(A) Suppose \( p + q = r \). Write \( p = \sum p_i \cdot 2^i \) where \( p_i \in \{0,1\} \) (\( q \) and \( r \) similarly.) Then for this addition

\[
\begin{align*}
\ldots p_4 & p_3 p_2 p_1 p_0 \\
+ & \ldots q_4 q_3 q_2 q_1 q_0 \\
\end{align*}
\]

we have \( r_i + 2c_i = p_i + q_i + c_{i-1} \) where \( c_i \) is the \( i \)'th carry.
(This inductively defines \( c_i \) subject to \( c_{-1} = 0 \).)

(B) Suppose still that \( p + q = r \). Then the values of \( i \) for which \( c_i = 1 \) separate into strings indexed by

\( 0 \leq a_0 < b_0 < a_1 < b_1 < \ldots < a_j < b_j \leq n \) so that

2. Subscripts on \( w \) and \( x \) refer to bits in their binary expansions.
\[ c_i = 1 \text{ iff } i \in \bigcup_{k=0}^{j} [a_k, b_k) \]

Now \( c_{a_k} = 1 \) and \( c_{a_k - 1} = 0 \) implies \( r_{a_k} = p_{a_k} + q_{a_k} + 2 = p_{a_k} + q_{a_k} \)

which means \( p_{a_k} = q_{a_k} = 1 \) and \( r_{a_k} = 0 \). Similarly, \( c_{b_k} = 0 \)

and \( c_{b_k - 1} = 1 \) implies \( r_{b_k} = p_{b_k} + q_{b_k} + 1 \) which means \( p_{b_k} = q_{b_k} = 0 \) and \( r_{b_k} = 1 \).

(C) Conversely for fixed \( w \), given any sequence \( \{a_i, b_i\} \) with \( 0 \leq a_0 < b_0 < \ldots < a_j < b_j \leq n \) and \( w_{a_k} = 0 \), \( w_{b_k} = 1 \), there exist integers \( p \) and \( q \) so that \( p + q = w \) realizing the given sequence of carries \( c_i \).

Moreover, if \( p \) and \( q \) satisfy \( p_i \geq q_i \), then \( p \) and \( q \) are uniquely determined. (Uniqueness is because the equation \( p_i + q_i = s \) (where \( p_i, q_i \in \{0, 1\} \) and \( s \in \{0, 1, 2\} \)) has a unique solution with \( p_i \geq q_i \). The conditions \( w_{a_k} = 0 \) and \( w_{b_k} = 1 \) insure we never have to try to solve the equations \( p_i + q_i = -1 \) or \( 3 \).)

Given the above considerations, the proof of IV.3 is now obvious upon observing:

1. \( b_{k-1} < 1 < a_k \) implies \( 2c_1 - c_{1-1} = 0 \).
2. \( 1 = a_k \) implies \( 2c_1 - c_{1-1} = 2 \).
3. \( a_k < 1 < b_k \) implies \( 2c_1 - c_{1-1} = 1 \).
4. \( 1 = b_k \) implies \( 2c_1 - c_{1-1} = -1 \).

We shall also need the following formula readily proven by induction on \( 1 \).

**Lemma IV.4** \[ \sum_{u=0}^{\infty} \binom{1}{u} \binom{N+1-u}{j-u} = \binom{N+1-1}{j} \]
V. The Basic Fixed Point Theorem

In this section, we shall show for the linear involutions of section III on $G_2(R^{N+1})$, that the equivariant cohomology uniquely determines the cohomology of the fixed point set. The key is the fixed point theorem of Wu - Yi Hsiang. (See Cohomology Theory of Topological Transformation Groups for details.) To fix notation, for $T$ a $Z_2$-torus, set $R = H^*(\text{point}) = H^*(B_T)$ and $R_0$ the quotient field of $R$. Also note that $H^*_T(F;Z_2) = H^*(F;Z_2) \otimes Z_2 R$ where $F$ is the fixed point set of the $T$ action on $X$.

Theorem V.1 (Hsiang, $Z_2$-torus case) Let $\xi_1, \ldots, \xi_k$ be a generator system of the $R_0$-algebra $H^*_T(X;Z_2) \otimes R R_0$, and $I$ be the ideal of defining relations; namely $I = \ker(\rho)$ of the following epimorphism:

$$\rho: A = R_0[x_1, \ldots, x_k] + H^*_T(X;Z_2) \otimes R R_0 \xrightarrow{x_j + \xi_j}$$

Then

(i) The radical of $I$, $\sqrt{I}$ decomposes into the intersection of $s$ maximal ideals $M_j = M(\alpha_j)$ whose varieties are respectively the rational points $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jk}) \in R_0^k$; i.e.

$$\sqrt{I} = M_1 \cap \ldots \cap M_k,$$

$$\nu(I) = \{\alpha_1, \ldots, \alpha_k\} \subseteq R_0^k$$

(ii) There is a one-to-one correspondence between the connected components of the fixed point set $F = F^1 + \ldots + F^k$ and the above points $\{\alpha_{j1}, \ldots, \alpha_{jk}\}$ such that the restriction
homomorphism of an arbitrary point \( q_j \in F_j \subseteq X \) maps \( \xi_j \in H^*_T(X; \mathbb{Z}_2) \) to \( \alpha_j \in H^*_T(q_j; \mathbb{Z}_2) \).

(iii) \( H^*(F_j; \mathbb{Z}_2) \otimes \mathbb{Z}_2 R_0 \cong A/l_j \) where \( l_j = l_{M_j} \cap A \) and \( l_{M_j} \) is the localization of \( l \) at \( M_j \).

(iv) \( l = l_1 \cap \ldots \cap l_k = l_1 \cdot l_2 \cdots \cdot l_k \)

For the remainder of this section, we shall use the following notations:

\[
T = \mathbb{Z}_2 \quad R = \mathbb{Z}_2[t]
\]

\( p \in F \) is an arbitrary point in the fixed point set \( \mathcal{F}_p \), \( p \rightarrow G_2(R^{N+1}) \) is the inclusion \( i_p: \mathcal{F}_p \rightarrow G_2(R^{N+1}) \) is the inclusion of the component \( F_p \) (containing \( p \)) of the fixed point set \( F \).

\[
f(n, l) = \sum_{j=0}^{l} \rho_j h_{n-j} \quad \text{where} \quad \rho_j = \binom{l}{j} t^j
\]

For the maps in \( H^*_T \) induced by \( i_p \) and \( j \):

\[
j* h_2 = n_2 + t n_1 + t^2 n_0 \quad i_p* h_2 = t^2 n_0
\]

\[
j* h_1 = m_1 + t m_0 \quad i_p* h_1 = t m_0
\]

Here \( m_k, n_k \in H^k(F_p; \mathbb{Z}_2) \) and we identify \( H^0(p; \mathbb{Z}_2) \cong H^0(F_p; \mathbb{Z}_2) \).

In assuming the equivariant cohomology of the action to be identical to that of a linear action, we are assuming that the relations are generated by \( f(N, l) \) and \( f(N+1, l) \) for some \( l \leq N+1 \).

The easiest information to obtain from the equivariant cohomology is the number of components of the fixed point set. By V.1, these are simply parameterized by the points in the variety of the defining ideal. Since there are only four
possible values of the pair \((i \star h_1, i \star h_2)\), it is immediate
that there can be at most four components of the fixed point
set. However, since \(\text{Sq}^1_T \) is zero on \(H^2_{\text{T}}(p;Z_2) = H^2(RP^\infty;Z_2)\),
we know \(i \star \text{Sq}^1_p h_2 = 0\). But 
\[i \star \text{Sq}^1_p h_2 = i \star h_1(h_1^2 + h_2) = m_0 t(m_0^2 t^2 + n_0 t^2)\]. Hence \(m_0 (m_0 + n_0) = 0\), and \((i \star h_1, i \star h_2) = (t, 0)\) is impossible. Geometrically, this shows that the
fixed point set can have at most three components (as was
the case for the linear models.)

**Lemma V.2** If \(p\) is a fixed point of the involution, then
there are the following three possibilities for the maps
induced on equivariant cohomology by \(i_p\) and \(j\):
(Recall \(i_p: p \to G_2(R^{N+1})\) and \(j: F_p \to G_2(R^{N+1})\).)

(i) \[(a) \quad (i \star h_1, i \star h_2) = (0, 0)\]

\[\quad (b) \quad i \star h_k = 0 \text{ for all } k\]

\[\quad (c) \quad j \star h_2 = n_2 \text{ where } n_2 \in H^2(F_p;Z_2)\]

\[\quad j \star h_1 = m_1 \text{ where } m_1 \in H^1(F_p;Z_2)\]

\[\quad (d) \quad \text{Sq}^1 n_2 = m_1(n_2 + m_1^2)\]

(ii) \[(a) \quad (i \star h_1, i \star h_2) = (0, t^2)\]

\[\quad (b) \quad i \star h_k = (k+1)t^k \text{ for all } k\]

\[\quad (c) \quad j \star h_2 = n_2 + tn_1 + t^2\]

\[\quad j \star h_1 = n_1\]

\[\quad (d) \quad \text{Sq}^1 n_2 = n_1(n_1 + n_2)\]

(iii) \[(a) \quad (i \star h_1, i \star h_2) = (t, t^2)\]

\[\quad (b) \quad i \star h_k = t^k \text{ for all } k\]
(c) \(j * h_2 = (n_1^2 + n_1 m_1 + m_1^2) + t n_1 + t^2\)
\[j * h_1 = m_1 + t\]

Proof: Claim (a) of each of the three parts simply represents the three remaining possibilities for \((i * h_1', i * h_2')\).

Claim (b) in all cases follows easily from the recurrence relation:
\[i * p_{h+2} = (i * p_{h1})(i * p_{h+k+1}) + (i * (h_1^2 + h_2))(i * p_{h+k})\]

The remaining claims all follow by applying \(j^*\) to the relationship \(S^1 h_2 = h_1(h_1^2 + h_2)\). In more detail:

**case (i):**
\[j * S^1 h_2 = S^1(n_1 + t n_1) = S^1 n_1 + t n_1^2 + t^2 n_1\]
\[j * (h_1(h_1^2 + h_2)) = m_1(m_1^2 + n_2 + t n_1)\]
\[= m_1(m_1^2 + n_2) + t n_1 m_1\]

Comparing the coefficients of \(t^0\) and \(t^2\) shows \(n_1 = 0\) and \(S^1 n_2 = m_1(m_1^2 + n_2)\)

**case (iii):**
\[j * S^1 h_2 = S^1(n_2 + tn_1 + t^2)\]
\[= S^1 n_2 + tn_1^2 + t^2 n_1\]
\[j * (h_1(h_1^2 + h_2)) = (m_1 + t)(m_1^2 + n_2 + t n_1)\]
\[= m_1(m_1^2 + n_2) + t(m_1^2 + m_1 n_1 + n_2)\]
\[+ t^2 n_1\]

Therefore \(n_2 = m_1^2 + m_1 n_1 + n_1^2\).

**case (ii):**
\[j * S^1 h_2 = S^1(n_2 + t n_1 + t^2)\]
\[= S^1 n_2 + t n_1^2 + t^2 n_1\]
\[j * (h_1(h_1^2 + h_2)) = m_1(m_1^2 + n_2 + t n_1 + t^2)\]
\[= m_1(m_1^2 + n_2) + t n_1 m_1 + t^2 m_1\]

Therefore \(n_1 = m_1\) and \(S^1 n_2 = n_1(n_1^2 + n_2)\).
Geometrically, cases (i) and (ii) above correspond to components of Grassman type, while case (iii) is a product of projective spaces. Note that we have already shown the cohomology of each component to be generated by elements of appropriate degree and Steenrod algebra structure.

The linear transformations
\[
\begin{pmatrix}
-1 & 0 \\
0 & l_{N+1-1}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
l_1 & 0 \\
0 & -l_{N+1-1}
\end{pmatrix}
\]
clearly induce the same maps on Grassmannians. This is reflected algebraically in the following lemma:

**Lemma V.3** The reparameterization \( \phi: H^*(BO(2); \mathbb{Z}_2) \otimes R \to H^*(BO(2); \mathbb{Z}_2) \otimes R \) defined by
\[
\begin{align*}
h_1 + h_1 & \\
h_2 + h_2 + th_1 + t^2
\end{align*}
\]
takes \( f(N,1) \) to \( f(N,N+1-1) \).

**Proof:** First we'll prove the case \( l = 0 \) by induction on \( N \). The cases \( N \leq 2 \) are clear. Note \( f(N,0) = h_N^N \). \[ \phi(h_N^N) = \phi(h_1^1 h_{N-1} + (h_1^2 + h_2^2) h_{N-2}) \]
\[
= h_1 \sum_{k=0}^{N-1} \binom{N}{k} t^k h_{N-1-k} + (h_1^2 + h_2^2 + th_1 + t^2) \sum_{k=0}^{N-2} t^k \binom{N-1}{k-1} h_{N-k-1}
\]
\[
= \sum_{k=0}^{N-1} \binom{N}{k} t^k h_1^1 h_{N-1-k} + \sum_{k=0}^{N-2} \binom{N-1}{k} t^k (h_{N-k} + h_1^1 h_{N-k-1} + \sum_{k=0}^{N-1} \binom{N-1}{k} t^k h_1^1 h_{N-k-1} + \sum_{k=0}^{N} \binom{N-1}{k-2} t^k h_{N-k}
\]
}\]
\[ = \sum_{k=0}^{N-1} t^k h_{N-1-k} \left[ \binom{N}{k} + \binom{N-1}{k} + \binom{N-1}{k-1} \right] + t^{N-1} h_1 \\
+ \sum_{k=0}^{N-2} t^k h_{N-k} \left[ \binom{N-1}{k} + \binom{N-1}{k-2} \right] + (N-1) t^{N-1} h_1 \\
+ \frac{(N-1) t^N}{N-2} \\
= \sum_{k=0}^{N-2} \binom{N+1}{k} t^k h_{N-k} + \frac{1}{2} \binom{N+1}{2} t^{N-1} h_1 + (N+1) t^N \\
= \sum_{k=0}^{N} \binom{N+1}{k} t^k h_{N-k} \\
= f(N, N+1) \]

This proves the case \( l = 0 \). To see the general case:

\[
(f(N,1)) = \phi \sum_{u=0}^{1} \binom{1}{u} t^u h_{N-u} \\
= \sum_{u=0}^{1} \binom{1}{u} t^u \phi(f(N-u, 0)) \\
= \sum_{u=0}^{1} \binom{1}{u} t^u f(N-u, N-u+1) \\
= \sum_{u=0}^{1} \binom{1}{u} t^u \sum_{k=0}^{N-u} \binom{N-u}{k} t^k h_{N-u-k} \\
= \sum_{u=0}^{1} \sum_{k=0}^{N-u} \binom{N-u}{k} \binom{1}{u} t^k h_{N-u-k} \\
= \sum_{j=k+u} \sum_{u=0}^{1} \binom{N}{k} \binom{1}{u}(N+1-u) h_{N-u-k} \\
= \sum_{u=0}^{1} \sum_{j=u}^{N} t^j h_{N-j} \binom{N+1-u}{j-u} \\
= \sum_{j=0}^{1} \sum_{u=0}^{1} \Theta_{j u} + \sum_{j=1+1} \sum_{u=0}^{1} \Theta_{j u} \\
= \sum_{j=0}^{1} t^j h_{N-j} \sum_{u=0}^{1} \binom{1}{u}(N+1-u) \\
+ \sum_{j=1+1} t^j h_{N-j} \sum_{u=0}^{1} \binom{1}{u}(N+1-u) \]

Let \( j = k + u \). Then this expression equals

\[
\sum_{u=0}^{1} \sum_{j=u}^{N} t^j \binom{1}{u}(N+1-u) h_{N-j} \\
= \sum_{j=0}^{1} \sum_{u=0}^{1} \Theta_{j u} + \sum_{j=1+1} \sum_{u=0}^{1} \Theta_{j u} \\
= \sum_{j=0}^{1} t^j h_{N-j} \sum_{u=0}^{1} \binom{1}{u}(N+1-u) \\
+ \sum_{j=1+1} t^j h_{N-j} \sum_{u=0}^{1} \binom{1}{u}(N+1-u) \]

Thus \( \phi(f(N,1)) = \sum_{j=0}^{1} t^j h_{N-j} \sum_{u=0}^{1} \binom{1}{u}(N+1-u) \\
+ \sum_{j=1+1} t^j h_{N-j} \sum_{u=0}^{1} \binom{1}{u}(N+1-u) \]
But \( \sum_{u=0}^{\infty} \binom{1}{u} \binom{N+1-u}{j-u} = \binom{N+1-1}{j} \) by IV.4 and hence

\[ \phi(f(N,1)) = f(N,N+1-1) \]

We are now ready to explicitly calculate the fixed point set.

**Proposition V.4** If \( H\pi(G_2(R^{N+1});Z_2) \cong Z_2[h_1,h_2,t]/<f(N,1), f(N+1,1)> \) and \( Sq^1h_2 = h_1(h_1^2 + h_2) \), then the fixed point set will have three \( 3 \) components of \( Z_2 \) cohomology types \( G_2(R^1), G_2(R^{N+1-1}) \), and \( RP^{1-1} \times RP^{N-1} \).

Proof: By lemma V.2, there are at most three components indexed by the possible values of \( (i^*h_1, i^*h_2) \).

**case (i)** \( (i^*h_1, i^*h_2) = (0,0) \)

Then if \( 1 < N, i^*f(N,1) = 0 = i^*f(N+1,1) \). Therefore \( (0,0) \) is a point in the variety of the ideal, and there really is a nonempty component of the fixed point set indexed by this possibility for \( i^* \). (If \( 1 = N \) or \( N+1, (0,0) \) is not in the variety of the ideal.) To calculate the cohomology of this component, we must localize \( I \) at \( (0,0) \).

\[ f(N,1) = \sum_{j=0}^{1} \rho_j h_{N-j} \text{ where } \rho_j = \binom{1}{j} t^j \]

But \( h_{N-1+j} = (h_{1-j})h_{N-1} + (h_{1-j-1})(h_1 h_{N-1} + h_{N-1+1}) \)

3. Note when \( 1 = 0,1,N \) or \( N+1, some of these components are \( \phi \).
Thus \( f(N,1) = h_{N-1}f(1,1) + (h_{N-1+1} + h_1h_{N-1})f(1-1,1) \). By identical reasoning, \( f(N+1,1) = h_{N-1}f(1+1,1) + (h_{N-1+1} + h_1h_{N-1})f(1,1) \). Then \( f(1,1)f(N,1) + f(1-1,1)f(N+1,1) \in \mathcal{L} \).

This is just \([f(1,1)^2 + f(1+1,1)f(1-1,1)]h_{N-1}\). Since \( i_p^*f(1,1) = t^1 \) and \( i_p^*f(1-1,1) = 0 \), we see that \([f(1,1)^2 + f(1+1,1)f(1-1,1)] \) is invertible at \((0,0)\) and

\[
h_{N-1} \in \mathcal{L}(0,0)
\]

Similarly \( f(1+1,1)f(N,1) + f(1,1)f(N+1,1) = [f(1+1,1)f(1-1,1) + f(1,1)^2] (h_1h_{N-1} + h_{N-1+1}) \) is in \( \mathcal{L} \) and hence

\[
h_1h_{N-1} + h_{N-1+1} \in \mathcal{L}(0,0)
\]

Since \( h_{N-1}, h_{N-1+1} \in \mathcal{L}(0,0) \), \( j^*h_{N-1} \) and \( j^*h_{N-1+1} \) will be relations in \( \text{H}^*(F_p; \mathbb{Z}_2) \otimes R_0 \). But \( j^*h_2 = n_2, j^*h_1 = m_1 \), and \( \text{Sq}^1n_2 = m_1(n_2 + m_1^2) \), so this is just the standard description of \( \text{H}^*(G_2(R^{N+1}); \mathbb{Z}_2) \). (Note by a dimension count comparison with the linear models, there cannot be any other relations in \( \text{H}^*(F_p; \mathbb{Z}_2) \).)

**case (ii)** \( (i_p^*h_1, i_p^*h_2) = (0, t^2) \)

The reparameterization of V.3 interchanges this case with (i) - replacing 1 by \( N+1-1 \). Thus for \( l > 1 \), we get a component of type \( G_2(R^1) \).
\text{case (iii)} \quad (i^*h_1, i^*h_2) = (t, t^2)

If \( l \neq 0 \) or \( N+1 \) then

\[ i^*f(N, l) = \sum_{j=0}^{l} \binom{l}{j} t^j = 2^l t^l = 0 \quad \text{and} \]

\[ i^*f(N+1, 1) = 2^l t^{l+1} \]

Let's first note by induction on \( k \) that

\[(h_1^2 + h_2)^k f(N-k, 1) \in I \quad \text{for} \quad k \leq N+1-1\]

Proof: Suppose this is true for \( k-1 \) and \( k-2 \) with \( k \leq N+1-1 \).

Then

\[ (h_1^2 + h_2) [(h_1^2 + h_2)^{k-2} f(N-k+2, 1)] + \]

\[ h_1 [(h_1^2 + h_2)^{k-1} f(N-k+1, 1)] \]

\[ = (h_1^2 + h_2)^{k-1} [h_1 f(N-k+1, 1) + (h_1^2 + h_2) f(N-k, 1)] \]

\[ + (h_1^2 + h_2)^{k-1} h_1 f(N-k+1, 1) \]

\[ = (h_1^2 + h_2)^k f(N-k, 1) \]

Now set \( k = N+1-1 \). We have \( f(N-k, 1) = f(l-1, 1) \) and

\[ i^*f(l-1, 1) = t^{l-1} \]

\[ j^*(h_1^2 + h_2) = n_1 (n_1 + m_1 + t) \]

Thus the coefficient of \( t^N \) in \( j^*[(h_1^2 + h_2)^{N+1-1} f(l-1, 1)] \) is \( n_1^{N+1-1} \). Therefore this is one of the relations in \( H^*(F_p; \mathbb{Z}_2) \).

Applying the reparameterization of V.3 replaces \( n_1 \) by \( n_1 + m_1 \) and \( N+1-1 \) by \( l \). Thus we also have \( (n_1 + m_1)^l = 0 \).

By dimension count, these two relations exhaust the relations in \( H^*(F_p; \mathbb{Z}_2) \) and show that \( F_p \) is \( \mathbb{R}P^{l-1} \times \mathbb{R}P^{N-1} \).
VI. Setting up of the Proof

For the remainder of this paper, we shall assume $X$ to be a space (with involution) of $\mathbb{Z}_2$ cohomology type (over the Steenrod algebra) $G_2(\mathbb{R}^{N+1})$ where $N$ is even and not congruent to 64 modulo 192.

Section III showed that the equivariant cohomology of a linear involution on $G_2(\mathbb{R}^{N+1})$ is given by

$Z_2[h_1, h_2, t]/\langle f, g \rangle$

where

$f = \sum_{j=0}^{N} \binom{1}{j} t^j h_{N-j}$

$g = Sq^1 f$

$0 \leq 1 \leq N+1$

Section V showed that any involution with the above equivariant cohomology would have the same fixed point structure as that of a linear involution. Thus the proof of theorem 0 is reduced to showing that the equivariant cohomology of an arbitrary involution of $X$ (with nonempty fixed point set) must be isomorphic to one of the linear possibilities.

Letting $T$ denote the group $\mathbb{Z}_2$, consider the canonical fibration associated to the Borel construction:

$X \rightarrow X_T \rightarrow \mathbb{S}_T$

In studying the Serre spectral sequence of $X_T$, the first important question is must we use local coefficient systems?
Lemma VI.1 If \( N \) is even and not congruent to 64 modulo 192, then the action of \( \mathcal{G}_1(B_T) \) on \( H^*(X;Z_2) \) is trivial.

Actually, in the dimension range stated, \( H^*(G_2(R^{N+1});Z_2) \) has no nontrivial automorphisms. This fact is proven in appendix A. It should be noted that \( H^*(G_2(R^k);Z_2) \) when \( k \) is a power of 4 does have a nontrivial involution.

The Borel criterion for the existence of a nonempty fixed point set is that the map \( \Phi^* : H^*(B_T) \to H^*(X_T) \) be injective. (This criterion is an immediate corollary of VI.1.) In our case, this criterion has the effect of collapsing the spectral sequence of \( X_T \).

Lemma VI.2 If \( N \) is even and the involution has a fixed point, then the spectral sequence of \( X_T \) collapses. Thus additively

\[
H^*_1(X;Z_2) = H^*(X;Z_2) \otimes Z_2[t]
\]

and \( i^* : H^*(X_T;Z_2) \to H^*(X;Z_2) \) is surjective.

Proof: Consider the \( E_2 \) term of the spectral sequence of \( X_T \).

\[
\begin{array}{cccc}
  & & & \\
  & h^2 \mathcal{G}_1, h_2 & \ldots & \\
  \vdots & \vdots & \vdots & \\
 h_1 & t h_1 & t^2 h_1 & \ldots \\
 1 & t & t^2 & t^3 \\
\end{array}
\]
Since \( \varphi^* \) is injective, \( d_2h_1 = 0 \). Suppose \( d_2h_2 \neq 0 \). Then 
\[ d_2h_2 = t^2h_1. \]

**Claim:** 
\[ d_2h_m = (m+1)t^2h_{m-1} \] 
for all \( m \geq 0 \).

**Proof of claim:** 
\[ h_{m+2} = h_1h_{m+1} + (h_1^2 + h_2)h_m \] 
yields 
\[ d_2h_{m+2} = h_1d_2h_{m+1} + t^2h_1h_m + (h_1^2 + h_2)d_2h_m. \] 
The assertion now follows trivially by induction on \( m \).

Since \( N \) is even, \( h_N = 0 \), but \( d_2h_N = t^2h_{N-1} \neq 0 \).

Therefore \( d_2h_2 = 0 \). Thus \( E_2 = E_3 \). Once again, because \( \varphi^* \) is injective, \( d_2h_2 = 0 \), and all higher differentials in the spectral sequence vanish.

The assumption in the above lemma that \( N \) be even is quite essential. An almost complex structure on \( R^{2n} \) will induce an involution of \( G_2(R^{2n}) \) whose spectral sequence does not collapse at the \( E_2 \) level.

Lemma VI.2 shows that we may choose lifts \( h_1^*, h_2^* \in H^*(X) \) of the generators \( h_1, h_2 \in H^*(X) \). However these lifts need not be compatible with the Steenrod algebra action.

**Lemma VI.3** The lifts \( h_1^* \) and \( h_2^* \) may be chosen so as to preserve the Steenrod algebra action on \( H^*(X;\mathbb{Z}_2) \).

**Proof:** All that need be shown is that \( h_1^* \) and \( h_2^* \) may be chosen so that 
\[ Sq^1h_2^* = h_1^*(h_1^2 + h_2^*). \]
Given arbitrary lifts $h_1^*$ and $h_2^*$, define $h_3^* = h_1^* h_2^*$. Then

$$\text{Sq}^1 h_2^* = a_3 h_3^* + a_{12} h_1^* h_2^* + a_2 t h_2^* + a_{11} t^2 h_1^* + a_0 t^3$$

where the $a$'s are in $\mathbb{Z}_2$.

(i) Restriction of this relation to $H^*(X;\mathbb{Z}_2)$ (i.e. modulo $t$) shows $a_3 = a_{12} = 1$.

(ii) The substitution $h_1^* + h_2^* + a_2 t$ shows we may assume $a_2 = 0$.

(iii) The substitution $h_2^* + h_2^* + a_1 t^2$ permits us to assume $a_1 = 0$.

(iv) $\text{Sq}^1 \text{Sq}^1 h_2^* = 0$ implies $a_0 = a_{11} = 0$.

Henceforth, and for all time we shall fix lifts of $h_1$ and $h_2$ into $H^*(X;\mathbb{Z}_2)$ satisfying the conclusion of VI.3. To avoid extra notation, we shall also denote these lifts by $h_1$ and $h_2$. Using the basic recursion formula $h_{k+2} = h_1 h_{k+1} + (h_1 + h_2) h_k$, we can define classes $h_k^* \in H^*(X;\mathbb{Z}_2)$ whose restriction to the fibre is just the usual $h_k^* \in H^k(X;\mathbb{Z}_2)$.

The classes $\{h_a h_b : 0 \leq a, b \leq N-1\}$ will be a basis for the $\mathbb{Z}_2[t]$ module $H_{\mathcal{T}}^*(X;\mathbb{Z}_2)$. Also the $h_k^* \in H^*_T(X;\mathbb{Z}_2)$ will satisfy the multiplication formula in 11.5 and the Steenrod algebra formulae of 11.7 and 11.9. Thus the filtrations $H_T^*$ and $S^*_a$ are defined and satisfy 11.8.

The class $h_N^* \in H^*_T(X;\mathbb{Z}_2)$ may be expressed via the module basis as $\sum_{v=0}^N t^v h_{N-v}$ where $f_{N-v}$ is a linear combination (over $\mathbb{Z}_2$) of terms $h_j h_{N-v-j}$. (Note that there is no $v = 0$ term since $h_N$ restricted to $H^*(X;\mathbb{Z}_2)$ is zero.) Accordingly, one
of the relations in $H^*_T(X;\mathbb{Z}_2)$ is $f = h_N + \sum_{v=1}^N t^v f_{N-v}$. The polynomial $g = Sq^1 f$ restricts to $h_1 h_N + h_{N+1} \in H^*(X;\mathbb{Z}_2)$ and thus determines the expansion of $h_{N+1} \in H^*(X;\mathbb{Z}_2)$ in terms of the module basis.

Using the multiplication formula 11.5 and the recursion formula 11.2, it is quite clear that the relations $f$ and $g$ suffice to express any product of module basis elements in terms of the module basis. Thus $f$ and $g$ completely determine the multiplicative structure of the equivariant cohomology; i.e.

$$H^*_T(X;\mathbb{Z}_2) = \mathbb{Z}_2[h_1, h_2, t]/\langle f, g \rangle$$

where $\langle f, g \rangle$ is the ideal generated by $f$ and $g$. So the proof of theorem 0 is reduced to proving:

$$f_{N-v} = \binom{1}{v} h_{N-v}$$

Since $f$ and $g$ generate the ideal of relations in $\mathbb{Z}_2[h_1, h_2, t]$, $Sq^m f$ and $Sq^m g$ must also lie in this ideal. We shall show that the conditions

$$Sq^m f = af + bg, \quad m = 2$$
$$Sq^2 g = df + eg$$

require $f$ to be $\sum_{j=0}^N \binom{1}{j} t^j h_{N-j}$ for some $1 \leq N+1$. 
VII. Inductive Hypothesis and Notation

The proof that \( f_{N-v} = \binom{1}{v} h_{N-v} \) will be by induction on \( v \). Denoting \( f \) by \( h_N + \sum_{w=1}^{N} t^w f_{N-w} \) where \( f_{N-w} \in Z_2[h_1, h_2] \subseteq Z_2[h_1, h_2, t] \), the inductive hypothesis at stage \( v \) will be:

**Inductive Hypothesis** For all \( w < v \)

1. \( f_{N-w} = F^w h_{N-w} \) where \( F^w \in Z_2 \).
2. If \( w = 2^i (2^i+1) \), then \( F^w = F^{2^i} F^{2^{i-1}} \).

This hypothesis is trivially satisfied for \( v = 1 \). By lemma IV.1, verifying the inductive hypothesis for \( v \leq N+1 \) (and the relations (2) for \( w > N+1 \) and \( i \geq 1 \)) will suffice to prove \( f_{N-v} = \binom{1}{v} h_{N-v} \) with \( i \leq N+1 \).

Assume now that the inductive hypothesis at stage \( v \) is satisfied. The remainder of this paper shall be concerned with verifying this hypothesis for stage \( v+1 \).

The proof of the inductive hypothesis will come from analyzing the conditions

- \( \text{Sq}^m f \in \langle f, g \rangle \), \( m = 2^k \)
- \( \text{Sq}^2 g \in \langle f, g \rangle \), \( g = \text{Sq}^1 f \)

In fact, the verification of the inductive hypothesis for \( v + 1 \) will only require studying the above conditions modulo \( t^{v+1} \). The filtrations \( H^r \) and \( S^b_a \) of section II will provide the framework for efficiently understanding these conditions.

More precisely, suppose \( \text{Sq}^m f = af + bg \). Write \( f = \sum_{j=0}^{N} t^j f_{N-j} \),
\[ g = \sum_{j=0}^{N+1} t_j g_{N+1-j}, \quad a = \sum_{j=0}^{m} t_j a_{m-j}, \quad \text{and} \quad b = \sum_{j=0}^{m-1} t_j b_{m-j-1} \]

where the subscripts indicate the degrees (in \( h_1 \) and \( h_2 \)) of the corresponding terms. Now the \( t^v \) term of \( S^m f = af + bg \) becomes:

\[
\sum_{k=0}^{m} \binom{v-k}{k} S^m f_{N-v+k} = \sum_{k=0}^{m} a_{m-k} f_{N-v+k} + \\
\sum_{k=0}^{m-1} b_{m-k-1} g_{N-v+k+1}
\]

However the inductive hypothesis says that for \( k \geq 1 \),
\( f_{N-v+k} \in S_0^0 \) and \( g_{N-v+k+1} \in S_0^1 \). Then 11.8 shows that modulo \( S_0^m \), this equation reduces to:

\[
S^m f_{N-v} = a_m f_{N-v} + b_{m-1} S^1 f_{N-v}
\]

This is a tremendous simplification! Here \( a_m \) and \( b_{m-1} \) are just the elements in \( H^*(BO(2); Z_2) \) that express \( S^m h_N \) as a combination of \( h_N \) and \( S^1 h_N = h_1 h_N + h_{N+1} \). (They are given explicitly by 111.6 with \( p_j = 0 \) for \( j \geq 1 \).)

Because the expression for \( S^m h_a h_b \) is combinatorially simpler than that for \( S^m h_a h_b \), it is often convenient to reformulate the conditions \( S^m f = a^m f + b^m g \) (here the superscripts index the square, not the degrees) in terms of the symmetric sums \( s_k \). Since multiplication by \( h_1^2 \) takes the module basis \( \{ h_a h_b \mid a, b \geq 0 \} \) of \( Z_2[h_1, h_2] \) into the module basis \( \{ s_c s_d \mid c, d \geq 1 \} \) of \( h_1^2 Z_2[h_1, h_2] \), it is quite easy to make this reformulation. Simply set \( F = s_2 f \) and \( G = s_2 g \). Then:
Lemma VII.1 \( Sq^m f = a^m f + b^m g \) for all \( m \) iff there exist polynomials \( A^m \) and \( B^m \) in \( h_1^2 Z_2 [h_1, h_2, t] \) so that
\[
Sq^m s_2 F = A^m F + B^m G
\]
for all \( m \). In fact
\[
A^m = s_2 a^m + s_2 s_4 a^{m-4}
\]
\[
B^m = s_2 b^m + s_2 s_4 b^{m-4}
\]

Proof: \( Sq^m s_2 F = Sq^m s_4 f \)
\[
= s_4 Sq^m f + s_4^2 Sq^{m-4} f
\]
\[
= s_4 (a^m f + b^m g) + s_4^2 (a^{m-4} f + b^{m-4} g)
\]
\[
= (s_2 a^m + s_2 s_4 a^{m-4}) F + (s_2 b^m + s_2 s_4 b^{m-4}) G
\]

We shall need a more explicit representation of \( f_{N-v'} \), namely \( f_{N-v'} = \sum_{j=0}^{\alpha} j^{1+1} J_{j} h_{N-v-j} \) where \( \alpha = [(N-v)/2] \). (Then the \( t^v \) term of \( F \) will be \( \sum_{j=1}^{\alpha+1} J_{j} h_{s_j} s_{N+2-v-j} \).) Since \( g_{N+1-v} = Sq^1 f_{N-v} + (v+1) f_{N-v+1} \), we can write \( g_{N+1-v} = \sum_{j=0}^{\alpha+1} \tilde{G}^{j+1} h_j h_{N+1-v} \) where \( \tilde{G}^{j} = (j+1) F^{j-1} + (j+v) F^j \) for \( j \geq 3 \).

Also when studying the condition \( Sq^2 f \in \langle f, g \rangle \), we shall set \( \delta = [\alpha/2] \). (This will be motivated in sections XI and XII.)
VIII. Relation Coefficients

Although the condition \( S_q^m f_{N-v} = a_m f_{N-v} + b_{m-1} g_{N-v+1} \mod S_{0}^{m-1} \) will be the workhorse of our proof, it is necessary for some purposes (such as identifying the \( F^v \) with binomial coefficients) to keep track of the other relation coefficients \( a_j \) and \( b_j \). First we need the following easy lemma:

**Lemma VIII.1** For all \( m > 0 \), \( h_m \) and \( h_{m+1} \) are coprime in the unique factorization ring \( Z_2[h_1, h_2] \).

**Proof:** Let \( m \) be the smallest \( k \) for which \( h_k \) and \( h_{k+1} \) have a nontrivial common factor. Then \( (h_1^2 + h_2) h_{m-1} = h_1 h_m + h_{m+1} \) shows that this common factor must be divisible by \( h_1^2 + h_2 \). But it is obvious by induction on \( r \) (using \( h_{r+1} = h_1 h_r + (h_1^2 + h_2) h_{r-1} \)) that \( h_r \) is never divisible by \( h_1^2 + h_2 \).

This lemma has the following important corollary. (Note \( S_{k+1}^{N-1} \cap H^{N+k} \) is just the complement in \( H^{N+k} \) of \( S_0^k \cap H^{N+k} \).)

**Corollary VIII.2**

1. If \( k < N \), the solution \( x_k, y_{k-1} \in Z_2[h_1, h_2] \) of the equation
   \[ x_k^N h_N + y_{k-1} h_{N+1} = c_{N+k} \mod S_{k+1}^{N-1} \cap H^{N+k} \]
is unique. (Here all subscripts indicate degrees.)

2. If \( k < N \), the solution \( x_{k+2}, y_{k+1} \in h_1^2 Z_2[h_1, h_2] \) of the equation
\[ X_{k+2}(s_1 s_{N+1}) + Y_{k+1}(s_1 s_{N+2}) = c_{N+k} \text{ modulo } S_{k+4}^N \cap H^{N+k+4} \]

is unique.

Proof: Uniqueness for the equation \( x_k h_N + y_{k-1} h_{N+1} = c_{N+k} \) is immediate from VII.1. Since the left hand side of this equation automatically lies in \( S_0^k \), we obtain uniqueness for the equation modulo \( S_{k+1}^{N-1} \cap H^N \).

Uniqueness for (2) follows from (1) applied to the equivalent equation \((s_2 x_{k+2}) h_N + (s_2 y_{k+1}) h_{N+1} = c_{N+k+4}\).

Recall from section VII that the \( t^v \) terms of \( Sq^m f = a f + bg \) are
\[
\sum_{k=0}^m \binom{v-k}{k} Sq^{m-k} f_{N-v+k} = \sum_{k=0}^m a_{m-k} f_{N-v+k} + \sum_{k=0}^{m-1} b_{m-k-1} g_{N-v+k+1}
\]

When \( v \leq m \), this may be rewritten
\[
a_{m-v} h_N + b_{m-v-1} (h_1 h_N + h_{N+1}) = \sum_{k=0}^v \binom{v-k}{k} Sq^{m-k} f_{N-v+k} + \sum_{k=0}^{v-1} a_{m-k} f_{N-v+k} + \sum_{k=0}^{v-1} b_{m-k} g_{N-v+k+1}
\]

Thus determining the right hand side of this equation modulo \( S_{m-v+1}^{N-1} \) (i.e. the coefficients of all terms \( h_r h_{N+m-v-r} \) for \( 0 \leq r \leq m-v \)) will uniquely determine \( a_{m-v} \) and \( b_{m-v-1} \).

Hence it is clear that our inductive hypothesis will allow us to recursively compute the relation coefficients \( a_j \) and \( b_j \). Fortunately, we shall be able to avoid detailed involvement with the combinatorics of this. Note that lemma III.6 explicitly calculates these relation coefficients for
the linear models. \( \rho_j = \binom{1}{j} \) in these formulae.)

The key to minimizing combinatorial computation is comparison with the linear models. The technique is illustrated in the proof of the following proposition.

**Proposition VIII.3** Suppose \( v = 2^r \) (\( r \geq 1 \)) and we have demonstrated part (1) of the inductive hypothesis at stage \( v+1 \); i.e. we know

1. For \( w \leq v \), \( f_{N-w}^v = F_{w}^n R_{N-w}^v \).
2. For \( w < v \), if \( w = 2^s \) (\( 2^s+1 \)) then \( F^w = F^2 F_{w-2^s}^v \).

Then

(a) In the relation \( S_q^v f = a f + b g \), we have \( a_0 = F^v \).

(b) \( \sum_{k=1}^{v} \binom{v-k}{k} S_q^{v-k} f_{N-v+k}^v + \sum_{k=1}^{v-1} [a_{v-k} f_{N-v+k}^v + b_{v-k-1} g_{N-v+k+1}^v + b_{v-1} f_{N-v+1}^v] = 0. \)

**Proof:** If \( 2v > N \), set \( N' = N + 2v \); otherwise \( N' = N \). (Note that the combinatorics of \( S_q^v \) on \( G_2(R^{N'+1}) \) is almost identical to that of \( S_q^v \) on \( G_2(R^{N+1}) \).)

The inductive hypothesis (2) together with lemma IV.1 shows that there is a positive integer \( 1^0 < v \) so that \( F^w = \binom{1^0}{w} \) for all \( w < v \). Let \( 1^1 = 1^0 + v \). Then \( F^w = \binom{1^1}{w} \) for all \( w < v \) also.

Consider the involutions

\[
\begin{pmatrix}
-1^i & 0 \\
0 & 1^{N'+1-i}
\end{pmatrix}
\]
\((i = 0, 1)\)
on $G_2(R^{N'+1})$. (Since $0 < 1^0 < N'+1$, these are honest geometrical involutions.) Denote the corresponding relations in $H^*$ by $f^i$, and consider the two equations:

$$\sum_{k=0}^{v} (v-k)Sq^{v-k}f^i_{N-v+k} = \sum_{k=0}^{v} a^i_{v-k}f^i_{N-v+k} + \sum_{k=0}^{v-1} b^i_{v-k-1}g^i_{N-v+k+1}$$

By lemma III.6 (or indirect argument):

$$a^i_0 = (F^v)^i$$

Hence it is easy to see that

$$Sq^v f^i_{N'-v} = a^i_0 h^i_{N'} + a^i_v f^i_{N'-v} + b^i_{v-1} Sq^v f^i_{N'-v}$$

(observe $g^i_{N'-v+1} = f^i_{N'-v+1} + Sq^v f^i_{N'-v}$.)

Thus for both involutions $0^0$ and $1^1$,

(a) $a^i_0 = (F^v)^i$

(b) $\sum_{k=0}^{v} (v-k) Sq^{v-k} f^i_{N'-v+k} + \sum_{k=1}^{v-1} a^i_{v-k} f^i_{N'-v+k} + b^i_{v-k-1} g^i_{N'-v+k+1} + b^i_{v-1} f^i_{N'-v+1} = 0$

Since $N' = N (2v)$ and since the relation coefficients are uniquely determined by $f$, it is immediate that the relation coefficients $a_j$ and $b_j$ of our involution on $G_2(R^{N+1})$ satisfy:

---

4. $a_v$ and $b_{v-1}$ were defined so that $Sq^v h^i_{N'} = a_v h^i_{N'} + b_{v-1}(h^i_{1N'} + h^i_{N'+1})$. By Wu's formula and the periodicity of binomial coefficients, it is immediate that $Sq^v h^i_{N'-v} = h^i_{N'} + a_v h^i_{N'-v} + b_{v-1}(h^i_{1N'-v} + h^i_{N'+1-v})$. 
(a) For either \( i = 0 \) or \( 1 \), \( b_j = b_j^i \) for all \( j \geq 0 \) and \( a_j = a_j^i \) for all \( j > 0 \).

\[
\sum_{k=1}^{v} \binom{v-k}{k} S_q^{v-k} f_{N-v+k} + \sum_{k=1}^{v-1} [a_{v-k} f_{N-v+k} + b_{v-k-1} f_{N-v+k+1}] + b_{v-1} f_{N-v+1} = 0
\]

The condition \( \sum_{k=0}^{v} \binom{v-k}{k} S_q^{v-k} f_{N-v+k} + \sum_{k=0}^{v-1} a_{v-k} f_{N-v+k} + \sum_{k=0}^{v-1} b_{v-k-1} f_{N-v+k+1} = 0 \) then simplifies to

\[
S_q^{v} f_{N-v} = a_0 h_N + a_v f_{N-v} + b_{v-1} S_q^{1} f_{N-v}
\]

or

\[
a_0 = f^{v}.
\]

The above proof really shows more than is asserted. It shows that except for the minor replacement of \( N \) by \( N+2^{r+1} \), at any stage of our induction, we already know that the relations \( S_q^{m} f = a f + b g \) modulo \( t^{v+1} \) look exactly like the corresponding relations for some linear involution.

**Corollary VIII.4** Suppose \( v = 2^{l} (2^{l+1}) \) (where \( l \geq 1 \)) and we have demonstrated part (1) of the inductive hypothesis at stage \( v+1 \). Then part (2) follows, i.e.

\[
F^{v} = f^{2^{l}} F^{v-2^{l}}
\]

**Proof:** Let \( m = 2^{l} \). Consideration of the \( t^{v} \) terms of \( S_q^{m} f \) gives

\[
(\ast) \sum_{k=0}^{m} \binom{v-k}{k} S_q^{m-k} f_{N-v+k} + \sum_{k=0}^{m} a_{m-k} f_{N-v+k} +
\]
\[ \sum_{k=0}^{m-1} b_{m-k-1} g_{N+1-v+k} = 0 \]

By inductive hypothesis, for \(0 \leq k < m\), \(F^{v-m+k} = F^k F^{v-m}\). Then multiplying conclusion (b) of VII.3 by \(F^{v-m}\), we see that

\[ \sum_{k=1}^{m} (v-k) S_{a-k}^{m-k} F_{N-v+k} + \sum_{k=1}^{m-1} [a_{m-k} f_{N-v+k} + b_{m-k-1} g_{N+1-v+k}] + b_{m-1} f_{N-v+1} = 0 \]

Comparison with (*) gives

\[ S_{a}^{m} f_{N-v} = F_{a}^{m} f_{N-v+m} + a_{m} f_{N-v} + b_{m-1} S_{a}^{1} f_{N-v} \]

As before, \(S_{a}^{m} f_{N-v} + a_{m} f_{N-v} + b_{m-1} S_{a}^{1} f_{N-v} = F^{v} h_{N-v+m}\). So we obtain \(F^{m} F^{v-m} = F^{v}\).

If \(v > N\) (and hence \(N+1\) since \(N\) and \(v\) are even), the above argument shows \(F^{m} F^{v-m} = 0\). In section IX, we'll show very directly that when \(v\) is odd, \(F^{v} = F F^{v-1}\). Thus the second part of our inductive hypothesis follows readily from the first, and the proof of theorem 0 is reduced to showing \(f_{N-v} = F^{v} h_{N-v}\).

We mentioned in section VII that most of our work would involve studying the equation

\[ S_{a}^{m} f_{N-v} = a_{m} f_{N-v} + b_{m-1} S_{a}^{1} f_{N-v} \text{ modulo } S_{0}^{m-1} \]

or its power sum reformulation

\[ S_{a}^{m} s_{2} F_{N-v+2} = A_{m+2} F_{N-v+2} + B_{m+1} S_{a}^{1} F_{N-v+2} \text{ modulo } S_{0}^{m+2} \]

To do this, it is useful to have more explicit representations of \(a_{m'}, b_{m-1}, A_{m+2}\), and \(B_{m+1}\). (Recall that these are just the coefficients in \(Z_2[h_1, h_2]\) for the equations \(S_{a}^{m} h_{N} = a_{m} h_{N} + \))
\[ b_{m-1}(h_1h_N + h_{N+1}) \text{ and } Sq^m s_2s_{N+1} = A_{m+2}(s_1s_{N+1}) + \\
B_{m+1}(s_1s_{N+2} + s_2s_{N+1}). \]

**Lemma VII.1.5.**

1. \( Sq^2f = (B_2^2h_2)f + (C_2^1h_1)g \mod t \) where \( B_2^2 = \binom{N}{2} \)
   and \( C_2^1 = \binom{N-2}{2} \).

2. \( Sq^4f = (B_4^4h_4 + B_4^{22}h_2)f + (C_4^3h_3 + C_4^{12}h_1h_2)g \mod t \)
   where \( B_4^4 = \binom{N-5}{4}, \ B_4^{22} = \binom{N-3}{2}, \ C_4^3 = \binom{N-1}{4}, \) and \( C_4^{12} = \binom{N-3}{3} \).

3. \( Sq^2g = (D_2^3h_3 + D_2^{12}h_1h_2)f + (E_2^2h_2 + E_2^{11}h_1^2)g \mod t \)
   where \( D_2^3 = D_2^{12} = \binom{N}{2} \) and \( E_2^2 = E_2^{11} = \binom{N-2}{2} \).

**Proof:** Modulo \( t \), \( f = h_N \) and \( g = h_1h_N + h_{N+1} \). Results (1) and (2) both follow easily from either VII.6 or direct calculation using II.7. We illustrate the latter method to prove (3). Modulo \( t \):

\[
Sq^2g = Sq^2(h_1h_N + h_{N+1}) = Sq^2(s_{N+1} + h_{N+1}) \\
= \binom{N+1}{2}h_1h_{N+2} + h_2h_{N+1} + h_1h_{N+2} + \binom{N}{2}h_{N+3} \\
= \binom{N-1}{2}h_1h_{N+2} + h_2h_{N+1} + \binom{N}{2}h_N \\
(D_2^3h_3 + D_2^{12}h_1h_2)h_N = D_2^3h_3h_N + D_2^{12}(h_{N+3} + h_2h_{N+1} + h_3h_N) \\
(E_2^2h_2 + E_2^{11}h_1^2)(h_1h_N + h_{N+1}) = E_2^2(h_{N+3} + h_3h_N) + E_2^{11}(h_3h_N + \\
\quad h_{N+3} + h_1h_{N+2} + h_2h_{N+1})
\]
Comparing coefficients of $\overline{h}_j = h_j h_{N+3-j}$ ($0 \leq j \leq 3$)

$\overline{h}_0: \binom{N}{2} = E_2^{11} + D_2^{12} + E_2^2$

$\overline{h}_1: \binom{N-1}{2} = E_2^{11}$

$\overline{h}_2: 1 = E_2^{11} + D_2^{12}$

$\overline{h}_3: 0 = E_2^{11} + D_2^{12} + E_2^2 + D_2^3$

As the system is triangular, it is easy to solve and obtain the answer given by (3).

Calculating $S\vec{q}_m^m s_2^m F = A_{m+2}^m F + B_{m+1}^m G$ modulo $t$ is most easily accomplished via the formula 5

$s_2^m s_k^m s_{N+m+1} = (s_k^m s_{m+1}^m)(s_1^m s_{N+1}^m) + (s_k^m s_{m+1}^m)(s_1^m s_{N+2}^m + s_2^m s_{N+1}^m)$

**Lemma VIII.6** Suppose $m$ is even and $m \geq 8$. In $S\vec{q}_m^m s_2^m F =$

$[\sum_{j=1}^{m/2+1} A_j^j s_j^m s_{m+2-j}^m F + [\sum_{j=1}^{m/2} B_j^j s_j^m s_{m+1-j}^m]G$ modulo $t$:

1. $A_j^j = B_j^j$ for all $j$
2. $A_j^j = B_j^j = 0$ for $j \geq 5$
3. $B_4^4 = \binom{N+1}{m-3}$
   $B_3^3 = \binom{N+1}{m-2}$
   $B_2^2 = \binom{N+2}{m-1} = 0$

5. This follows from $h_{N+m}^m = h_m^m h_N^m + h_{m-1}^m (h_1 h_N^m + h_{N+1}^m)$ upon multiplication by $s_4 = s_1^4$. 
\[- \frac{1}{B} = \binom{N+3}{m}^{-1} \]

**Proof:** Modulo \( t \)

\[
\text{Sq}^m s_2 F = \text{Sq}^m s_2 s_1 s_{N+1}^N
\]

\[
= s_2 \left( \binom{N+1}{m} s_1 s_{N+m+1} + \binom{N+1}{m-1} s_2 s_{N+m}^N \right) + \]

\[
s_2^2 \left[ \binom{N+1}{m-2} s_1 s_{N+m-1} + \binom{N+1}{m-3} s_2 s_{N+m-2}^N \right]
\]

\[
= s_2 \left[ \binom{N+1}{m} s_1 s_{N+m+1} + \binom{N+1}{m-1} s_2 s_{N+m} + \binom{N+1}{m-2} s_1 s_{N+m+1} \right.
\]

\[
+ s_2 s_{N+m} + s_3 s_{N+m-1} \right) + \binom{N+1}{m-3} s_4 s_{N+m-2}^N \right]
\]

\[
= s_2 \left[ s_1 s_{N+m+1} \left( \binom{N+1}{m} + \binom{N+1}{m-2} \right) + s_2 s_{N+m} \left( \binom{N+1}{m-1} + \binom{N+1}{m-2} \right) \right]
\]

\[
+ s_3 s_{N+m-1} \left( \binom{N+1}{m-2} + s_4 s_{N+m-2} \right)
\]

\[
= s_2 \left[ \binom{N+3}{m} s_1 s_{N+m+1} + \binom{N+2}{m-1} s_2 s_{N+m} + \binom{N+1}{m-2} s_3 s_{N+m-1} + \right.
\]

\[
\left. \binom{N+1}{m-3} s_4 s_{N+m-2} \right]
\]

\[
= \left[ \sum_{j=1}^{4} \bar{A}_j s_j s_{m+2-j} \right] s_1 s_{N+1}^N + \left[ \sum_{j=1}^{4} \bar{B}_j s_j s_{m+1-j} \right] (s_1 s_{N+2}^N
\]

\[
+ s_2 s_{N+1} \right)
\]

**Remark:** When \( m = 4 \), the above argument shows \( \text{Sq}^4 s_2 F = \)

\[
\left[ \binom{N+1}{2} s_3^2 + s_2 s_4 + \binom{N+3}{4} s_1 s_5 \right] F + \left[ \binom{N+1}{2} s_2 s_3^2 + \binom{N+1}{4} s_1 s_4 \right] G \mod t.
\]

When \( m = 2 \), we have \( \text{Sq}^2 s_2 F = \binom{N+1}{2} s_2^2 F + \binom{N-1}{2} s_1 s_2 G \mod t. \)
IX. The \( v \) odd case

When \( v \) is odd, showing that the inductive hypothesis for \( v \) implies the hypothesis for \( v+1 \) requires only the efforts of \( \text{Sq}^2 \mathfrak{f} \in \langle f, g \rangle \) and \( \text{Sq}^2 \mathfrak{g} \in \langle f, g \rangle \). This part of the proof will be described in this section.

The most basic result is the inductive \( \text{Sq}^2 \mathfrak{f} \) lemma. It is really just a more explicit version of the equation

\[
\text{Sq}^2 \mathfrak{f}_{N-v} = a_2 \mathfrak{f}_{N-v} + b_1 \text{Sq}^1 \mathfrak{f}_{N-v} \mod S^1_0
\]

Note that this lemma holds for arbitrary \( v \).

**Lemma IX.1** \((\text{Sq}^2 \mathfrak{f})\) For \( 2 \leq r < \left[ (N-v+1)/2 \right] \) or \( r = (N-v+2)/2 \),

\[
\phi(r) = 0
\]

where

\[
\phi(r) = \sum_{u=0}^{2} \alpha_u(r) F^{r-u+1}
\]

\[
\alpha_2 = \binom{r+1}{2} + \binom{N}{2} + \binom{r+1}{2} \binom{N+2}{2}
\]

\[
\alpha_1 = r(r+v) + \binom{N}{2} + v \binom{N+2}{2}
\]

\[
\alpha_0 = \binom{N-r-v-1}{2} + \binom{N}{2} + \binom{r+v+1}{2} \binom{N+2}{2}
\]

Also \( \phi\left(\left[ (N-v+1)/2 \right]\right) = \phi(\alpha+2) \) (if \( \left[ (N-v+1)/2 \right] > 1 \))

(Recall \( \alpha = \left[ (N-v)/2 \right] \).)

**Proof:** Modulo \( S^1_0 \), using II.7, and considering only \( t^v \) terms: (also set \( h_j = h_j h_{N-v+j+2} \).)

\[
\text{Sq}^2 \mathfrak{f} = \text{Sq}^2 t^v \mathfrak{f}_{N-v}
\]

\[
= \text{Sq}^2 t^v \sum_{j=0}^{\alpha} \bar{p}^{j+1} h_j h_{N-v-j}
\]
\[ t^\alpha \left( \sum_{j=0}^{N-1} \bar{F}^{j+1} \binom{j+1}{2} \bar{h}_{j+2} + (j+1)(j+\nu+1) \bar{h}_{j+1} \right) \] 
\[ + \binom{N-\nu-1}{2} \bar{h}_{\nu} \right) \] 
\[ = t^\alpha \binom{\alpha+2}{2} \bar{h} \left[ \binom{r-3}{2} \bar{F}^{r-1} + r(r+\nu) \bar{F}^r + \binom{N-\nu-r-1}{2} \bar{F}^{r+1} \right] \]

Observe \( \bar{h}_{\alpha+2} = \bar{h} \left( N-\nu+1 \right) / 2 \)

By lemma VIII.5
\[ a = \left( \frac{N}{2} \right) \bar{h}_{2} \text{ and } b = \left( \frac{N+2}{2} \right) \bar{h}_{1} \text{ modulo } t \]

Thus modulo \( S^1_0 \), the \( t^\gamma \) part of \( a f + bg \) is
\[ t^\gamma [a_2 f_{N-\nu} + b_1 g_{N-\nu+1}] = t^\gamma \left[ a_2 f_{N-\nu} + b_1 S^1 f_{N-\nu} \right] \]
\[ = t^\gamma \sum_{j=0}^{\alpha} \bar{F}^{j+1} \left[ \binom{N}{2} (\bar{h}_j + \bar{h}_{j+1} + \bar{h}_{j+2}) + \right. \\
\left. \binom{N+2}{2} \left( (j+1) \bar{h}_{j+2} + \nu \bar{h}_{j+1} + (j+\nu+1) \bar{h}_j \right) \right] \]
\[ = t^\gamma \binom{\alpha+2}{2} \bar{h} \left[ \binom{N}{2} + (r+\nu) \binom{N+2}{2} \bar{F}^{r-1} + \binom{N+2}{2} \bar{F}^r + \binom{N+2}{2} \bar{F}^{r+1} \right] \]

Comparing coefficients of \( \bar{h}_r \) completes the proof.

By the same method, we can prove:

**Lemma IX.2 (Sq^2 g)** For \( 3 \leq r < \left[ \left( N-\nu+1 \right) / 2 \right] \) or \( r = \left( N-\nu+3 \right) / 2 \)
\[ \phi(r) = 0 \]

where
\[ \phi(r) = \sum_{u=0}^{3} \beta_u(r) \bar{F}^{r+1-u} \]
\[ \beta_3 = r \binom{r-3}{2} \]
\[ \beta_2 = (r+v+1) \binom{r+1}{2} + \binom{N}{2} + (r+1) \binom{N+2}{2} \]
\[ \beta_1 = r \binom{N-v-r}{2} + \binom{N}{2} + (r+v) \binom{N+3}{2} \]
\[ \beta_0 = (r+v+1) \binom{N-r-v}{2} \]

Also if \([\lceil (N-v+1)/2 \rceil \geq 3\), then \(\phi([\lceil (N-v+1)/2 \rceil]) = \phi(\alpha+3)\)

and if\([\lceil (N-v+1)/2 \rceil \geq 2\) and \(v\) is even, then \(\phi([\lceil (N-v+3)/2 \rceil]) = \phi(\alpha+2)\)

We'll now use these two lemmas to show that \(F_j = 0\) for all \(j > 1\). In doing this, it is helpful to have a table of the different equations \(\phi(r)\). Since all binomial coefficients involved have period 4, there are 4 possible values of \(r\) and 2 each of \(N\) and \(v\). Below is the \(N = 0 (4)\) table:

A. \(v = 1 (4)\) and \(N = 0 (4)\)

<table>
<thead>
<tr>
<th>(r)</th>
<th>(\phi(r)) for (S)</th>
<th>(\phi(r)) for (S^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4k</td>
<td>(F^{-4k+1} + F^{-4k} + F^{-4k+1})</td>
<td>(F^{-4k-1} + F^{-4k})</td>
</tr>
<tr>
<td>4k+1</td>
<td>(F^{-4k} + F^{-4k+1} + F^{-4k+2})</td>
<td>(F^{-4k-1} + F^{-4k} + F^{-4k+1} + F^{-4k+2})</td>
</tr>
<tr>
<td>4k+2</td>
<td>(F^{-4k+2})</td>
<td>(F^{-4k+1} + F^{-4k+2})</td>
</tr>
<tr>
<td>4k+3</td>
<td>(F^{-4k+3})</td>
<td>0</td>
</tr>
</tbody>
</table>

B. \(v = 3 (4)\) and \(N = 0 (4)\)

<table>
<thead>
<tr>
<th>(r)</th>
<th>(\phi(r)) for (S)</th>
<th>(\phi(r)) for (S^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4k</td>
<td>(F^{-4k-1} + F^{-4k})</td>
<td>(F^{-4k-1} + F^{-4k})</td>
</tr>
<tr>
<td>4k+1</td>
<td>(F^{-4k} + F^{-4k+1})</td>
<td>(F^{-4k-1} + F^{-4k})</td>
</tr>
<tr>
<td>4k+2</td>
<td>(F^{-4k+2} + F^{-4k+3})</td>
<td>(F^{-4k+1} + F^{-4k+2})</td>
</tr>
<tr>
<td>4k+3</td>
<td>(F^{-4k+3} + F^{-4k+4})</td>
<td>(F^{-4k+3} + F^{-4k+4})</td>
</tr>
</tbody>
</table>

Lemma IX.3 Suppose \(v = 1 (2)\) and \(v \leq N-2\). Then the inductive
$S^q_f$ and $S^g \text{ lemmas show } F^j = 0 \text{ for all } j \geq 2. \text{ (i.e. } 2 \leq j \leq \alpha+1\text{)}$

Proof: \text{ We'll only discuss the case } N = 0 \text{ (4). The case } N = 2 \text{ (4) is summarized in appendix B.}

1. $N = 0 \text{ (4) and } \nu = 1 \text{ (4)}$

Then $\alpha = [(N-\nu)/2] = 1 \text{ (2)}$.

Suppose $\alpha = 1 \text{ (4)}$. (The case $\alpha = 3 \text{ (4)}$ is virtually identical.)

Let $\delta = (\alpha-1)/4$.

(1) $S^q_f, r = 4k+2, 0 \leq k \leq \delta-1 \rightarrow F^i_k = 0$

(2) $S^q_f, r = 4k+3, 0 \leq k \leq \delta+1 \rightarrow F^{i+1}_k = 0$

(3) $S^q_f, r = 4k, 1 \leq k \leq \delta \rightarrow F^{\alpha}_k = F^{\alpha+1}_k$

(4) $S^g_f, r = 4k, 1 \leq k \leq \delta \rightarrow F^{\alpha+1}_k = 0$

The only $F^j \text{ (2} \leq j \leq \alpha+1\text{)}$ not covered by the above is $F^{\alpha+1}_N$. $S^q_f \text{ for } r = \alpha+1 = (N-\nu+1)/2 \text{ shows } F^{\alpha+1}_N = 0$.

The case $\alpha = 3 \text{ (4)}$ is almost identical, though we do also need $S^q_f \text{ for } r = \alpha+1$.

II. $N = 0 \text{ (4) and } \nu = 3 \text{ (4)}$

Then $\alpha = [(N-\nu)/2] = 0 \text{ (2)}$.

Suppose $\alpha = 0 \text{ (4)}$. ($\alpha = 2 \text{ (4)}$ is again almost the same.)

Set $\delta = \alpha/4$.

(1) $S^q_f, r = 4k, 1 \leq k \leq \delta \rightarrow F^{i+1}_k = F^{i+1}_k$

(2) $S^q_f, r = 4k+1, 1 \leq k \leq \delta-1 \rightarrow F^{i+1}_k = F^{i+1}_k$

(3) $S^q_f, r = 4k-2, 1 \leq k \leq \delta \rightarrow F^{i+1}_k = F^{i+1}_k$

(4) $S^q_f, r = 4k-2, 2 \leq k \leq \delta \rightarrow F^{i+1}_k = F^{i+1}_k$

Thus for $1 \leq k \leq \delta$

$F^{i+1}_k = F^{i+1}_k = F^{i+1}_k = F^{i+1}_k = F^{i+1}_k$

(if $k \geq 2$)
and hence $F^j = F^{j'}$ if $2 \leq j, j' \leq \alpha$.

$\text{Sq}^2 f, \quad r = \alpha + 1 = (N-v+1)/2 \rightarrow F^{\alpha+1} = F^x$ and thus all $F^j$ are equal for $j \geq 2$.

$\text{Sq}^2 g, \quad r = \alpha + 1 = (N-v+3)/2 \rightarrow F^{\alpha+1} = 0 \rightarrow F^j = 0$ for all $j \geq 2$.

Note that there was a major difference between the cases $v = 1$ (4) and $v = 3$ (4). In the latter, we had to show all $F^j$ were equal before demonstrating that they were 0. This difference is typical and will be handled somewhat more systematically in sections XI and XII.

The only remaining part of the inductive hypothesis to be proven is $F^v = F^{v-1}$. This falls out of either $\text{Sq}^2 g$ or $\text{Sq}^2 f$, depending upon whether $N = 0$ (4) or $N = 2$ (4).

**Corollary IX.4**

(1) In $\text{Sq}^2 g = d^2 f + e^2 g$,

$$d^2 = \binom{N}{2}(h_3 + h_1 h_2) + t\binom{N}{2} F^1 h_1 \mod t^2$$

$$e^2 = \binom{N+2}{2}(h_2 + h_1^2) + t\binom{N+2}{2} F^1 h_1 \mod t^2$$

(2) In $\text{Sq}^2 f = a^2 f + b^2 g$,

$$a^2 = \binom{N}{2} h_2 + t\binom{N}{2} F^1 h_1 \mod t^2$$

$$b^2 = \binom{N+2}{2} h_1 \mod t^2$$

**Proof:** Modulo $t$, the above results are contained in VIII.5. The $t$ terms follow by an easy calculation similar to that of VIII.5.
Corollary IX.5  If $v$ is odd and $v \leq N-1$, then $F^1 F^{v-1} = F^v$.

Proof: If $N = 0 \ (4)$, look at the $t^{\nu h_0}$ or $t^{\nu h_2}$ part of $\text{Sq}^2 g = d^2 f + e^2 g$. If $N = 2 \ (4)$, look at the $t^{\nu h_1}$ part of $\text{Sq}^2 f = a^2 f + b^2 g$. 
X. \( v = 2 \) (4)

By methods analogous to the last section, we can establish the following lemmas:

**Lemma X.1** (Sq \( 4 f \)) For \( 4 \leq r < \alpha \) or \( r = (N-v+4)/2 \),

\[
\phi(r) = 0
\]

where

\[
\phi(r) = \sum_{u=0}^{4} \tau_u(r) r^{r+1-u}
\]

\[
\tau_4 = \binom{r-5}{4} + \binom{N-5}{4} + \binom{N-3}{2} + (r+1) \binom{N-1}{4} + \binom{N-3}{3}
\]

\[
\tau_3 = (r+v) \binom{r-4}{3} + \binom{N-5}{4} \binom{N-1}{4} + (r+v) \binom{N-3}{3}
\]

\[
\tau_2 = \binom{r-3}{2} \binom{N-r-v+1}{2} + 1 + \binom{N-5}{4} \binom{N-3}{4} + v \binom{N-1}{4}
\]

\[
\tau_1 = \binom{N-5}{4} v \binom{N-1}{4} + r \binom{N-3}{3} + r \binom{N-v-r}{3}
\]

\[
\tau_0 = \binom{N-r-v-1}{4} \binom{N-5}{4} + \binom{N-3}{2} + (r+v+1) \binom{N-1}{4} + \binom{N-3}{3}
\]

Also \( \phi(\alpha) = \phi(\alpha+4) \) and \( \phi(\alpha+1) = \phi(\alpha+3) \).

**Lemma X.2** In Sq \( 2 f = a^2 f + b^2 g \),

\[
a^2 = \binom{N}{2} h_2 + t \binom{1}{1} h_1 + t^2 F^2 \quad \text{and} \quad b^2 = \binom{N-2}{2} h_1.
\]

**Lemma X.3** In Sq \( 4 f = a^4 f + b^4 g \),

\[
a^4 = \binom{N+2}{4} h_4 + \binom{N}{2} h_2^2 + t \binom{N+2}{4} F^1 h_3 + t^2 \binom{N+2}{2} F^2 h_2 + \binom{N}{2} F^1 h_2^2
\]

\[
+ t^3 \binom{N+2}{2} F^3 h_1 + t^4 F^4
\]

\[
b^4 = \binom{N-2}{4} h_3 + \binom{N}{2} h_1 h_2 + t \binom{N}{2} F^1 h_2 + t^2 \binom{N}{2} h_1
\]
Lemma X.4 (Sq²f) The \( t^\overline{h}_0 \) and \( t^\overline{h}_1 \) equations of \( Sq^2f = a^2f + b^2g \) are:

1. \( t^\overline{h}_0 (v > 2) : \binom{N-2}{2} (v\sum_{j=1}^{\alpha+1} \overline{F}^j + (v+1)\overline{F}^1) = \binom{v-2}{2} F^v - \binom{N-v-1}{2} F^1 + F^2 F^v - \binom{N}{2} \sum_{j=1}^{\alpha+1} \overline{F}^j + \overline{F}^1 \)

2. \( t^\overline{h}_1 (v > 1) : \binom{N}{2} (F^1 F^v - 1 + (\sum_{j=1}^{\alpha+1} \overline{F}^j + \overline{F}^1 + \overline{F}^2)) = (v+1) F^v - 1 + \sum_{j=1}^{\alpha+1} \overline{F}^j + (v+1) \overline{F}^1 + \binom{N-v+2}{2} F^2 + \binom{N-2\chi(v+1)}{2} F^v - 1 + v(\overline{F}^1 + \overline{F}^2) \)

Lemma X.5 (Sq⁴f) The \( t^\overline{h}_3 (v \geq 2) \) equation of \( Sq^4f = a^4f + b^4g \) is:

\[
\binom{N+2}{4} (\sum_{j=1}^{\alpha+1} \overline{F}^j + F^4 + \overline{F}^3 + F^2 + \overline{F}^1) + \binom{N}{2} (\sum_{j=1}^{\alpha+1} \overline{F}^j + \overline{F}^4 + \overline{F}^2) + \binom{N-2}{4} F^v - 1 + \binom{N}{2} (F^v - 1 + \overline{F}^1 + F^3) + \sum_{j=1}^{\alpha+1} \overline{F}^j + (v+1) \overline{F}^1 + \overline{F}^2 + \binom{N-v-3}{3} F^3 + \binom{N-v-4}{4} F^4 = (v+1) F^v - 1 + \binom{N+2}{4} F^1 F^v - 1 + \binom{N}{2} F^1 F^v - 1
\]

By comparison with the linear models, we know in fact that most of the terms in X.4 and X.5 will cancel out leaving much more compact statements. For example when \( N = 0 \) (8) and \( v = 2 \) (8),

\( Sq^2f \overline{h}_1 \) becomes \( \sum_{j=2}^{\alpha+1} \overline{F}^j = 0 \)

\( Sq^4f \overline{h}_3 \) becomes \( \sum_{j=4}^{\alpha+1} \overline{F}^j = 0 \)
Proposition X.6

(1) When \( v = 2 \) (4) and either \( v \neq 2 \) or \( N \neq 6 \) (8), the inductive \( \text{Sq}^2 f \) and \( \text{Sq}^4 f \) lemmas show \( \overline{F}^j = 0 \) for all \( j \geq 2 \).

(2) When \( v = 2 \) and \( N = 6 \) (8), then the lemmas show \( \overline{F}^j = 0 \) for \( 2 \leq j \leq \alpha \), but \( \overline{F}^{\alpha+1} \) is undetermined.

The proof is summarized in appendix C. The reason for (2) in X.6 is that equation \( h_0 \) of \( \text{Sq}^2 f \) must be used to show \( \overline{F}^{\alpha+1} = 0 \). When \( v = 2 \), this equation is needed to determine the relation coefficient \( a_0 \) and hence cannot eliminate any \( \overline{F}^j \). However it is quite easy to simply carry this extra term along in our induction for \( v = 3 \) and 4. At \( v = 4 \), it will be eliminated.
XI. **Inductive $S^{m}s_{2}F$ Lemma**

Because of the increasing complexity of the formula for $S^{m}a_{h}b_{b}$ as $m$ gets larger, it is more convenient to work with the condition $S^{m}s_{2}F = A^{m}F + B^{m}G$ for $m \geq 8$.

The lemma below is the major technical tool for completing the proof of theorem 0. (Recall that VIII.6 determined the values of $A^{m}$ and $B^{m}$ modulo $t$.)

**Lemma XI.1 (Inductive $S^{m}s_{2}F$ lemma)**

Suppose $m$ is even. For $m+2 < r < \alpha+1$ or $r = (N-v+m+4)/2$, 

$$\phi(r) = 0$$

and for $\alpha+1 \leq r < (N-v+m+4)/2$,

$$\phi(r) = \phi(N+m+4-v-r)$$

where

$$\phi(r) = \sum_{u=0}^{m+2} r^{u} \gamma^{u}_{F} - r^{u}$$

and

$$\gamma^{u}_{F}(r) = \delta^{u}(r-u) + \delta^{u-2}(r-u+2) + \zeta_{1}^{u} + \zeta_{2}^{u}$$

$$\delta^{u}(a) = \binom{a}{u} \binom{N-v-a+4}{m-u}$$

$$\zeta_{1}^{u} = 0 \text{ for } u > 5$$

$$\zeta_{2}^{u} = 0 \text{ for } u < m-3$$

$$\zeta_{1}^{u} = A^{u} + (r+u)B^{u-1} + (r+u+v)B^{u} \text{ for } u \leq 5$$

$$B^{0} = \sum_{j=1}^{4} B^{j} \text{ and } A^{0} = \sum_{j=1}^{4} A^{j}$$

$$\zeta_{2}^{m+2-u} = A^{u} + (r+u+v)B^{u-1} + (r+u)B^{u} \text{ for } u \leq 5$$
Proof: Modulo $s_0^{m+2}$, considering only terms of degree $v$ in $t$:

1) $s_2 F = s_2 t^V F_{N+2-v}$
   
   $= t^V \sum_{j=1}^{\alpha+1} \overline{F^j} (\overline{s_j s_{N-j-v+4}} + \overline{s_j s_{N-j-v+2}})$
   
   $= t^V \sum_{j=1}^{\alpha+3} (\overline{F^j + F^{j-2}}) s_j s_{N-j-v+4}$

   Set $\overline{s_j} = s_j s_{N+m-j-v+4}$

   Since $S_{q}^m s_j s_{N-j-v+4} = \sum_{u=0}^{m} \delta^u(j) \overline{s_j + u}$,

   $S_{q}^m s_2 F = t^V \sum_{j=1}^{\alpha+3} (\overline{F^j + F^{j-2}}) \sum_{u=0}^{m} \delta^u(j) \overline{s_j + u}$
   
   $= t^V \sum_{r=m+3}^{\alpha+m+3} \overline{s_r} [\sum_{u=0}^{m} \delta^u(r-u)(\overline{F^r - u + F^{r-u-2}})]$
   
   $= t^V \sum_{r=m+3}^{\alpha+m+3} \overline{s_r} [\sum_{u=0}^{m+2} (\delta^u(r-u) + \delta^{u-2}(r-u+2)) \overline{F^r - u}]$

2) $A_{m}^F = A_{m+2} t^V F_{N+2-v}$
   
   $= t^V \sum_{j=1}^{4} \sum_{k=1}^{\alpha+1} \overline{A^j F^k} (\overline{s_k} + \overline{s_{k+j}} + \overline{s_{k+m+2-j}} + \overline{s_{k+m+2}})$
   
   $= t^V \sum_{r=m+3}^{\alpha+m+3} \overline{s_r} [\sum_{j=1}^{4} \overline{A^j} (\overline{F^r} + \overline{F^{r+j}} + \overline{F^{r+j-m-2}} + \overline{F^{r-m-2}})]$

   Set $\overline{A^0} = \sum_{j=1}^{4} \overline{A^j}$. Then

   $A_{m}^F = t^V \sum_{r=m+3}^{\alpha+m+3} \overline{s_r} [\sum_{j=0}^{4} \overline{A^j} (\overline{F^r} + \overline{F^{r-m-2+j}})]$

3) $B_{m}^G = B_{m+1} t^V G_{N+3-v}$
   
   $= t^V \sum_{r=m+3}^{\alpha+m+3} \overline{s_r} [\sum_{j=1}^{4} \overline{B^j} (\overline{G^r} + \overline{G^{r-j}} + \overline{G^{r+j-m-1}} + \overline{G^{r-m-1}})]$

   Letting $\overline{B^0} = \sum_{j=1}^{4} \overline{B^j}$, we have:

   $B_{m}^G = \sum_{r=m+3}^{\alpha+m+3} \overline{s_r} \sum_{j=0}^{4} \overline{B^j} (\overline{G^{r-j}} + \overline{G^{r-m-1+j}})$
To relate this expression to the $\overline{F}^j$, note

$$\overline{G}^j = (j+1)\overline{F}^{j-1} + (j+v)\overline{F}^j.$$ Hence

$$B^mG = t^v \sum_{r=m+3}^{\infty} \sum_{\ell=0}^{4} \overline{B}^r \overline{F}^{-r-j-1} + (r+j+v)\overline{F}^{-r-j} +$$

$$(r+j)\overline{F}^{-r-m-2+j} + (r+j+v+1)\overline{F}^{-r-m-1+j}$$

$$= t^v \sum_{r=m+3}^{\infty} \sum_{\ell=0}^{5} \overline{F}^{-r-k} [(r+k)\overline{B}^{-k-1} + (r+k+v)\overline{B}^{-k}] +$$

$$[r-m-2+k] [(r+k+v)\overline{B}^{-k-1} + (r+k)\overline{B}^{-k}]$$

Notice how the formula $\phi(r) = \sum_{u=0}^{m+2} \gamma^r \overline{F}^{-r-u}$ is directly traceable to the filtration properties of $S^b_a$ and $H^r$. In applying XI.1, it is useful to simplify the expressions for $\xi_1^u$ and $\xi_2^u$. The table below shows all nonzero $\xi_1^u$ for $m \geq 8$. (Also $\overline{B}^{0} = \frac{(N+1)}{m-3} \frac{(N+1)}{m-2} \frac{(N+2)}{m-1} \frac{(N+3)}{m}$ $= \frac{(N+2)}{m-2} \frac{(N+2)}{m}$ $= \frac{(N+4)}{m}$.)

**Nonzero $\xi_1^u$ for $m \geq 8$**

1. $r$ and $v$ even

$$\xi_1^0 = \xi_2^1 = \xi_2^2 = \overline{B}^{0} = \frac{(N+4)}{m}$$

$$\xi_1^1 = \xi_2^2 = \overline{B}^{0}$$

$$\xi_1^2 = \xi_2^3 = 0$$

$$\xi_1^3 = \xi_2^4 = 0$$

$$\xi_1^4 = \xi_2^5 = \overline{B}^{4} = \frac{(N)}{m-4}$$

$$\xi_1^5 = \xi_2^6 = \overline{B}^{4}$$
II. r odd and v even

\[
\begin{align*}
\zeta_1 &= \zeta_2 = 0 \\
\zeta_1 &= \zeta_2 = \frac{m+1}{B^1} = \binom{N+2}{m} \\
\zeta_1 &= \zeta_2 = \frac{m}{B^3} = \binom{N}{m-2} \\
\zeta_1 &= \zeta_2 = 0
\end{align*}
\]

**Lemma XI.2** For \( m^3-v < r \lesssim m+2 \), the inductive square lemma remains valid (subject to proviso (\*) provided the following terms are added to \( \Phi(r) \) for \( r = 1, 2, 3, 4, m-2, m-1, m+1 \), or \( m+2 \):

- \( r=1: \quad -B^1 W \)
- \( r=2: \quad \binom{N-v+2}{m} W \)
- \( r=3: \quad -B^3 W \)
- \( r=4: \quad \left( \binom{N-v+2}{m-2} + B^4 \right) W \) \quad \( m \geq 8 \) and \( v \) even
- \( r=m-2: \quad B^4 W \)
- \( r=m-1: \quad -B^3 W \)
- \( r=m+1: \quad -B^1 W \)
- \( r=m+2: \quad B^0 W \)

where \( W = \sum_{j=1}^{\alpha+1} \overline{F}^j \)

**Proviso (\*)** Any expression \( \Phi(r) \) whose coefficient of \( \overline{F}^1 \) (=\( F^v \)) is 1 will have an extra collection of terms in \( \{ F^w : w < v \} \) which will cancel with \( F^v \) once the second part of our in-
ductive hypothesis for $F^V$ is known.

Proof: The extra terms are simply the intrinsically $S_{0}^{m+2}$ part of $S_{n}^{m} \cdot s_{2}^{F_{N-v+2}} = A_{m+2}^{F_{N-v+2}} + B_{m+1}^{1} F_{N-v+2}$. All other terms were kept track of in XI.1 and show up in $\phi(r)$. (Since $v$ is even, $B_{m+1}^{1} F_{N-v+2}$ contributes nothing extra.) Of course this equation is just part of the basic equation

$$\sum_{k=0}^{m} (v-k) S_{k}^{m-k} s_{2}^{F_{N+2-v+k}} = \sum_{k=0}^{m} A_{m+2-k} F_{N+2-v+k} + \sum_{k=0}^{m-1} B_{m+1-k} G_{N+3-v+k}$$

However, by the argument of VIII.3, (possibly replacing $N$ by $N$ plus a power of 2 larger than $m$) we can find a linear involution on $G_{2}(R^{N+1})$ with the same relation coefficients and $F^{W}$ (for $w < v$ at least; $w = v$ too given the second part of our inductive hypothesis) as our given involution. Then the cancellation in the linear model of the terms involving $F_{N+2-w}$ ($w < v$) will imply their cancellation in the involution we're studying. The remaining terms are exactly what lemmas XI.1 and XI.2 compute.

In all of our applications of XI.2, $W + F^{1}$ will already be known to be 0 or $F^{2}$. Thus little use will be made of the table of extra terms. Similarly, proviso (*) will not affect us much since the recursion relation argument VIII.4 can be carried out as soon as $W + F^{1}$ is known to be zero.
When \( j < 2^1 \), we noted in section IV that \( \binom{a+2^1}{j} = \binom{a}{j} \) and when \( j = 2^1 \), \( \binom{a+2^1}{j} = \binom{a}{j} + 1 \). This simple observation halves the work involved in applying the inductive square lemma. For fixed \( N \), let us explicitly note the dependence of \( \gamma^u_v(r) \) on \( v \) by adding \( v \) as a subscript.

**Lemma XI.3** (v-duality) If \( m = 2^1 \) and \( v - v' = 2^1 (2^{1+1}) \), then
\[
\gamma^u_v(r) = \gamma^u_{v'}(r) \text{ for } u \neq 0 \text{ or } 2 \\
\gamma^u_v(r) = \gamma^u_{v'}(r) + 1 \text{ for } u = 0 \text{ or } 2
\]

Proof: Replacement of \( v \) by \( v' \) clearly leaves the \( \zeta^{u}_i \) unchanged. By the trivial observation above on binomial coefficients, \( \delta^{u}_i \) is also unchanged for \( u \geq 1 \), but \( \delta^{0}_v = \delta^{0}_v + 1 \).

We also have the following parallel lemma which is useful for studying relations \( \Phi(r) = \Phi(N+m-v-r+4) \) in XI.1.

**Lemma XI.4** (r,v-duality) If \( m = 2^1 \), \( v - v' = 2^1 (2^{1+1}) \), and \( r - r' = 2^1 (2^{1+1}) \), then
\[
\gamma^u_v(r) = \gamma^u_{v'}(r') \text{ for } u \neq m \text{ or } m+2 \\
\gamma^u_v(r) = \gamma^u_{v'}(r') + 1 \text{ for } u = m \text{ or } m+2
\]

Because \( \binom{2}{1} = 0 \), we have the following observation:

**Lemma XI.5** If \( r \) and \( u \) are odd, then \( \delta^u(r-u) = 0 \).
XII. The Core of the Proof

In this section, we'll show how the inductive square lemma (XI.1) may be used to show that \( f_{N-v} \) is a multiple of \( h_{N-v} \). If \( v = 2^1 (2^{l+1}) \), then we shall need to apply this lemma for \( m \) equal to all powers of 2 up to and including \( 2^{l+1} \).

Because there is a \( 2^{l+1} \) fold periodicity for arguments involving \( \text{Sq}^2 \), we shall use the notation \((2^{l+1}, j, k)\) to indicate the coefficient \( \overline{u}_k \cdot 2^{l+1+j} \) of \( f_{N-v} \). (When the periodicity \( 2^{l+1} \) is clear from context, this will be further abbreviated \((j, k)\).) Also the residue classes of \( N \) and \( \alpha \) modulo \( 2^{l+1} \) \((i=0, 1, \text{ and } 2)\) are relevant and will be denoted by \( N_i \) and \( \alpha_i \) respectively. \((\text{where } 0 \leq N_i, \alpha_i < 2^{l+1})\)

The pattern of the proof has three general stages. Recall that the character of the inductive \( \text{Sq}^2 \) lemma is determined by residue classes modulo \( 2^{l+1} \). Thus if \( v = 2^1 (2^{l+1}) \), then all \( \text{Sq}^2 \) \((r < l)\) will be operating as though \( v = 0 \). Two applications of each of these squares will demonstrate inductively that most \( \overline{u}_j \) are zero. Then \( \text{Sq}^2 \) will tie some of the remaining coefficients together. Finally \( \text{Sq}^{2^{l+1}} \) eliminates all \( \overline{u}_j \) sometimes pausing to tie them together first.

Proposition XII.1 represents the first stage. Unfortunately, the inductive square lemma equations do change their form somewhat when \( l \) is small or \( N_0 = 0, 2, 2^{l-4}, \text{ or } 2^{l-2}. \) To avoid unenlightening technical details, the treatment of these cases is summarized in appendix D.
**Proposition XII.1** Suppose \( v = 0 \) (2\(^{1}\)) and \( N = N_0 \) (2\(^{1}\)). Then \( Sq^2, Sq^4, \ldots, Sq^{2^{1-1}} \) show:

1. \( (2^{1},j,k) = 0 \) unless \( j = 0, 1, N+1, \) or \( N+2 \) (2\(^{1}\)).
2. If \( N_0 \neq 0 \) or 2\(^{1}-2\), then:
   1. (a) \( (2^{1},0,k) = (2^{1},1,k) \) for \( k \geq 1 \).
   2. (b) \( (2^{1},N_0+1,k) = (2^{1},N_0+2,k) \) for \( k \geq 0 \).
3. If \( N_0 = 0 \), then \( \sum_{j=0}^{2} (2^{1},j,k) = 0 \) for \( k \geq 1 \). If \( N_0 = 2^{1}-2 \), then \( \sum_{j=-1}^{1} (2^{1},j,k) = 0 \) for \( k \geq 1 \).

**Proof:** The proof is by induction on \( l \). Assuming the proposition for \( l \geq 4 \), we shall show that two applications of the \( Sq^{2^{l}} \) lemma (\( r = k \cdot 2^{l+1} \) and \( r = k \cdot 2^{l+1} + N_{l+1} + 1 \)) suffice to prove the \( l+1 \) case. Because XII.1 occasionally concludes \( \phi(r) = \phi(N+m+4-v-r) \), we shall also have to analyze \( \phi(r) \) for \( r = N_{l+1}+4+2^{l} \) and \( r = 2^{l+3} \).

Since \( N_0 \) is assumed to be unequal to \( 0, 2, 2^{1}-4, \) or \( 2^{1}-2 \), it follows that \( B^2 = B^3 = B^4 = 0 \) and hence only \( \xi_{1}^{1} \) and \( \xi_{2}^{m+2-1} \) for \( i \leq 2 \) can be nonzero.

(1) Equation \( r = k \cdot 2^{l+1} \)
\[ \ln \phi(r) = \sum_{u=0}^{2^{l+2}} \gamma_{F^u-r-u} \] 

the only relevant \( u \) (i.e. \( u \) for which \( F^u-r-u \) is not known by induction to be zero) are of the form \( a \cdot 2^l, a \cdot 2^{l-1}, a \cdot 2^{l-N_0-1}, \) and \( a \cdot 2^{l-N_0-2} \).

More precisely, since \( N_0 \neq 0, 2^{l-4}, \) or \( 2^{l-2} \), the relevant \( u \) are \( 0, 2^{l-1}, 2^l, 2^{l-N_0-1}, \) and \( 2^{l-N_0-2} \). Note \( N+4-v-r = N_1+4 \cdot (2^{l+1}) \).

\[ \gamma^0 = \binom{N+4}{2^l} + \binom{0}{2^l} = 0 \]

\[ \gamma^{2^{l-1}} = \binom{1-2^l}{2^{l-1}} + \binom{3-2^l}{2^{l-3}} = 0 \]

\[ \gamma^{2^l} = \binom{-2^l}{2^l} + \binom{2-2^l}{2^{l-2}} = 1 \]

\[ \gamma^{2^l-N_0-1} = \binom{N_1+4+2^l-N_0-1}{N_0+1} + \binom{N_1+4+2^l-N_0-3}{N_0+3} \]

\[ = \binom{3}{N_0+1} + \binom{1}{N_0+3} = 0 \]

\[ \gamma^{2^l-N_0-2} = \binom{2}{N_0+2} + \binom{0}{N_0+4} = 0 \]

Hence \( \phi(k \cdot 2^{l+1}) = (2^{l+1}, -2^l, k) \).

\[ (2) \text{ Equation } r = k \cdot 2^{l+1} + N_1 + 1 \]

By inductive hypothesis, the relevant \( u \) in \( \sum_{u} \gamma_{F^u-r-u} \) must
be of the form $a \cdot 2^1, a \cdot 2^1-1, a \cdot 2^1+N_0$, and $a \cdot 2^1+N_0+1$. Since $N_0 \neq 0$ or 2, the relevant $u$ are $0, 2^1-1, 2^1, N_0$, and $N_0+1$.

Note $N+4-v-r = 3(2^1+1)$. Also, since $\binom{2}{1} = 0$ (X1.5), we have $\gamma^{2^1-1} = \gamma^{N_0+1} = 0$.

\[
\gamma^0 = \binom{3}{2^1} = 0
\]

\[
\gamma^{2^1} = \binom{N+2}{2^1} + \binom{N_1+1-2^1}{2^1} \binom{0}{2^1} + \binom{3+2^1-2}{2^1} = \binom{N}{2^1} + \binom{N-2^1}{2^1} = 1
\]

\[
\gamma^{N_0} = \binom{N_1+1-N_0}{N_0} + \binom{N_1+3-N_0}{N_0-2} = \binom{1}{N_0} + \binom{3}{N_0-2} = 0
\]

Therefore $\phi(k \cdot 2^{1+1} + N_1+1) = (2^{1+1}, N_1+1-2^1, k)$.

(3) Equation $r = k \cdot 2^1+1 + N_1+4+2^1$

Since $N_0 \neq 0, 2^1-4$, or $2^1-2$, the relevant $u$ are 2, 3, $N_0+3$, $N_0+4$, and $2^1+2$. Also $N+4-v-r = 2^1(2^{1+1})$.

\[
\gamma^2 = \binom{2^1+2}{2^1-2} + \binom{2^1}{0} = 1
\]
\[ \gamma^3 = \begin{pmatrix} 3 \\ 2^1 - 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2^1 - 1 \end{pmatrix} = 0 \]

\[ \gamma^{2^1+2} = \begin{pmatrix} \frac{N_1 + 4}{2^1} \\ 2^1 \end{pmatrix} + \begin{pmatrix} \frac{N_1 + 2^1+4-2^1}{2^1} \\ 2^1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = 0 \]

\[ \gamma^{N_0+3} = \begin{pmatrix} \frac{N_1 + 4+2^1-N_0-3}{N_0+3} \\ N_0+3 \end{pmatrix} + \begin{pmatrix} \frac{N_1 + 4+2^1-N_0-1}{N_0+1} \\ N_0+1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = 0 \]

\[ \gamma^{N_0+4} = \begin{pmatrix} 0 \\ N_0+4 \end{pmatrix} + \begin{pmatrix} 2 \\ N_0+2 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = 0 \]

And \( \Phi(r) = (2^{1+1}, N_1 + 2^1 + 2, k) \).

(4) Equation \( r = k \cdot 2^{1+1} + 2^1 + 3 \)

Now \( N+4-v-r = N_1 + 1 + 2^1 \), \( 2^{1+1} \) and (as \( N_0 > 2 \)) the relevant \( u \) are 2, 3, \( 2^1+2 \), \( 2^1-N_0+2 \), and \( 2^1-N_0+1 \). By X1.5, \( \gamma^3 \) and \( \gamma^{2^1-N_0+1} \) are immediately seen to be zero.

\[ \gamma^2 = \begin{pmatrix} N+2 \\ 2^1 \end{pmatrix} + \begin{pmatrix} 2^{1+1} \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1 \]

\[ \gamma^{2^1+2} = \begin{pmatrix} 2^1+3-2^1 \\ 2^1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = 0 \]
\[ \gamma^{2^{1}+N_0+2} = \binom{N_1+1+2^{1}+2^{1}-N_0+2}{N_0-2} + \binom{N_1+1+2^{1}+2^{1}-N_0}{N_0} \]

\[ = \binom{3}{N_0-2} + \binom{1}{N_0} = 0 \]

Thus \( \phi(r) = (2^{1+1},2^{1+1},k) \).

To complete the proof, let \( \alpha = \alpha_1 \cdot 2^{1+1} \) \((0 \leq \alpha_1 < 2^{1+1})\)
and \( \delta = (\alpha - \alpha_1)/2^{1+1} \). Then \( N_1 = 2\alpha_1 \cdot 2^{1+1} \) and there are two cases:

**Case 1** \( \alpha_1 < 2^{1} \) and \( N_1 = 2\alpha_1 \)

Equation (1), \( r = k \cdot 2^{1+1}, 1 \leq k \leq \delta \) shows \((2^{1+1},-2^{1},k) = 0 \)
for all \( k \geq 1 \).

Equation (2), \( r = k \cdot 2^{1+1} + N_1 + 1, k \leq \delta - 1 \) shows all \((2^{1+1}, \ldots, N_1 + 1 - 2^{1}, k)\) equal zero except possibly \((2^{1+1}, N_1 + 1 - 2^{1}, \delta)\). Then

\[ \phi(\alpha - \alpha_1 + N_1 + 1) = \phi(\alpha - \alpha_1 + 2^{1} + 3) \]
shows \((2^{1+1}, N_1 + 1 - 2^{1}, \delta) = 0 \).

**Case 2** \( \alpha_1 > 2^{1} \) and \( N_1 = 2\alpha_1 - 2^{1+1} \)

Equation (1), \( r = k \cdot 2^{1+1}, 1 \leq k \leq \delta \) shows all \((2^{1+1}, -2^{1}, k) = 0 \) except possibly \((2^{1+1}, -2^{1}, \delta + 1) \). \( \phi(\alpha - \alpha_1 + 2^{1+1} = \phi(\alpha - \alpha_1 + N_1 + 4 + 2^{1}) \) shows this coefficient to be zero.

Finally, equation (2), \( r = k \cdot 2^{1+1} + N_1 + 1 \) shows \((2^{1+1}, N_1 + 1 - 2^{1}, k) = 0 \) for all \( k \).

This completes the induction.
Understanding the implications of $\text{Sq}^{2^1}$ when $v = 2^1 (2^{1+1})$ is now quite easy. The principles of $v$ and $r, v$ duality (lemmas XI.3 and XI.4) eliminate the need for further explicit computations of $\phi(r)$.

The cases $N_0 = 0, 2, 2^{1-4}$, or $2^{1-2}$ in proposition XII.2 below are summarized in appendix E. Also note that statements (1), (2), and (3) of XII.2 are merely restatements of what was proven in XII.1.

**Proposition XII.2** Suppose $v = 2^1 (2^{1+1}), N = N_1 (2^{1+1}),$ and $\alpha = \alpha_1 (2^{1+1})$. Then $\text{Sq}^2, \ldots, \text{Sq}^{2^1}$ show:

1. $(2^{1+1}, j, k) = 0$ unless $j = 0, 1, N+1$ or $N+2 (2^1)$.
2. If $N \neq 0$ or $-2 (2^1)$, then $(2^1, 0, k) = (2^1, 1, k)$ and $(2^1, N_0+1, k) = (2^1, N_0+2, k)$.
3. If $N = 0 (2^1)$ then $\sum_{j=0}^{2} (2^1, j, k) = 0$.
   If $N = -2 (2^1)$ then $\sum_{j=-1}^{1} (2^1, j, k) = 0$.
4. If $0 < k \cdot 2^{1+1} \leq \alpha$, then $(2^{1+1}, 0, k) = (2^{1+1}, -2^1, k)$.
   If $k \cdot 2^{1+1}+N_1+1 \leq \alpha$, then $(2^{1+1}, N_1+1, k) = (2^{1+1}, N_1+1-2^1, k)$.
5. Suppose $N_0 \neq 0$ or $2^{1-2}$.
   a. If $2^1 \leq \alpha_1 < 3 \cdot 2^{1-1}$, then for $j = 0$ or $1$: 

\[(2^{1+1}, 2^1 j, \delta) = (2^{1+1}, N_1 + 2 - 2^1 - j, \delta)\]

(b) If \(3 \cdot 2^1 - 1 \leq \alpha_1\), then for \(j = 0\) or 1:
\[(2^{1+1}, 2^1 j, \delta) = (2^{1+1}, N_1 + 2^1 j, \delta)\]

(When \(N_0 = 0\), the \(j = 0\) statements are true, while when \(N_0 = 2^1 - 2\), the \(j = 1\) statements are ok.)

**Remark** Roughly, this proposition can be rephrased as follows:

(4') Whenever both coefficients are defined (i.e. \(\leq \alpha + 1\)), we have
\[(2^{1+1}, 0, k) = (2^{1+1}, 1, k)\] and
\[(2^{1+1}, N_1 + 1, k) = (2^{1+1}, N_1 + 1 - 2^1, k)\].

(5') When \(\alpha_1 \geq 2^1\), then (unless \(N_0 \not= 0\) or \(2^1 - 2\)) there remain two sets of coefficients unlinked by (4). These coefficients are now asserted to be equal.

Note also that when \(2 \leq N_1 < 2^1\), (4) implies \(F_{N_1 + 1} = F_{N_1 + 2} = 0\).

**Proof:** By \(v\)-duality and the computations in the argument of XII.1, we see immediately:

(1) \(\phi(k \cdot 2^{1+1}) = (2^{1+1}, 0, k) + (2^{1+1}, -2^1, k)\)

(2) \(\phi(k \cdot 2^{1+1 + N_1 + 1}) = (2^{1+1}, N_1 + 1, k) + (2^{1+1}, N_1 + 1 - 2^1, k)\)
Also, to handle situations when we need consider \( \phi(r) = \phi(N+m+4-v-r) \), by \( r,v \)-duality:

\[
(3) \quad \phi(k+2^{l+1}N_1+4) = (2^{l+1},N_1+2,k) + (2^{l+1},N_1+2-2^l,k)
\]

Equations (1) and (2) immediately give statement (4) of the proposition.

To see (5), suppose first that \( 2^l < x_1 < 3 \cdot 2^{l-1} \). Then \( N_1 = 2\alpha_1 - 2^l \) and \( 2^l < N_1 \). Now \( N_1+2 = \alpha_1 + (\alpha_1-2^l) + 2 > x_1 + 1 \) while \( N_1+2-2^l < 2^l < \alpha_1 \). Thus \( \phi((\delta+1)2^{l+1}) = \phi(\delta \cdot 2^{l+1} + N_1 + 4) \) gives: (recall \( \delta = (\alpha - \alpha_1)/2^{l+1} \))

\[
(2^{l+1},-2^l,\delta+1) = (2^{l+1},N_1+2-2^l,\delta) \text{ or }
\]

\[
(2^{l+1},2^l,\delta) = (2^{l+1},N_1+2-2^l,\delta)
\]

The other case of (5) is \( 3 \cdot 2^{l-1} \leq \alpha_1 < 2^{l+1} \). Then:

\[
N_1 = 2\alpha_1 - 3 \cdot 2^{l-1}
\]

\[
0 < N_1 < 2^l
\]

\[
2^{l+1} + N_1 + 2 > 2^{l+1} > \alpha_1 + 1
\]

\[
2^{l+1} + N_1 + 2 - 2^l = N_1 + 2 + 2^l = 2\alpha_1 - 2^{l+1} < \alpha_1
\]

Thus \( \phi((\delta+1)2^{l+1}) = \phi((\delta+1)2^{l+1} + N_1 + 4) \) gives:

\[
(2^{l+1},2^l,\delta) = (2^{l+1},N_1+2+2^l,\delta)
\]

We are now ready to complete the proof at stage \( v \) that \( f_{N-v} \) is a multiple of \( h_{N-v} \). As \( Sq^{2^{l+1}} \) is the key tool, the
proof branches into two possibilities depending upon whether \( v = \pm 2^1 (2^{1+2}) \). When \( v = -2^1 (2^{1+2}) \), the induction works for all even values of \( N \).

The cases \( 1 \leq 3 \) or \( N_0 = 0, 2, 2^{1-4}, \) or \( 2^{1-2} \) are summarized in appendix F.

**Proposition XII.3** If \( v = -2^1 (2^{1+2}) \), then \( Sq^2, \ldots, Sq^{2+1} \) show that \( F^j = 0 \) for all \( j > 1 \).

**Proof:** The basic method of proof will be to analyze the inductive \( Sq^{2+1} \) \( s_2 F \) lemma for four values of \( r: k \cdot 2^{1+2}, k \cdot 2^{1+2} + 2^1, k \cdot 2^{1+2} + N_2 + 1, \) and \( k \cdot 2^{1+2} + N_2 + 2^1 + 1 \). Because of the need to apply \( \phi(r) = \phi(N + 4 + 2^{1+1} - r - v) \), we shall also study \( \phi(r) \) for \( r = N_2 + 3 \cdot 2^1 + 4, N_2 + 2^{1+1} + 4, 3 \cdot 2^1 + 3, \) and \( 2^{1+1} + 3 \).

As all coefficients \( (2^{1+2}, j, k) \) will have periodicity \( 2^{1+2} \), we shall abbreviate this coefficient by \((j, k)\).

Because \( N_0 \neq 0, 2, 2^{1-4}, \) or \( 2^{1-2} \), the only possible non-zero \( \zeta^u \) are \( u \leq 2 \), or \( u \geq 2^{1+1} \).

(1) **Equation** \( r = k \cdot 2^{1+2} \)

By XII.2, the relevant \( u \) in \( \phi(r) = \sum_{u=0}^{2^{1+1}+2} \gamma_u F^{r-u} \) are all of the form \( a \cdot 2^1, a \cdot 2^{1-1}, a \cdot 2^{1-N_0-1}, \) and \( a \cdot 2^{1-N_0-2} \). Since \( N_0 \neq 0, 2^{1-4}, \) or \( 2^{1-2} \), the complete list is \( u = 0, 2^{1-1}, 2^1, \)
Also $N+4-v-r = N+4+2^1 (2^1+2)$.

\[
\gamma^0 = \binom{N+4}{2^1+1} + \binom{N+4+2^1}{0} = \binom{N+4}{2^1}
\]

\[
\gamma^{2^1-1} = \binom{1-2^1}{2^1-1} + \binom{3-2^1}{2^1-3} = 0
\]

\[
\gamma^{2^1} = \binom{-2^1}{2^1} \binom{N+4+2^1-2^1}{2^1} + \binom{2-2^1}{2^1-2} = \binom{N+4}{2^1}
\]

\[
\gamma^{2^1+1-1} = \binom{1-2^1+1}{2^1+1-1} + \binom{3-2^1+1}{2^1+1-3} = 0
\]

\[
\gamma^{2^1+1} = \binom{-2^1+1}{2^1+1} + \binom{2-2^1+1}{2^1+1-2} = 1
\]

\[
\gamma^{2^1-N_0-2} = \binom{N+4+2^1+2^1-N_0-2}{2^1+N_0+2} + \binom{N+4+2^1+2^1-N_0-2}{2^1+N_0+4}
\]

\[
= \binom{N-N_0}{2^1} \binom{2}{N_0+2} + \binom{N-N_0}{2^1} \binom{0}{N_0+4} = 0
\]

\[
\gamma^{2^1-N_0-1} = \binom{N-N_0}{2^1} \binom{3}{N_0+1} + \binom{N-N_0}{2^1} \binom{1}{N_0+3} = 0
\]

\[
\gamma^{2^1+1-N_0-2} = \binom{N-N_0+2^1}{2^1} \binom{2}{N_0+2} + \binom{N-N_0+2^1}{2^1} \binom{0}{N_0+4} = 0
\]

\[
\gamma^{2^1+1-N_0-1} = \binom{N-N_0+2^1}{2^1} \binom{3}{N_0+1} + \binom{N-N_0+2^1}{2^1} \binom{1}{N_0+3} = 0
\]
Thus \( \phi(k \cdot 2^{1+2}) = \binom{N+4}{2^1} \left[ (0, k) + (-2^1, k) \right] + (-2^{1+1}, k) \).

(2) Equation \( r = k \cdot 2^{1+2} + 2^1 \)

As in (1), the relevant \( u \) are 0, \( 2^1-1, 2^1, 2^{1+1-1}, 2^{1+1}, 2^1-N_0-2, 2^1-N_0-1, 2^{1+1}-N_0-2, \) and \( 2^{1+1}-N_0-1 \).

Now \( N+4-\nu-r = N+4 \left( 2^{1+2} \right) \). It is immediate that the same argument as for equation (1) shows \( \gamma^{2^1-N_0-1} = \gamma^{2^1-N_0-2} = \gamma^{2^{1+1}-N_0-2} = \gamma^{2^{1+1}-N_0-1} = 0 \).

\[
\begin{align*}
\gamma^0 &= \binom{N+4}{2^{1+1}} + \binom{N+4}{2^1} = 0 \\
\gamma^{2^1-1} &= \binom{2^1-2^1+1}{2^1-1} + \binom{3}{2^1-3} = 0 \\
\gamma^{2^1} &= \binom{0}{2^1} + \binom{2}{2^1-2} = 0 \\
\gamma^{2^{1+1}-1} &= \binom{2^1-2^{1+1+1}}{2^{1+1}-1} + \binom{3-2^1}{2^{1+1}-3} = 0 \\
\gamma^{2^{1+1}} &= \binom{-2^1}{2^{1+1}} + \binom{2-2^1}{2^{1+1}-2} = 1
\end{align*}
\]

Thus \( \phi(k \cdot 2^{1+2} + 2^1) = (-2^1, k) \).

(3) Equation \( r = k \cdot 2^{1+2} + N_2 + 1 \)
The relevant $u$ are $0$, $2^1-1$, $2^1$, $2^1+1-1$, $2^1+1$, $N_0$, $N_0+1$, $2^1+N_0$, and $2^1+N_0+1$. Because of X1.5, $\gamma_{2^1-1} = \gamma_{N_0+1} = \gamma_{N_0+2^1+1} = 0$.

Also, $N+4-v-r = 2^1+3 (2^1+2)$.

\[
\gamma^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2^1+3 \\ 2^1+1 \end{pmatrix} = 0
\]

\[
\gamma^{2^1} = \begin{pmatrix} \begin{pmatrix} 3 \\ 2^1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2^1+2 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 2^1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2^1+2 \end{pmatrix} \end{pmatrix} = 0
\]

\[
\gamma^{2^1+1-1} = \begin{pmatrix} N \\ 2^1+1-2 \end{pmatrix} = 0
\]

\[
\gamma^{2^1+1} = \begin{pmatrix} N+2 \\ 2^1+1 \end{pmatrix} + \begin{pmatrix} N_2+1-2^1+1 \\ 2^1+1 \end{pmatrix} \begin{pmatrix} 0 \\ 2^1+1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2^1+1 \end{pmatrix} = 1
\]

\[
\gamma^N = \begin{pmatrix} N_2+1-N_0 \\ N_0 \end{pmatrix} \begin{pmatrix} 3 \\ N_0-2 \end{pmatrix} + \begin{pmatrix} N_2+1-N_0+2 \\ N_0-2 \end{pmatrix} \begin{pmatrix} 3 \\ N_0-2 \end{pmatrix} = 0
\]

\[
\gamma^{N_0+2^1} = \begin{pmatrix} N_2+1-N_0-2^1 \\ N_0+2^1 \end{pmatrix} \begin{pmatrix} 3 \\ N_0-2 \end{pmatrix} + \begin{pmatrix} N_2+1-N_0-2^1+2 \\ N_0+2^1-2 \end{pmatrix} \begin{pmatrix} 3 \\ N_0-2 \end{pmatrix} = 1
\]
\[ = \begin{pmatrix} N_2 - N_0 - 2^1 \\ 2^1 \end{pmatrix} \begin{pmatrix} 1 \\ N_0 \end{pmatrix} + \begin{pmatrix} N_2 - N_0 - 2^1 \\ 2^1 \end{pmatrix} \begin{pmatrix} 3 \\ N_0 - 2 \end{pmatrix} = 0 \]

In short, \( \phi(k \cdot 2^{1+2} + N_2 + 1) = (N_2 + 1 - 2^{1+1}, k) \).

(4) Equation \( r = k \cdot 2^{1+2} + N_2 + 1 + 2^1 \)

The relevant \( u \) are identical to (3). Also, it is immediate from the same considerations as (3) that \( \gamma^{2^1 - 1} = \gamma^{2^1+1-1} = \gamma^{N_0 + 1} = \gamma^{N_0} = \gamma^{N_0 + 2^1 + 1} = N_0 + 2^1 = 0. \)

Now \( N + 4 - v - r = 3 \cdot 2^{1+2} \).

\[ \gamma^0 = \begin{pmatrix} 3 \\ 2^1 + 1 \end{pmatrix} = 0 \]

\[ \gamma^{2^1} = \begin{pmatrix} N_2 + 2^1 + 1 - 2^1 \\ 2^1 \end{pmatrix} \begin{pmatrix} 3 + 2^1 \\ 2^1 \end{pmatrix} + \begin{pmatrix} 1 + 2^1 \\ 2^1 + 2 \end{pmatrix} = \begin{pmatrix} N \\ 2^1 \end{pmatrix} \]

\[ \gamma^{2^1+1+1} = \begin{pmatrix} N + 2 \\ 2^1 + 1 \end{pmatrix} + \begin{pmatrix} N_2 + 2^1 + 1 - 2^1 + 1 \\ 2^1 + 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2^1 + 1 \end{pmatrix} + \begin{pmatrix} 1 + 2^1 + 1 \\ 2^1 \end{pmatrix} \]

\[ = \begin{pmatrix} N + 2 \\ 2^1 + 1 \end{pmatrix} + \begin{pmatrix} N + 1 - 2^1 \\ 2^1 + 1 \end{pmatrix} = \begin{pmatrix} N + 2^1 \\ 2^1 \end{pmatrix} \]

Hence \( \phi(k \cdot 2^{1+2} + N_2 + 1 + 2^1) = \begin{pmatrix} N \\ 2^1 \end{pmatrix} (N_2 + 1, k) + \begin{pmatrix} N + 2^1 \\ 2^1 \end{pmatrix} (N_2 + 1 - 2^1, k). \)

(5) Equation \( r = k \cdot 2^{1+2} + N_2 + 3 \cdot 2^1 + 4 \)

Since \( N_0 \neq 0, 2^1 - 2, \) or \( 2^1 - 4, \) the relevant \( u \) are 2, 3,
\[ 2^{1+2}, 2^{1+3}, 2^{1+1+2}, N_0+3, N_0+4, N_0+2^{1+3}, \text{ and } N_0+2^{1+4}. \]

Now \( N-v-r+4 = 2^{1+1} \left( 2^{1+2} \right) \).

\[
\gamma^2 = \left( \begin{array}{c}
\binom{2^{1+1}+2}{2^{1+1}-2} + \binom{2^{1+1}}{2^{1+1}}
\end{array} \right) = 1
\]

\[
\gamma^3 = \left( \begin{array}{c}
\binom{2^{1+1}+3}{2^{1+1}-3} + \binom{2^{1+1}+1}{2^{1+1}-1}
\end{array} \right) = 0
\]

\[
\gamma^{2^{1+2}} = \left( \begin{array}{c}
\binom{3 \cdot 2^{1+2}}{2^{1}-2} + \binom{N_2+4+3 \cdot 2^{1+2}}{2^{1}}
\end{array} \right) = \left( \begin{array}{c}
\binom{N+4}{2^{1}}
\end{array} \right)
\]

\[
\gamma^{2^{1+3}} = \left( \begin{array}{c}
\binom{3 \cdot 2^{1+3}}{2^{1}-3} + \binom{3 \cdot 2^{1+1}}{2^{1}-1}
\end{array} \right) = 0
\]

\[
\gamma^{2^{1+1+2}} = \binom{N+4}{2^{1+1}} + \binom{N_2+3 \cdot 2^{1}+4-2^{1+1}}{2^{1+1}} = \binom{N+4}{2^{1}}
\]

\[
\gamma^{N_0+3} = \left( \begin{array}{c}
\binom{N_2+3 \cdot 2^{1+4}-N_0-3}{N_0+3}
\end{array} \right) + \left( \begin{array}{c}
\binom{N_2+3 \cdot 2^{1+4}-N_0-1}{N_0+1}
\end{array} \right) = \left( \begin{array}{c}
1
\end{array} \right)
\]

\[
\gamma^{N_0+3} = \left( \begin{array}{c}
\binom{1}{N_0+3}
\end{array} \right) + \left( \begin{array}{c}
3
\end{array} \right) = 0
\]

\[
\gamma^{N_0+4} = \left( \begin{array}{c}
\binom{0}{N_0+4}
\end{array} \right) + \left( \begin{array}{c}
2
\end{array} \right) = 0
\]

\[
\gamma^{N_0+2^{1+3}} = \left( \begin{array}{c}
\binom{N_2+3 \cdot 2^{1+4}-N_0-2^{1}-3}{2^{1}+N_0+3}
\end{array} \right) + \left( \begin{array}{c}
\binom{N_2+3 \cdot 2^{1}+4-N_0-2^{1}-1}{2^{1}+N_0+1}
\end{array} \right)
\]

\[
\gamma^{N_0+2^{1+3}} = \frac{N_2-N_0}{2^{1}} \left( \begin{array}{c}
1
\end{array} \right) + \left( \begin{array}{c}
N_2-N_0
\end{array} \right) = 0
\]

\[
\gamma^{N_0+2^{1+3}} = \left( \begin{array}{c}
\binom{N_2-N_0}{N_0+3}
\end{array} \right) + \left( \begin{array}{c}
3
\end{array} \right) = 0
\]

\[
\gamma^{N_0+2^{1+3}} = \left( \begin{array}{c}
\binom{N_2-N_0}{N_0+3}
\end{array} \right) + \left( \begin{array}{c}
3
\end{array} \right) = 0
\]
\( \gamma^{2+0+3} = \left( \begin{array}{c} N_2 - N_0 \\ 2^1 \end{array} \right) \left( \begin{array}{c} 0 \\ N_0 + 4 \end{array} \right) + \left( \begin{array}{c} N_2 - N_0 \\ 2^1 \end{array} \right) \left( \begin{array}{c} 2^1 \\ N_0 + 2 \end{array} \right) \) = 0

Thus \( \phi(k \cdot 2^1+2 + N_2 + 3 \cdot 2^1+4) = (N_2 + 3 \cdot 2^1+2, k) + \binom{N+4}{2^1}[(N_2 + 2^1+1+2, k) + (N_2 + 2^1+2, k)] \).

(6) Equation \( r = k \cdot 2^1+2 + N_2 + 2^1+1 + 4 \)

The list of relevant \( u \) is the same as (5). The argument of (5) immediately shows \( \gamma^{N_0+3} = \gamma^{N_0+4} = \gamma^{N_0+2^1+3} = \gamma^{N_0+2^1+4} = 0 \). In this case, \( N+4 - v - r = 3 \cdot 2^1 (2^1+2) \).

\[ \gamma^2 = \left( \binom{3 \cdot 2^1+2}{2^1+1-2} \right) + \left( \binom{3 \cdot 2^1}{2^1+1} \right) = 1 \]

\[ \gamma^3 = \left( \binom{3 \cdot 2^1+3}{2^1+1-3} \right) + \left( \binom{3 \cdot 2^1+1}{2^1+1} \right) = 0 \]

\[ \gamma^{2^1+2} = \left( \binom{3 \cdot 2^1+2}{2^1-2} \right) + \left( \binom{3 \cdot 2^1+2}{2^1} \right) = 0 \]

\[ \gamma^{2^1+3} = \left( \binom{3}{2^1-3} \right) + \left( \binom{1}{2^1-1} \right) = 0 \]

\[ \gamma^{2^1+1+2} = \left( \binom{N+4}{2^1+1} \right) + \left( \binom{N_2 + 2^1+1+4-2^1+1}{2^1+1} \right) \left( \binom{0}{0} \right) = 0 \]

So we can conclude \( \phi(k \cdot 2^1+2 + N_2 + 2^1+1+4) = (N_2 + 2^1+1+2, k) \).
(7) Equation \( r = k \cdot 2^{1+2} + 3 \cdot 2^1 + 3 \)

Since \( N_0 \neq 0 \) or 2, the relevant \( u \) are \( 2, 3, 2^{1+2}, 2^{1+3}, 2^{1+1}+2, 2^1-N_0+1, 2^1-N_0+2, 2^{1+1}-N_0+1, \) and \( 2^{1+1}-N_0+2. \)

By XI.5, \( \gamma^3 = \gamma^{2^{1+3}} = \gamma^{2^1-N_0+1} = \gamma^{2^{1+1}-N_0+1} = 0. \) Also 
\( N-v-r+4 = N_2+2^{1+1}+1 \) \( (2^{1+2}). \)

\[
\gamma^2 = \binom{N+2}{2^{1+1}} + \binom{3 \cdot 2^{1+1}}{2^{1+2}} \binom{1}{2^1} = \binom{N+2^{1+1}}{2^{1+1}} + \binom{N_2+2^{1+1}+1}{2^{1+1}} = 1
\]

\[
\gamma^{2^{1+2}} = \binom{3 \cdot 2^{1+3}-2^1-2}{2^{1+2}} + \binom{3 \cdot 2^{1+3}-2^1}{2^1} = 0
\]

\[
\gamma^{2^{1+1}+2} = \binom{3 \cdot 2^{1+3}-2^{1+1}}{2^{1+1}} = 0
\]

\[
\gamma^{2^1-N_0+2} = \binom{N_2+2^{1+1}+1+2^1-N_0+2}{2^{1+N_0}-2} + \binom{N_2+2^{1+1}+1+2^1-N_0}{2^{1+N_0}} = 0
\]

\[
\gamma^{2^{1+1}-N_0+2} = \binom{3}{N_0-2} + \binom{1}{N_0} = 0
\]
Thus $\phi(k \cdot 2^{1+2} + 3 \cdot 2^1 + 3) = (3 \cdot 2^1 + 1, k)$.

(8) Equation $r = k \cdot 2^{1+2} + 2^{1+1} + 3$

The relevant $u$ are the same as (7) and we immediately have $\gamma^3 = \gamma^{2^{1+3}} = \gamma^{2^{1-N_0+1}} = \gamma^{2^{1+1-N_0+1}} = \gamma^{2^{1-N_0+2}} = \gamma^{2^{1+1-N_0+2}} = 0$. Now $N-v-r+4 = N_2^2 + 3 \cdot 2^{1+1} (2^{1+2})$.

$$\gamma^2 = \binom{2^1+1}{2^1+1} + \binom{2^1+1+3-2}{2^1+1} + \binom{N_2^2+3 \cdot 2^1+1}{2^1+1}$$

$$= \binom{N}{2^1+1} + \binom{N-2^1}{2^1+1} = \binom{N+2^1}{2^1+1}$$

$$\gamma^{2^{1+2}} = \binom{2^{1+1+3 \cdot 2^1+1}-2}{2^{1+2}} + \binom{2^{1+1+3-2}}{2^1} (N_2^2+3 \cdot 2^{1+1}+1+2^1) = \binom{N}{2^1}$$

$$\gamma^{2^{1+1+2}} = \binom{2^{1+1+3-2+1}}{2^1+1} = 0$$

Therefore $\phi(k \cdot 2^{1+2} + 2^{1+1} + 3) = \binom{N+2^1}{2^1} (2^{1+1+1}+1, k) + \binom{N}{2^1} (2^{1+1}, k)$.

Except for extreme values of $k$ (i.e. $k$ near 0 or $2^1$), the inductive square lemma says $\phi(r) = 0$. Since we know by XII.2 that $(0, k) = (-2^1, k)$ and $(N_2^2+1, k) = (N_2^2+2^1, k)$, it is immediate that equations (1) - (4) say that most $\bar{F}_j$ are zero.

To make this conclusion more precise, it seems helpful to divide the final step of the argument into eight cases indexed by $x_2$. Since $N = 2a+v$, $N_2 = 2x_2-2^1(2^{1+2})$ and the
cases will correspond to different ranges in the application of XII.2 part (5).

A few remarks about these arguments may aid the reader in following them. Firstly, the form of equations (1)-(8) clearly depends only on whether $0 \leq N_1 < 2^1$ or $2^1 \leq N_1 < 2^{1+1}$.

By narrowing $\alpha_2$ to a range of width $2^{1-1}$, we will narrow $N_2$ and hence $N_1$ to a range of width $2^1$. Secondly, the actual content of proposition XII.2 part (5) (about the $\overline{F}_j$ when $j$ is near $\alpha$) depends somewhat delicately on the relative positions of $N_2$ (and its translates by $2^1$) and $\alpha_2$. So we shall immediately convert the narrowness of the range of $\alpha_2$ into a statement about which translates of $N_2$ are bigger and smaller than $\alpha_2$. (Generally only one will be nonobvious, and we'll just write that one down.)

case 1  $\alpha_2 < 2^{1-1}$

Then $N_2 = 2\alpha_2 + 3 \cdot 2^1$, $N_2 > 3 \cdot 2^1$, and $N_2 - 3 \cdot 2^1 > \alpha_2$. (since $N_2 - 3 \cdot 2^1 = \alpha_2 + \alpha_2$.)

Equations (1) - (4) immediately show all $\overline{F}_j = 0$ except possibly $(0, \delta) = (-2^1, \delta)$ and $(N_2 + 1, \delta - 1) = (N_2 + 1 - 2^1, \delta - 1)$.

(4) $= (8) \ (\phi((\delta - 1) \cdot 2^{1+2} + N_2 + 1 + 2^1) = \phi(\delta \cdot 2^{1+2} + 2^{1+1+3}))$ shows $\ (N_2 + 1, \delta - 1) = 0$.

(2) $= (6) \ (\phi(\delta \cdot 2^{1+2} + 2^1) = \phi((\delta - 1) \cdot 2^{1+2} + N_2 + 2^{1+1+4})$ shows $(-2^1, \delta) = 0$.

case 2  $2^{1-1} < \alpha_2 < 2^1$

Then $N_2 = 2\alpha_2 - 2^1$, $N_2 < 2^1$ and $N_2 < \alpha_2$ (since $N_2 = \alpha_2 + (\alpha_2 - 2^1)$.)

6. i.e. the real arena of $N_2$ and $\alpha_2$ is the circle of integers modulo $2^{1+2}$. 
Equations (1)-(4) immediately show all $F^j = 0$ except possibly
$(0, \delta) = (-2^1, \delta)$ and $(N_2+1, \delta) = (N_2+1-2^1, \delta)$.
(2) = (6) ($\delta = \delta - 1$) shows $(-2^1, \delta) = 0$.
(4) = (8) ($\delta = \delta$) shows $(N_2+1-2^1, \delta) = 0$.

**Case 3** $2^1 < \alpha_2 < 3 \cdot 2^{1-1}$

Then $N_2 = 2\alpha_2 - 2^1$, $2^1 < N_2 < 2^{1+1}$, and $N_2 > \alpha_2$.

Equations (1)-(4) show $F^j = 0$ for all $j > 1$ except possibly
$(N_2+1-2^{1+1}, \delta) = (N_2+1-3 \cdot 2^1, \delta)$ and $(N_2+1-2^1, \delta) = (2^1+1, \delta)$.
(3) = (7) ($\delta = \delta$) gives $(N_2+1-2^{1+1}, \delta) = 0$.
(4) = (8) ($\delta = \delta$) gives $0 = (2^1+1, \delta)$.

**Case 4** $3 \cdot 2^{1-1} < \alpha_2 < 2^{1+1}$

Then $N_2 = 2\alpha_2 - 2^1$, $2^{1+2} < N_2 < 3 \cdot 2^1$, and $N_2-2^1 > \alpha_2$.

Equations (1)-(4) eliminate all $F^j$ except possibly $(N_2+1-2^{1+1}, \delta) = (N_2+1-3 \cdot 2^1, \delta)$ and $(N_2+1-2^1, \delta) = (2^1+1, \delta)$.
(3) = (7) ($\delta = \delta$) gives $(N_2+1-2^{1+1}, \delta) = 0$.
(4) = (8) ($\delta = \delta$) gives $(N_2+1-2^1, \delta) = 0$.

**Case 5** $2^{1+1} < \alpha_2 < 5 \cdot 2^{1-1}$

Then $N_2 \neq 2\alpha_2 - 2^1$, $3 \cdot 2^1 < N_2$, and $N_2-2^1 > \alpha_2$.

Equations (1)-(4) eliminate all $F^j$ except possibly $(2^{1+1}, \delta) = (2^1, \delta)$ and $(N_2+1-2^{1+1}, \delta) = (N_2+1-3 \cdot 2^1, \delta)$.
(1) = (5) ($\delta + 1 = \delta - 1$) yields $(-2^{1+1}, \delta+1) = (N_2+2+2^{1+1}, \delta-1) + (N_2+2+2^1, \delta-1) = (N_2+2-2^{1+1}, \delta) + (N_2+2-3 \cdot 2^1, \delta) = 0$.
(3) = (7) ($\delta = \delta$) yields $(N_2+1-2^{1+1}, \delta) = 0$.

---

7. This is just a shorthand for $k = \delta$ in equation (2) and $k = \delta - 1$ in equation (6).
case (6) \( 5 \cdot 2^{l-1} < \alpha_2 < 3 \cdot 2^l \)

Then \( N_2 = 2\alpha_2 - 5 \cdot 2^l, N_2 < 2^l \) and \( N_2 + 2^{l+1} < \alpha_2 \).

Equations (1)-(4) eliminate all \( \overline{F}^j \) except possibly \( (N_2 + 2^l + 1, \delta) = (N_2 + 1 + 2^l, \delta) \) and \( (2^l + 1, \delta) = (2^l, \delta) \).

(1) = (5) \( (\delta + 1 = \delta) \) yields \( -2^l + 1, \delta + 1 \) = 0.

(3) = (7) \( (\delta + 1 = \delta) \) yields \( (N_2 + 1 + 2^l + 1, \delta + 1) = 0 \).

case (7) \( 3 \cdot 2^l < \alpha_2 < 7 \cdot 2^{l-1} \)

Then \( N_2 = 2\alpha_2 - 5 \cdot 2^l, 2^l < N_2 < 2^l + 1 \), and \( N_2 + 2^{l+1} > \alpha_2 \).

Equations (1)-(4) eliminate all \( \overline{F}^j \) except possibly \( (3 \cdot 2^l, \delta) = (N_2 + 2^l + 2, \delta) \) and \( (2^l + 1, \delta) = (2^l, \delta) \).

(1) = (5) \( (\delta + 1 = \delta) \) yields \( -2^l + 1, \delta + 1 \) = \( (N_2 + 2^l + 2, \delta) + (-2^l, \delta + 1) \) or \( (-2^l + 1, \delta + 1) = (2^l + 1, \delta) = 0 \).

(2) = (6) \( (\delta + 1 = \delta) \) yields \( -2^l, \delta + 1 \) = 0.

case (8) \( 7 \cdot 2^{l-1} < \alpha_2 \)

Then \( N_2 = 2\alpha_2 - 5 \cdot 2^l, 2^l + 1 < N_2 < 3 \cdot 2^l \) and \( N_2 + 2^l < \alpha_2 \).

Equations (1)-(4) eliminate all \( \overline{F}^j \) except possibly \( (3 \cdot 2^l, \delta) = (N_2 + 2^l + 2, \delta) \) and \( (2^l + 1, \delta) = (2^l, \delta) \).

(1) = (5) \( (\delta + 1 = \delta) \) yields \( -2^l + 1, \delta + 1 \) = 0.

(2) = (6) \( (\delta + 1 = \delta) \) yields \( -2^l, \delta + 1 \) = 0.

Thus in all eight cases, \( \overline{F}^j = 0 \) for all \( j > 1 \).

The above proposition shows that the induction always works when \( v = -2^l (2^l + 2) \). As long as \( v \neq 2^l \), a parallel argument to XII.3 (XII.4 below) does indeed complete the induction. However, when \( v = 2^l \) the coefficients of \( \overline{s}_r \) for \( r \leq 2^l + 3 \) are needed to determine relation coefficients in \( Sq^{2^l + 1} \).

Thus, for example, we can't apply the inductive square lemma.
for \( r = 2^1 \) as we did in XII.3. It is this problem which accounts for the dimension restriction on \( N \) in theorem 0.

The cases \( 1 \leq 3 \) or \( N_0 = 0, 2, 2^1-4 \) or \( 2^1-2 \) are in appendix G.

**Proposition XII.4**

1. Suppose \( v = 2^1 (2^1+2) \) and either \( v \neq 2^1 \) or \( N_2 < 3 \cdot 2^1 \), then \( \overline{F}^j = 0 \) for all \( j > 1 \).

2. If \( v = 2^1 \) and \( 3 \cdot 2^1 < N_2 < 2^{1+2-2} \), then \( \overline{F}^j = \overline{F}^{j'} \) for all \( j, j' = 0, 1, N+1, \) or \( N+2 (2^1) \) and \( 1 < j, j' \leq \alpha+1 \).

(A similar statement holds when \( N_2 = 3 \cdot 2^1 \) (respectively \( 2^{1+2-2} \) except that all \( \overline{F}^{\text{odd}} \) (resp. \( \overline{F}^{\text{even}} \)) besides \( \overline{F}^1 \) are now zero.)

Proof: By \( v \) and \( r, v \)-duality applied to the equations analyzed in XII.3, we immediately obtain the following results about \( \phi(r) \). (We'll drop the implicit \( k \cdot 2^1+2 \) part of \( r \) and abbreviate \( (2^1+2, j, k) \) as \( (j, k) \).)

1. \( \phi(0) = \binom{N+4+2^1}{2^1} (0, k) + \binom{N+4}{2^1} (-2^1, k) + (-2^{1+1}, k) \)

2. \( \phi(2^1) = (2^1, k) + (-2^1, k) \)

3. \( \phi(N_2+1) = (N_2+1, k) + (N_2+1-2^{1+1}, k) \)

4. \( \phi(N_2+1+2^1) = (N_2+1+2^1, k) + \binom{N}{2^1} (N_2+1, k) + \binom{N+2^1}{2^1} (N_2+1-2^1, k) \)
(5) \( \phi(N_2 + 2^1 + 4) = (N_2 + 2^2 + 2^1, k) + \binom{N+4}{2^1} (N_2 + 2, k) + \binom{N+4+2^1}{2^1} (N_2 + 2 - 2^1, k) \)

(6) \( \phi(N_2 + 4) = (N_2 + 2, k) + (N_2 + 2 - 2^1 + 1, k) \)

(7) \( \phi(2^1 + 3) = (2^1 + 1, k) + (1 - 2^1, k) \)

(8) \( \phi(3) = \binom{N+2^1}{2^1} (1, k) + \binom{N}{2^1} (1 - 2^1, k) + (1 - 2^1 + 1, k) \)

Inspection of (1)-(4) immediately shows generically that the \( \bar{F}_j \) which survived in proposition XII.2 are now being linked up and forced to equal each other. As in the proof of XII.3, a detailed analysis seems easiest upon division into eight cases indexed by the values of \( \alpha_2 \).

**case 1** \( \alpha_2 < 2^1 - 1 \)

Then \( N_2 = 2\alpha_2 + 2^1, 2^1 < N_2 < 2^1 + 1, \) and \( N_2 - 2^1 > \alpha_2 \).

Equations (1)-(4) become:

(1) \( 1 \leq k \leq \delta \) \( (-2^1, k) + (-2^1 + 1, k) = 0 \)

(2) \( 1 \leq k \leq \delta - 1 \) \( (2^1, k) + (-2^1, k) = 0 \)

(3) \( 0 \leq k \leq \delta - 1 \) \( (N_2 + 1, k) + (N_2 + 1 - 2^1 + 1, k) = 0 \)

(4) \( 0 \leq k \leq \delta - 1 \) \( (N_2 + 1 + 2^1, k) + (N_2 + 1, k) = 0 \)

Together with XII.2, it is clear that:

(1) and (2) show \( \bar{F}_j = \bar{F}_{j'} \forall j, j' < 0 \) \( 2^1 \) and \( \alpha + 1 \geq j, j' > 1 \).

(3) and (4) show \( \bar{F}_j = \bar{F}_{j'} \forall j, j' = N + 1 \) \( 2^1 \) and \( j, j' > 1 \).

Now (2) = (6) \((\delta = \delta). shows (-2^1, 6) = (N_2 + 1 - 2^1 + 1, 6) which means \( \bar{F}_j = \bar{F}_{j'} \forall j, j' = 0, 1, N + 1, \) or \( N + 2 \) \( 2^1 \).

\( \phi(N_2 + 1) = 0 \) gives \( (N_2 + 1, 0) = 0 \) and hence \( \bar{F}_j = 0 \forall j > 1 \).
**case 2** \(2^{l-1} < \alpha_2 < 2^l\)

Then \(N_2 = 2x_2 + 2^l, 2^{l+1} < N_2 < 3 \cdot 2^l\), and \(N_2 - 2^{l+1} < \alpha_2\).

Equations (1)-(4) become:

1. \(1 \leq k \leq \delta\) \((0, k) + (-2^{l+1}, k) = 0\)
2. \(1 \leq k \leq \delta - 1\) \((2^l, k) + (-2^l, k) = 0\)
3. \(0 \leq k \leq \delta - 1\) \((N_2 + 1, k) + (N_2 + 1 - 2^{l+1}, k) = 0\)
4. \(0 \leq k \leq \delta - 1\) \((N_2 + 1 + 2^l, k) + (N_2 + 1 - 2^l, k) = 0\)

(1) and (2) show \(\overline{F}^j = \overline{F}^{j'}\) \(\forall j, j' > 0\) \((2^l)\) and \(j, j' > 1\).

(3) and (4) show \(\overline{F}^j = \overline{F}^{j'}\) \(\forall j, j' = N + 1\) \((2^l)\) and \(j, j' > 1\).

(2) = (6) \((\delta = \delta)\) gives \((-2^l, \delta) + (N_2 + 2 - 2^{l+1}, \delta)\).

Since XII.2 showed \((N_2 + 1 - 2^{l+1}, 0) = 0\) (this followed from \(S + 2^l, r = N_2 + 1\)), we conclude \(\overline{F}^j = 0\) for all \(j > 1\).

**case 3** \(2^l < \alpha_2 < 3 \cdot 2^{l-1}\)

Then \(N_2 = 2\alpha_2 + 2^l, 3 \cdot 2^l < N_2\), and \(N_2 - 2^{l+1} > \alpha_2\).

Equations (1)-(4) become:

1. \(1 \leq k \leq \delta\) \((-2^l, k) + (-2^{l+1}, k) = 0\)
2. \(1 \leq k \leq \delta\) \((2^l, k) + (-2^l, k) = 0\)
3. \(0 \leq k \leq \delta - 1\) \((N_2 + 1, k) + (N_2 + 1 - 2^{l+1}, k) = 0\)
4. \(0 \leq k \leq \delta - 1\) \((N_2 + 1 + 2^l, k) + (N_2 + 1, k) = 0\)

(1) and (2) show \(\overline{F}^j = \overline{F}^{j'}\) \(\forall j, j' = 0\) \((2^l)\) and \(j, j' > 1\).

(3) and (4) show \(\overline{F}^j = \overline{F}^{j'}\) \(\forall j, j' = N + 1\) \((2^l)\) and \(j, j' > 1\).

We also know by XII.2 that \((2^l, \delta) = (N_2 + 2 - 3 \cdot 2^l, \delta)\). If \(v \neq 2^l\), then (2) (for \(k = 0\)) shows \((2^l, 0) = 0\) which gives \(\overline{F}^j = 0\) for all \(j > 1\).

**case 4** \(3 \cdot 2^{l-1} < \alpha_2 < 2^{l+1}\)

Then \(N_2 = 2\alpha_2 - 3 \cdot 2^l, N_2 < 2^l\), and \(N_2 + 2^l < \alpha_2\).

Equations (1)-(4) become:
(1) \( 1 \leq k \leq \delta \) \( (0, k) + (-2^{1+1}, k) = 0 \)
(2) \( 1 \leq k \leq \delta \) \( (2^1, k) + (-2^1, k) = 0 \)
(3) \( 1 \leq k \leq \delta \) \( (N_2 + 1, k) + (N_2 + 1 - 2^{1+1}, k) = 0 \)
(4) \( 0 \leq k \leq \delta \) \( (N_2 + 1 + 2^1, k) + (N_2 + 1 - 2^1, k) = 0 \)

(1) and (2) show \( \overline{F}^j = \overline{F}^{j'} \forall j, j' = 0 \) \( (2^1) \) and \( j, j' > 1 \).
(3) and (4) show \( \overline{F}^j = \overline{F}^{j'} \forall j, j' = N + 1 \) \( (2^1) \) and \( j, j' > N_2 + 1 \).

By XII.2, \( (N_2 + 1, 0) = 0 \) and \( (2^1, \delta) = (N_2 + 2^1 + 2, \delta) \).

(4) for \( k = 0 \) shows \( (N_2 + 1 + 2^1, 0) = 0 \) and hence \( \overline{F}^j = 0 \) for all \( j > 1 \).

**case 5** \( 2^{1+1} < \alpha_2 < 5 \cdot 2^{1-1} \)

Then \( N_2 = 2\alpha_2 - 3 \cdot 2^1, 2^1 < N_2 < 2^{1+1}, \) and \( N_2 + 2^1 > \alpha_2 \).

Equations (1)-(4) become:

(1) \( 1 \leq k \leq \delta \) \( (-2^1, k) + (-2^{1+1}, k) = 0 \)
(2) \( 1 \leq k \leq \delta \) \( (2^1, k) + (-2^1, k) = 0 \)
(3) \( 0 \leq k \leq \delta \) \( (N_2 + 1, k) + (N_2 + 1 - 2^{1+1}, k) = 0 \)
(4) \( 0 \leq k \leq \delta \) \( (N_2 + 1 + 2^1, k) + (N_2 + 1, k) = 0 \)

(1) and (2) show \( \overline{F}^j = \overline{F}^{j'} \forall j, j' = 0 \) \( (2^1) \) and \( j, j' > 1 \).
(3) and (4) show \( \overline{F}^j = \overline{F}^{j'} \forall j, j' = N + 1 \) \( (2^1) \) and \( j, j' > 1 \).

(1) = (5) \( (\delta + 1 = \delta) \) gives \( (2^{1+1}, \delta) = (N_2 + 2, \delta) \).
(3) for \( k = 0 \) shows \( (N_2 + 1, 0) = 0 \) which implies \( \overline{F}^j = 0 \) \( \forall j > 1 \).

**case 6** \( 5 \cdot 2^{1-1} < \alpha_2 < 3 \cdot 2^1 \)

Then \( N_2 = 2\alpha_2 - 3 \cdot 2^1, 2^{1+1} < N_2 < 3 \cdot 2^1, \) and \( N_2 < \alpha_2 \).

Equations (1)-(4) become:

(1) \( 1 \leq k \leq \delta \) \( (0, k) + (-2^{1+1}, k) = 0 \)
(2) \( 1 \leq k \leq \delta \) \( (2^1, k) + (-2^1, k) = 0 \)
(3) \( 0 \leq k \leq \delta \) \( (N_2 + 1, k) + (N_2 + 1 - 2^{1+1}, k) = 0 \)
(4) \( 0 \leq k \leq \delta - 1 \) \( (N_2 + 1 + 2^1, k) + (N_2 + 1 - 2^1, k) = 0 \)
(1) and (2) show $F^j = F^{j'}$ ∀ $j, j' = 0 \ (2^1)$ and $j, j' > 1$.

(3) and (4) show $F^j = F^{j'}$ ∀ $j, j' = N+1 \ (2^1)$ and $j, j' > 1$.

(1) = (5) ($\delta+1 = \delta$) shows $(2^1+1, \delta) = (N_2+2-2^1, \delta)$.

But XII.2 showed $(N_2+1-2^1+1, 0) = 0$ which implies $F^j = 0 \ \forall j > 1$.

**case 7.3.2** $2^1 \ < \ \alpha_2 \ < \ 7 \cdot 2^{1-1}$

Then $N_2 = 2\alpha_2-3 \cdot 2^1$, $3 \cdot 2^1 < N_2$, and $N_2 > \alpha_2$.

Equations (1)-(4) become:

(1) $1 \leq k \leq \delta$ \quad \quad $0, k + (-2^1+1, k) = 0$

(2) $1 \leq k \leq \delta$ \quad \quad $2^1, k + (-2^1, k) = 0$

(3) $0 \leq k \leq \delta-1$ \quad $(N_2+1, k) + (N_2+1-2^1+1, k) = 0$

(4) $0 \leq k \leq \delta-1$ \quad $(N_2+1+2^1, k) + (N_2+1, k) = 0$

(1) and (2) show $F^j = F^{j'}$ ∀ $j, j' = 0 \ (2^1)$ and $\alpha-\alpha_2 + 3 \cdot 2^1 > j, j' > 1$.

(3) and (4) show $F^j = F^{j'}$ ∀ $j, j' = N+1 \ (2^1)$ and $\alpha-\alpha_2 + N_2+1-2^1$

$> j, j' > 1$.

(1) = (5) ($\delta+1 = \delta$) gives $(-2^1, \delta+1) + (-2^1+1, \delta+1) = 0$.

By XII.2, $(3 \cdot 2^1, \delta) = (N_2+2-2^1, \delta)$.

If $\nu \neq 2^1$, (2) for $k = 0$ shows $(2^1, 0) = 0$ which implies $F^j = 0$

for all $j > 1$.

**case 8.** $7 \cdot 2^{1-1} < \alpha_2$

Then $N_2 = 2\alpha_2-7 \cdot 2^1$, $N_2 < 2^1$, and $N_2 + 3 \cdot 2^1 < \alpha_2$.

Equations (1)-(4) become:

(1) $1 \leq k \leq \delta$ \quad \quad $0, k + (-2^1+1, k) = 0$

(2) $1 \leq k \leq \delta$ \quad \quad $2^1, k + (-2^1, k) = 0$

(3) $1 \leq k \leq \delta$ \quad $(N_2+1, k) + (N_2+1-2^1+1, k) = 0$

(4) $0 \leq k \leq \delta$ \quad $(N_2+1+2^1, k) + (N_2+1-2^1, k) = 0$

(1) and (2) show $F^j = F^{j'}$ ∀ $j, j' = 0 \ (2^1)$ and $1 < j, j' <$ \alpha-\alpha_2 + 3 \cdot 2^1.
(3) and (4) show \( \overline{F}^j = F^{j'} \forall j, j' = N+1(2^1) \) and \( N_2 + 1 < j, j' < \alpha - \alpha_2 + N_2 + 1 + 3 \cdot 2^1 \).

(1) = (5) \((\delta + 1 = \delta + 1)\) gives \((-2^1 + 1, 6 + 1) = (N_2 + 2 - 2^1, \delta + 1)\).

(3) = (7) \((\delta + 1 = \delta + 1)\) gives \((N_2 + 1 - 2^1 + 1, \delta + 1) = (1 - 2^1, \delta + 1)\).

By XII.2, \((-2^1, \delta + 1) = (N_2 + 2 - 2^1, \delta + 1)\) and \((N_2 + 1, 0) = 0\).

(4) for \( k = 0 \) shows \((N_2 + 1 + 2^1, 0) = 0\) which implies \( \overline{F}^j = 0 \) for all \( j > 1 \).

The preceding lemma shows that when \( v = 2^1(2^1 + 2) \) (but \( v \neq 2^1 \)), the induction works. When \( v = 2^1 \) and \( N_2 \geq 3 \cdot 2^1 \), the following proposition sometimes provides an easy way to complete the induction.

**Proposition XII.5** Suppose \( v = 2^1, 3 \cdot 2^1 \leq N_2 \), and there exist \( C \) and \( k \) so that:

- (a) \( N = C(2^k + 1) \)
- (b) \( N > C \)
- (c) \( 2^k - 2^1 \leq C \leq 2^k \)

Then \( \overline{F}^j = 0 \) for all \( j > 1 \).

**Proof:** Consider equation \( r = C + 4 \) for \( S_{2^k} s_2 F \). Suppose \( N_0 \neq 2^1 - 4 \) or \( 2^1 - 2 \). (Equation \( 2^k \) works similarly for \( N_0 = 2^1 - 4 \) or \( 2^1 - 2 \).) Then:

\[ 2^k - 2^1 + 4 \leq C + 4 < 2^k \]

Since \( N > C \), \( C + 4 < \alpha \).

Since \( C < 2^k - 4 \), \( \overline{F}^j = 0 \) for all \( j \).
Now examine the coefficient \( \delta^u \) = \( \binom{r-u}{u} \binom{N-v-r+u+4}{2^k-u} \) 

= \( \binom{C+4-u}{u} \binom{u-2^1}{2^k-u} \).

If \( 2^1 \leq u \leq 2^k \), then \( C+4-u \), \( u-2^1 \), \( u \), and \( 2^k-u \) are all non-negative but \( (C+4-u) + (u-2^1) < (u) + (2^k-u) \). Hence either \( C+4-u < u \) or \( u-2^1 < 2^k-u \). This means \( \delta^u = 0 \).

Thus only the values \( u = 2, 3, N_0+3 \), and \( N_0+4 \) are relevant to the value of \( \phi(r) \).

\[
\gamma^2 = \binom{C+2}{2} \binom{2-2^1}{2^k-2} + \binom{0}{2^k} = 1
\]

\[
\gamma^3 = \binom{3-2^1}{2^k-3} + \binom{1-2^1}{2^k-1} = 0
\]

\[
\gamma^{N_0+3} = \binom{C+4-N_0-3}{N_0+3} + \binom{C+4-N_0-1}{N_0+1} = 0
\]

\[
\gamma^{N_0+4} = \binom{0}{N_0+4} + \binom{2}{N_0+2} = 0
\]

So \( \phi(C+4) = 0 \) shows \( \overline{F^{C+2}} = 0 \) which implies \( F^d = 0 \) \( \forall j > 1 \).

Note that this proposition completes the proof of theorem 0.
XIII. **Conclusion**

While the proof of theorem 0 is methodologically quite sound, the need for numerous special case arguments in sections IX, X, and XII is unfortunate. It is entirely reasonable to try to extend theorem 0 to arbitrary Grassmannians, but this will presumably require somewhat cleverer technique.

Even without improved technique, a number of extensions of theorem 0 are quite close. Firstly, the requirement that the second binary bit (from the left) of \( N \) be 0 was required only to finish the induction when \( v \) is a power of two. A number of specific examples suggest that a relatively simple modification of the inductive hypothesis at powers of two will allow us to carry along a few extra terms and eliminate them later. As of this writing, this has not been carried out, but it should not be very difficult.

The condition that \( N \) be even is also probably unnecessary. For \( N \) odd, \( \text{Sq}^1 h_{N+1} = (h_1^2 + h_2) h_N \) in \( H^*G_2(R^{N+1}); Z_2 \). Hence roughly, \( \text{Sq}^1 g = (h_1^2 + h_2) f \) may be used to express all coefficients of \( f \) in terms of those of \( g \). Inductive square lemmas may be readily formulated, and a parallel procedure to the \( N \) even case may be applied. For example, it is quite easy to use \( \text{Sq}^1 g \) and \( \text{Sq}^2 f \) to show that the induction always works when \( v = 1 \) (4).

Also, almost all of the arguments used work equally well for \( Z_2 \)-tori acting on \( G_2(R^{N+1}) \). In particular, lemma XI.1 is true without modification, and hence so is most of section XII. The essential extra complication is that the symmetric functions
of the weights satisfy more complicated recursion relations than the binomial coefficients do, and hence the technique of avoiding combinatorics by comparison with linear models does not so clearly work. Satisfactory resolution of this problem will allow the study of other Lie group actions by restriction to maximal $\mathbb{Z}_2$-tori.
Bibliography


Appendix I

The purpose of this appendix is to indicate the proof of the following proposition which implies VI.1.

**Proposition A.1** If $N$ is even and $\neq 64$ (192), then $H^*(G_2^c (R^{N+1}; Z_2)$ admits no nontrivial (degree preserving) automorphisms.

The condition $N \neq 64$ (192) is believed superflous. However, when $N = 4^a - 1$, the cohomology of the Grassmannian has a nontrivial involution.

Since $H^1(G_2^c (R^{N+1}); Z_2)$ has only one nonzero element ($h_1$), any automorphism will preserve $h_1$. $H^2(G_2^c (R^{N+1}); Z_2)$ has two nonzero elements other than $h_1^2$; namely $h_2$ and $h_1^2 + h_2 = \sigma_2$.

Thus a priori, the possibility of an automorphism exchanging $h_2$ and $\sigma_2$ does exist. Such an involution would have to leave the ideal of relations $\langle h_N, h_{N+1} \rangle$ invariant. We'll show this to be impossible in the dimension ranges stated.

The technique will be the usual brutal force; namely expand both $h_N$ and $\phi(h_N)$ in terms of $h_1$ and $h_2$ and then compare coefficients. Since $h_N$ is the only relation of degree $N$, a necessary condition for the existence of $\phi$ is that $\phi(h_N) = h_N$.

It turns out that expanding $\phi(h_N)$ in terms of $h_1$ and $h_2$ is easy.

**Lemma A.2**

1. $h_N = \sum_{j=0}^{[N/2]} \binom{N-j}{j} \sigma_1^{N-2j} \sigma_2^j$

2. $\phi(h_N) = \sum_{j=0}^{[N/2]} \binom{N-j}{j} h_1^{N-2j} h_2^j$
Proof: Since $\phi(\sigma_i) = h_i$ ($i = 1$ or 2), (2) obviously follows from (1). But (1) comes immediately from considering the generating function $H$.

$$H = 1/[(1+t_1)(1+t_2)] = 1/(1+\sigma_1+\sigma_2) = \sum_{r=0}^{\infty} (\sigma_1+\sigma_2)^r$$

$$= \sum_{r=0}^{\infty} \sum_{j=0}^{r} \left( \begin{array}{c} r \\ j \end{array} \right) \sigma_1^{r-j} \sigma_2^j.$$

Collecting terms of degree $N$ yields the result.

Since $\sigma_1+\sigma_2 = h_1+h_1^2+h_2$, the same argument together with the multinomial theorem yields the following unwieldy formula for $h_N$ in terms of $h_1$ and $h_2$.

**Lemma A.3** $h_N = \sum_{j=0}^{[N/2]} a(N,j) h_1^{N-2j} h_2^j$ where $a(N,k) = \sum_{j=k}^{[N/2]} \left( \begin{array}{c} N-j \\ j \end{array} \right) \left( \begin{array}{c} j \\ k \end{array} \right)$.

A more efficient means of calculating the numbers $a(N,j)$ comes from the recursive formula $h_{N+2} = h_1 h_{N+1} + (h_1^2+h_2) h_N$.

**Lemma A.4** $a(N+2,j) = a(N+1,j) + a(N,j) + a(N,j-1)$.

It turns out that for fixed $j$, the $a(N,j)$ are periodic in $N$. Note that if $a(N,j-1)$ has period $p_{j-1}$, $a(N_0,j) = a(N_0+p_{j-1},j)$, $a(N_0+1,j) = a(N_0+p_{j}+1,j)$, and $p_{j-1}$ divides $p_{j}$, then A.4 shows that $a(N,j)$ has period $p_{j}$ for all $N \geq N_0$. Using this, we can readily establish the following lemma.

**Lemma A.5**

(1) $a(N,0)$ has period 3.
a(N,0) = 1 iff N = 0 or 1 (3)
(2) a(N,1) has period 6
a(N,1) = 1 iff N = 2 or 4 (6)
(3) a(N,2) has period 12
a(N,2) = 1 iff N = 4, 5, 6, 8, 9, or 10 (12)
(4) a(N,3) has period 12
a(N,3) = 1 iff N = 6 or 10 (12)
(5) a(N,4) has period 24
a(N,4) = 1 iff N = 8, 9, 11, 13, 14, 16, 17, 19, 21, or 22 (24)
(6) a(N,5) has period 24
a(N,5) = 1 iff N = 10, 12, 14, 18, 20, r 22 (24)

In order that \( \phi(h_N) = h_N \), it is necessary that \( a(N,j) = \binom{N-j}{j} \) for all \( j \leq N/2 \). It is fairly immediate from A.5 that for \( N \neq 4 \) even, this can only happen when \( N = 16 \) (24).
(For \( N = 4^a \), \( \phi(h_{N+1}) \notin \langle h_N, h_{N+1} \rangle \).

However, we can also play off lemmas A.2 and A.3. It follows from these two that \( \phi(h_N) = h_N \) implies
\[
\sum_{j=k+1}^{N/2} \binom{N-j}{j}(j) = 0 \text{ for } k \leq N/2
\]

Setting \( M = N/2 \) and \( r = M-j \), this may be converted to:

\[
(*) \quad \sum_{r=0}^{M-1-k} \left( \frac{M+r}{2r} \right) \left( \frac{M-r}{M-r-k} \right) = 0
\]

We already know that \( \phi(h_N) = h_N \) for \( N \) even implies \( N \) is divisible by 8, and hence \( M \) is divisible by 4. Applying \( (*) \) for \( k = M-4 \) gives
\[
0 = \left( \frac{M}{4} \right) + \left( \frac{M+1}{2} \right) \left( \frac{M-1}{3} \right) + \left( \frac{M+2}{4} \right) \left( \frac{M-2}{2} \right) + \left( \frac{M+3}{6} \right) \left( \frac{M-3}{1} \right)
\]
\[ = \binom{M}{4} + 0 + \binom{M+2}{6} + \binom{M+3}{6} = \binom{M}{4} \]

Thus \( M \) is divisible by 8, \( N \) is divisible by 16, and \( N = 16 \) (48).

Further extension of A.5 will show that \( a(N,17) \) has period 96 and is zero when \( N = 16 \) (96). By A.2, this means \( \phi(n) \neq h \) if \( N = 16 \) (96) \((N > 16)\). Application of (*) for \( k = M-16 \) then requires \( \binom{M}{16} = 0 \) and thus eliminates \( N = 160 \) (192).
### Appendix B

**A. v = 1 (4) and N = 2 (4)**

<table>
<thead>
<tr>
<th>r</th>
<th>( \Phi(r) ) for ( Sq^2f )</th>
<th>( \Phi(r) ) for ( Sq^2g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4k</td>
<td>( \overline{F}^{4k+1} + \overline{F}^{4k} + \overline{F}^{4k-1} )</td>
<td>( \overline{F}^{4k} + \overline{F}^{4k-1} )</td>
</tr>
<tr>
<td>4k+1</td>
<td>( \overline{F}^{4k+1} )</td>
<td>( \overline{F}^{4k+1} + \overline{F}^{4k-1} )</td>
</tr>
<tr>
<td>4k+2</td>
<td>( \overline{F}^{4k+2} )</td>
<td>( \overline{F}^{4k+2} + \overline{F}^{4k+1} )</td>
</tr>
<tr>
<td>4k+3</td>
<td>( \overline{F}^{4k+4} + \overline{F}^{4k+3} + \overline{F}^{4k+2} )</td>
<td>( \overline{F}^{4k+4} + \overline{F}^{4k+2} )</td>
</tr>
</tbody>
</table>

**B. v = 3 (4) and N = 2 (4)**

| \( 4k \) | \( \overline{F}^{4k} + \overline{F}^{4k-1} \) | \( \overline{F}^{4k} + \overline{F}^{4k-1} \) |
| \( 4k+1 \) | \( \overline{F}^{4k+2} + \overline{F}^{4k+1} \) | \( \overline{F}^{4k+2} + \overline{F}^{4k-1} \) |
| \( 4k+2 \) | \( \overline{F}^{4k+3} + \overline{F}^{4k+2} \) | \( \overline{F}^{4k+2} + \overline{F}^{4k+1} \) |
| \( 4k+3 \) | \( \overline{F}^{4k+3} + \overline{F}^{4k+2} \) | \( \overline{F}^{4k+3} + \overline{F}^{4k+2} \) |

It is quite easy using the above tables to verify IX.3 for \( N = 2 (4) \).
Appendix C (X.6; ν = 2 (4))

Major Equations to be Used: (It is suggested that the reader prepare complete $S_{q^2}f$ and $S_{q^4}f$ tables so as to check the $\Phi(r) = \Phi(r')$ type equations.)

1A. $N = 0 \ (8)$ and $\nu = 2 \ (8)$

<table>
<thead>
<tr>
<th>Square</th>
<th>$r$</th>
<th>$\Phi(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $S_{q^2}f$</td>
<td>4k</td>
<td>$\frac{F^{4k+1}}{F} + \frac{F^{4k-1}}{F}$</td>
</tr>
<tr>
<td>2. &quot;</td>
<td>4k+1</td>
<td>$\frac{F^{4k+1}}{F} + \frac{F^{4k}}{F}$</td>
</tr>
<tr>
<td>3. $S_{q^4}f$</td>
<td>8k</td>
<td>$\frac{F^{8k-1}}{F} + \frac{F^{8k+3}}{F}$</td>
</tr>
<tr>
<td>4. &quot;</td>
<td>8k+2</td>
<td>$\frac{F^{8k+3}}{F} + \frac{F^{8k+1}}{F}$</td>
</tr>
<tr>
<td>5. &quot;</td>
<td>8k-1</td>
<td>$\frac{F^{8k}}{F} + \frac{F^{8k-1}}{F} + \frac{F^{8k-2}}{F} + \frac{F^{8k-3}}{F}$</td>
</tr>
<tr>
<td>6. &quot;</td>
<td>8k+3</td>
<td>$\frac{F^{8k+3}}{F} + \frac{F^{8k+2}}{F} + \frac{F^{8k+1}}{F} + \frac{F^{8k}}{F}$</td>
</tr>
<tr>
<td>7. $S_{q^2}f$</td>
<td>1</td>
<td>$\sum_{j=2}^{q+1} \frac{F^j}{F} = 0$</td>
</tr>
<tr>
<td>8. $S_{q^4}f$</td>
<td>3</td>
<td>$\sum_{j=4}^{q+1} \frac{F^j}{F} = 0$</td>
</tr>
</tbody>
</table>

1B. $N = 0 \ (8)$ and $\nu = 6 \ (8)$

1, 2, and 7 exactly as in 1A.

3. $S_{q^4}f$ | 8k+2 | $\frac{F^{8k+1}}{F}$ |
| 4. " | 8k+1 | $\frac{F^{8k-2}}{F}$ |
| 5. " | 8k | $\frac{F^{8k+1}}{F} + \frac{F^{8k-1}}{F} + \frac{F^{8k-3}}{F}$ |
| 6. " | 8k+3 | $\frac{F^{8k+4}}{F} + \frac{F^{8k+3}}{F} + \frac{F^{8k+2}}{F} + \frac{F^{8k+1}}{F} + \frac{F^{8k}}{F}$ |
| 8. " | 3 | $\sum_{j=5}^{q+1} \frac{F^j}{F} = 0$ |

11A. $N = 2 \ (8)$ and $\nu = 2 \ (8)$

1. $S_{q^2}f$ | 4k-2 | $\frac{F^{4k-1}}{F} + \frac{F^{4k-2}}{F}$ |
| 2. " | 4k-1 | $\frac{F^{4k}}{F} + \frac{F^{4k-2}}{F}$ |
| 3. $S_{q^4}f$ | 8k-3 | $\frac{F^{8k-3}}{F} + \frac{F^{8k-4}}{F}$ |
| 4. " | 8k+2 | $\frac{F^{8k+2}}{F} + \frac{F^{8k}}{F}$ |
| 5. " | 8k+1 | $\frac{F^{8k+2}}{F} + \frac{F^{8k+1}}{F} + \frac{F^{8k}}{F} + \frac{F^{8k-2}}{F}$ |
6. $\text{Sq}^4 f$ 8k \[ \bar{F}^{8k} + \bar{F}^{8k-1} + \bar{F}^{8k-2} + \bar{F}^{8k-3} \]

7. " 3 \[ \sum_{j=5}^{\alpha+1} \bar{F}^j + \bar{F}^3 = 0 \]

**IIB. N = 2 (8) and v = 6 (8)**

1 and 2 exactly as in IIA.

3. $\text{Sq}^4 f$ 8k+1 \[ \bar{F}^{8k+1} + \bar{F}^{8k} + \bar{F}^{8k-2} \]

4. " 8k-4 \[ \bar{F}^{8k-4} + \bar{F}^{8k-5} + \bar{F}^{8k-6} \]

5. " 8k-1 \[ \bar{F}^{8k} + \bar{F}^{8k-2} + \bar{F}^{8k-3} \]

6. " 8k \[ \bar{F}^{8k+1} + \bar{F}^{8k} + \bar{F}^{8k-1} + \bar{F}^{8k-2} + \bar{F}^{8k-3} \]

7. " 3 \[ \sum_{j=3}^{\alpha+1} \bar{F}^j = 0 \]

**IIIA. N = 4 (8) and v = 2 (8)**

1. $\text{Sq}^2 f$ 4k \[ \bar{F}^{4k+1} + \bar{F}^{4k-1} \]

2. " 4k+1 \[ \bar{F}^{4k+1} + \bar{F}^{4k} \]

3. $\text{Sq}^4 f$ 8k-1 \[ \bar{F}^{8k} + \bar{F}^{8k-4} \]

4. " 8k+2 \[ \bar{F}^{8k+2} + \bar{F}^{8k} \]

5. " 8k-4 \[ \bar{F}^{8k-4} + \bar{F}^{8k-6} \]

6. " 8k-2 \[ \bar{F}^{8k-1} + \bar{F}^{8k-2} + \bar{F}^{8k-4} + \bar{F}^{8k-5} \]

7. $\text{Sq}^2 f$ 1 \[ \sum_{j=2}^{\alpha+1} \bar{F}^j = 0 \]

**IIB. N = 4 (8) and v = 6 (8)**

1, 2, and 7 exactly as in IIA.

3. $\text{Sq}^4 f$ 8k+1 \[ \bar{F}^{8k+1} + \bar{F}^{8k} + \bar{F}^{8k-1} \]

4. " 8k \[ \bar{F}^{8k} + \bar{F}^{8k-2} + \bar{F}^{8k-3} \]

5. " 8k-1 \[ \bar{F}^{8k-4} \]

6. " 8k-4 \[ \bar{F}^{8k-3} + \bar{F}^{8k-4} + \bar{F}^{8k-6} \]

**IV A. N = 6 (8) and v = 2 (8)**

1. $\text{Sq}^2 f$ 4k-2 \[ \bar{F}^{4k-1} + \bar{F}^{4k-2} \]

2. " 4k-1 \[ \bar{F}^{4k} + \bar{F}^{4k-2} \]

3. $\text{Sq}^4 f$ 8k \[ \bar{F}^{8k+1} + \bar{F}^{8k-3} \]
<table>
<thead>
<tr>
<th>square</th>
<th>r</th>
<th>$\phi(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. Sq$^4$f</td>
<td>8k-1</td>
<td>$\frac{1}{F}8k-1 + \frac{1}{F}8k-4$</td>
</tr>
<tr>
<td>5. &quot;</td>
<td>8k-2</td>
<td>$\frac{1}{F}8k-3 + \frac{1}{F}8k-5$</td>
</tr>
<tr>
<td>6. &quot;</td>
<td>8k+1</td>
<td>$\frac{1}{F}8k+2 + \frac{1}{F}8k-1$</td>
</tr>
<tr>
<td>7. &quot;</td>
<td>3</td>
<td>$\frac{1}{F}^3 = \frac{1}{F}^4$</td>
</tr>
<tr>
<td>8. Sq$^2$f</td>
<td>0 (v&gt;2)</td>
<td>$\sum_{j=2}^{\infty} \frac{1}{F}^j = 0$</td>
</tr>
</tbody>
</table>

IVB. N = 6 (8) and v = 6 (8)

1, 2, and 8 as in IVA.

<table>
<thead>
<tr>
<th>square</th>
<th>r</th>
<th>$\phi(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Sq$^4$f</td>
<td>8k</td>
<td>$\frac{1}{F}8k-3$</td>
</tr>
<tr>
<td>4. &quot;</td>
<td>8k+1</td>
<td>$\frac{1}{F}8k-1$</td>
</tr>
<tr>
<td>5. &quot;</td>
<td>8k+2</td>
<td>$\frac{1}{F}8k+1$</td>
</tr>
<tr>
<td>6. &quot;</td>
<td>8k-2</td>
<td>$\frac{1}{F}8k-1 + \frac{1}{F}8k-3 + \frac{1}{F}8k-5$</td>
</tr>
<tr>
<td>7. &quot;</td>
<td>3</td>
<td>$\frac{1}{F}^3 = 0$</td>
</tr>
</tbody>
</table>
Appendix D (Proposition XII.1; \( v = 0 \ (2^1+1 \))

Major Equations Needed: (Occasionally partial analysis of other equations is needed, but this is generally quite clear from the rare term not eliminated by the "major equations!"

For \( l = 1 \) or 2, it's just as easy to prepare a complete table of \( \phi(r) \), so equations of the form \( \phi(N+m+4-v-r) \) are not specifically indicated.)

\[
1.1 \geq 3 (s_{1}^{2} s_{2} F)
\]

A. \( N_0 = 0 \)

\[
\begin{align*}
\phi(r) & \quad r = k \cdot 2^1+1 \\
1. & \quad 0 \quad (-2^1, k) \\
2. & \quad N_1+1 \quad (N_1+1-2^1, k) + \binom{N}{2^1} [(N_1, k) + (N_1-2^1, k)] \\
3. & \quad N_1+4+2^1 \quad (N_1+2^1) \\
4. & \quad 2^1+3 \quad (2^1+1, k) + \binom{N}{2^1} [(2^1+2, k) + (2, k)] \\
\end{align*}
\]

B. \( N_0 = 2 \)

\[
\begin{align*}
\phi(r) & \quad r = k \cdot 2^1 \\
1. & \quad 0 \quad (-2^1, k) \\
2. & \quad N_1+1 \quad (N_1+1-2^1, k) + \binom{N}{2^1} [(N_1-1, k) + (N+2^1)(N_1-1-2^1, k)] \\
3. & \quad N_1+4+2^1 \quad (N_1+2^1, k) \\
4. & \quad 2^1+3 \quad (N+2^1)(2^1+3, k) + (2^1+1, k) + \binom{N}{2^1} (3, k) \\
\end{align*}
\]

C. \( N_0 = 2^1-4 \)

\[
\begin{align*}
\phi(r) & \quad r = k \cdot 2^1-1 \\
1. & \quad 0 \quad (N+4)(-2, k) + (-2^1, k) + \binom{N}{2^1} (-2^1-2, k) \\
2. & \quad N_1+1 \quad (N_1+1-2^1, k) \\
\end{align*}
\]
\[ r = k \cdot 2^{l+1} \]

3. \( N_1 + 4 + 2^1 \)

\[ \binom{N}{2^1} (N_1 + 4 + 2^1, k) + (N_1 + 2 + 2^1, k) \]

4. \( 2^1 + 3 \)

\[ (2^1 + 1, k) \]

D. \( N_0 = 2^1 - 2 \)

1. 0

\[ (-2^1, k) + \binom{N+4}{2^1} [(-1, k) + (-1 - 2^1, k)] \]

2. \( N_1 + 1 \)

\[ (N_1 + 1 - 2^1, k) \]

3. \( N_1 + 4 + 2^1 \)

\[ (N_1 + 2 + 2^1, k) + \binom{N+4}{2^1} [(N_1 + 3 + 2^1, k) + (N_1 + 3, k)] \]

4. \( 2^1 + 3 \)

\[ (2^1 + 1, k) \]

11. 1 = 2 \( \{S_{4}, s_{2}, F\} \)

A. \( N = 0 \) (8)

\[ r = 8k + \]

nonzero \( \overline{F^j} \) in \( \phi(r) \) are \( j = 8k + \)

1. 0

-2, -4

2. 4

4, -2

B. \( N = 2 \) (8)

1. 0

-1, -3

2. 5

4, 3, 1, 0

C. \( N = 4 \) (8)

1. 2

2, -4

2. 6

4, 2

D. \( N = 6 \) (8)

1. 1

-3, -5

2. 3

3, -3
III. \( l = 1 (Sg^2 \ell) \)

A. \( N = 0 \) (4)

<table>
<thead>
<tr>
<th>( r = 4k^+ )</th>
<th>Nonzero ( F^j ) in ( \Phi(r) ) are ( j = 4k^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 0</td>
<td>-1</td>
</tr>
<tr>
<td>2. 1</td>
<td>2, 1, 0</td>
</tr>
</tbody>
</table>

B. \( N = 2 \) (4)

<table>
<thead>
<tr>
<th>( r = 4k^+ )</th>
<th>Nonzero ( F^j ) in ( \Phi(r) ) are ( j = 4k^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1</td>
<td>2</td>
</tr>
<tr>
<td>2. 0</td>
<td>1, 0, -1</td>
</tr>
</tbody>
</table>
Appendix E (Proposition XII.2; ν = 2^1 (2^1+1))

Major Equations Needed:

\[ 1. \quad 1 \geq 3 (S_{2}^{s_{2}}F) \]

V-duality applied to (1) and (2) in appendix D (part I) and \( r, v \)-duality applied to (3) and (4) give essentially all equations needed.

\[ 11. \quad 1 = 2 (S_{4}^{q}) \]

A. \( N = 0 \) (8)

\( r = 8k^+ \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>1, -1, -3</td>
</tr>
<tr>
<td>( 1 )</td>
<td>1, 0, -2</td>
</tr>
</tbody>
</table>

B. \( N = 2 \) (8)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>0, -1, -2, -3</td>
</tr>
<tr>
<td>( 3 )</td>
<td>3, 2, 1, 0</td>
</tr>
</tbody>
</table>

C. \( N = 4 \) (8)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>0, -2, -3</td>
</tr>
<tr>
<td>( 4 )</td>
<td>5, 4, 2</td>
</tr>
</tbody>
</table>

D. \( N = 6 \) (8)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>1, -3</td>
</tr>
<tr>
<td>( 6 )</td>
<td>7, 3</td>
</tr>
</tbody>
</table>
Appendix F (Proposition XII.3; \(v = -2^1 (2^1 + 2^1)\))

Major Equations Needed:

\[ 1 + 1 \geq \frac{3}{s_2} \left( s_2^2 + 1 \right) \]

**A. \(N_0 = 0\)**

- \(r = k \cdot 2^1 + 2^1\)
- \(\Phi(r)\) (relevant)

1. 0

\[
\begin{pmatrix}
N+4 \\
N+1
\end{pmatrix}
\left[(0, k) + (-2^1, k)\right] + (-2^1 + 1, k)
\]

2. \(2^1\)

\((-2^1, k)\)

3. \(N_2 + 1\)

\[
(N_2 + 1 - 2^1 + 1, k) + \begin{pmatrix}
N+2 \\
N+1
\end{pmatrix}
\left[(N_2, k) + (N_2 - 2^1 + 1, k)\right]
\]

4. \(N_2 + 1 + 2^1\)

\[
\begin{pmatrix}
N \\
N+2
\end{pmatrix}
\left[(N_2 + 1, k) + (N_2 + 1 - 2^1, k)\right] + \begin{pmatrix}
2^1 \\
N+2
\end{pmatrix}
\left[(N_2 + 2^1, k) + (N_2 - 2^1, k)\right]
\]

5. \(N_2 + 3 \cdot 2^1 + 4\)

\[
(N_2 + 3 \cdot 2^1 + 2, k) + \begin{pmatrix}
N+4 \\
N+1
\end{pmatrix}
\left[(N_2 + 2^1 + 1 + 2, k)\right]
\]

6. \(N_2 + 2^1 + 1 + 4\)

\((N_2 + 2^1 + 1 + 2, k)\)

7. \(3 \cdot 2^1 + 3\)

\[
(3 \cdot 2^1 + 1, k) + \begin{pmatrix}
N+2 \\
N+1
\end{pmatrix}
\left[(3 \cdot 2^1 + 2, k) + (2^1 + 2, k)\right]
\]

8. \(2^1 + 1 + 3\)

\[
\begin{pmatrix}
N+2 \\
N+1
\end{pmatrix}
\left[(2^1 + 1 + 1, k)\right] + \begin{pmatrix}
N \\
N+1
\end{pmatrix}
\left[(2^1 + 1, k)\right]
\]

\[
\begin{pmatrix}
N+2 \\
N+1
\end{pmatrix}
\left[(2^1 + 1 + 2, k) + (2, k)\right]
\]

**B. \(N_0 = 2\)**

1. 0

\[
\begin{pmatrix}
N+4 \\
N+1
\end{pmatrix}
\left[(0, k) + (-2^1, k)\right] + (-2^1 + 1, k)
\]

\[
\begin{pmatrix}
N+4 \\
N+1
\end{pmatrix}
\left[(0, k) + (-2^1, k)\right]
\]
2. $2^1$  
   \((-2^1, k)\)

3. $N_2+1$  
   \((N_2+1-2^1+1, k) + \left( \begin{array}{c} N \\ 2^1+1 \end{array} \right) (N_2-1, k) + \left( \begin{array}{c} N+2^1+1 \\ 2^1+1 \end{array} \right) (N_2-1-2^1+1, k) \)

4. $N_2+1+2^1$  
   \(
   \left( \begin{array}{c} N \\ 2^1 \end{array} \right) (N_2+1, k) + \left( \begin{array}{c} N+2^1 \\ 2^1 \end{array} \right) (N_2+1-2^1, k) + \left( \begin{array}{c} N \\ 2^1+1 \end{array} \right) (N_2-1+2^1, k) + \left( \begin{array}{c} N+2^1 \\ 2^1+1 \end{array} \right) (N_2-1-2^1, k) \)

5. $N_2+3\cdot2^1+4$  
   \((N_2+3\cdot2^1+2, k) + \left( \begin{array}{c} N+4 \\ 2^1 \end{array} \right) [(N_2+2^1+1+2, k) + (N+2^1+2, k)]\)

6. $N_2+2^1+1+4$  
   \((N_2+2^1+1+2, k)\)

7. $3\cdot2^1+3$  
   \((3\cdot2^1+1, k) + \left( \begin{array}{c} N+2^1+1 \\ 2^1+1 \end{array} \right) (3\cdot2^1+3, k) + \left( \begin{array}{c} N \\ 2^1+1 \end{array} \right) (2^1+3, k)\)

8. $2^1+1+3$  
   \(
   \left( \begin{array}{c} N+2^1 \\ 2^1 \end{array} \right) (1+2^1+1, k) + \left( \begin{array}{c} N \\ 2^1 \end{array} \right) (1+2^1, k) + \left( \begin{array}{c} N \\ 2^1+1 \end{array} \right) (3, k) + \left( \begin{array}{c} N-2^1 \\ 2^1+1 \end{array} \right) (3+2^1+1, k) + \left( \begin{array}{c} N \\ 2^1 \end{array} \right) (3+2^1, k) \)

\text{C. } N_0 = 2^1-4

1. $0$  
   \(
   \left( \begin{array}{c} N+4 \\ 2^1 \end{array} \right) [(0, k) + (-2^1, k)] + (-2^1+1, k) + \left( \begin{array}{c} N+4+2^1 \\ 2^1+1 \end{array} \right) (-2, k) + \left( \begin{array}{c} N+4+2^1+1 \\ 2^1+1 \end{array} \right) (-2-2^1+1, k) + \left( \begin{array}{c} N+4+2^1 \\ 2^1+1 \end{array} \right) (-2, k) + \left( \begin{array}{c} N+4+2^1+1 \\ 2^1+1 \end{array} \right) (-2-2^1+1, k) \)

\[
\begin{align*}
2. \ 2^1 & \quad (N+4) \left(2^1, k \right) + \binom{N+4}{2^1} \left(2^1-2, k \right) + \binom{N+4+2^1+1}{2^1+1} \left(2^1-2, k \right) \\
3. \ N_2+1 & \quad (N_2+1-2^1+1, k) \\
4. \ N_2+1+2^1 & \quad \binom{N}{2^1} \left(N_2+1, k \right) + \binom{N+2^1}{2^1} \left(N_2+1-2^1, k \right) \\
5. \ N_2+3 \cdot 2^1+4 & \quad \binom{N_2+3 \cdot 2^1+2}{2^1} \left(N_2+3 \cdot 2^1+2, k \right) + \binom{N+4}{2^1} \left[N_2+2^1+1, k \right] + \\
& \quad \binom{N_2+2^1+2}{2^1} \left(N_2+2^1+2, k \right) + \binom{N_2+2^1+1+4}{2^1+1} \left(N_2+2^1+1+4, k \right) \\
6. \ N_2+2^1+1+4 & \quad \binom{N_2+2^1+1+2}{2^1} \left(N_2+2^1+1+2, k \right) + \binom{N+4+2^1+1}{2^1+1} \left(N_2+2^1+1+4, k \right) + \\
& \quad \binom{N+4}{2^1+1} \left(N_2+4, k \right) \\
7. \ 3 \cdot 2^1+3 & \quad (3 \cdot 2^1+1, k) \\
8. \ 2^1+1+3 & \quad \binom{N+2^1}{2^1} \left(2^1+1+1, k \right) + \binom{N}{2^1} \left(2^1+1, k \right)
\end{align*}
\]

D. \( N_0 = 2^1-2 \)

\[
\begin{align*}
1. \ 0 & \quad \binom{N+4}{2^1} \left[(0, k) + (-2^1, k) \right] + \binom{N+4}{2^1} \left(-2^1, k \right) + \\
& \quad \binom{N+4}{2^1+1} \left[(-1, k) + (-1-2^1+1, k) \right] \\
2. \ 2^1 & \quad (-2^1, k) + \binom{N+4}{2^1+1} \left[(-1+2^1, k) + (-1-2^1, k) \right]
\end{align*}
\]
3. \( N_2 + 1 \)  
\[
\binom{N}{2^1} [ (N_2 + 2^1 + 1, k) + (N_2 + 3 - 2^1 - 1, k) ] + \\
\binom{N+2}{2^1} (N_2 + 1 - 2^1 + 1, k)
\]

4. \( N_2 + 1 + 2^1 \)  
\[
(N_2 + 1 - 2^1, k) + \binom{N}{2^1} [(N_2 + 1, k) + (N_2 + 2 - 2^1, k) + (N_2 + 3 - 2^1, k)]
\]

5. \( N_2 + 3 \cdot 2^1 + 4 \)  
\[
(N_2 + 3 \cdot 2^1 + 2, k) + \binom{N+4}{2^1} [(N_2 + 2^1 + 1 + 2, k) + (N_2 + 2^1 + 2, k)] + \\
\binom{N+4}{2^1+1} [(N_2 + 2^1 + 2, k) + (N_2 + 3 \cdot 2^1 + 3, k) + (N_2 + 2^1 + 3, k)]
\]

6. \( N_2 + 2^1 + 1 + 4 \)  
\[
(N_2 + 2^1 + 1 + 2, k) + \binom{N+4}{2^1+1} [(N_2 + 2^1 + 1 + 3, k) + (N_2 + 3, k)]
\]

7. \( 3 \cdot 2^1 + 3 \)  
\[
\binom{N+2}{2^1} (3 \cdot 2^1 + 1, k) + \binom{N}{2^1} [(3 \cdot 2^1, k) + (3 \cdot 2^1 - 1, k)]
\]

8. \( 2^1 + 1 + 3 \)  
\[
(2^1 + 1 + 1, k) + \binom{N}{2^1} [(2^1 + 1, k) + (2^1 + 1 - 1, k) + (2^1 + 1, k)]
\]

11. \( 1 = 2^{(S_q\Sigma s_2F)} \)

A. \( N = 0 \) (16)

\( r = 16k + \) nonzero \( F_j \) in \( \phi(r) \) for \( j = 16k + \)

1. 0  
0, -2, -4, -6, -8, -10

2. 1  
-7, -9

3. 5  
-3, -5

4. 8  
4, 2
<table>
<thead>
<tr>
<th>B, ( N = 2 ) (16)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1</td>
<td>1, -1, -3, -5, -7, -9</td>
<td></td>
</tr>
<tr>
<td>2. 3</td>
<td>-5, -7</td>
<td></td>
</tr>
<tr>
<td>3. 7</td>
<td>-1, -3</td>
<td></td>
</tr>
<tr>
<td>4. 9</td>
<td>5, 3</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>C, ( N = 4 ) (16)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 0</td>
<td>-1, -6, -8, -9</td>
<td></td>
</tr>
<tr>
<td>2. 4</td>
<td>3, 2, 0, -5</td>
<td></td>
</tr>
<tr>
<td>3. 10</td>
<td>10, 9, 7, 3, 1, 0</td>
<td></td>
</tr>
<tr>
<td>4. 14</td>
<td>12, 11, 10, 8, 7, 4</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>D, ( N = 6 ) (16)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 0</td>
<td>0, -1, -8, -9</td>
<td></td>
</tr>
<tr>
<td>2. 4</td>
<td>2, 1, 0, -1, -2, -3, -4, -5</td>
<td></td>
</tr>
<tr>
<td>3. 7</td>
<td>7, 6, 4, 3, 1, 0, -2, -3</td>
<td></td>
</tr>
<tr>
<td>4. 11</td>
<td>10, 9, 8, 7, 5, 4, 3, 2</td>
<td></td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>E, ( N = 8 ) (16)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 0</td>
<td>0, -1, -4, -6, -8, -9</td>
<td></td>
</tr>
<tr>
<td>2. 4</td>
<td>3, 0, -2, -5</td>
<td></td>
</tr>
<tr>
<td>3. 9</td>
<td>8, 7, 1, 0</td>
<td></td>
</tr>
<tr>
<td>4. 13</td>
<td>12, 11, 5, 4</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>F, ( N = 10 ) (16)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1</td>
<td>0, -1, -3, -5, -8, -9</td>
<td></td>
</tr>
<tr>
<td>2. 3</td>
<td>3, 2, -6, -7</td>
<td></td>
</tr>
<tr>
<td>3. 5</td>
<td>5, 4, 3, 1, -1, -3, -4, -5</td>
<td></td>
</tr>
<tr>
<td>4. 11</td>
<td>10, 9, 3, 2</td>
<td></td>
</tr>
</tbody>
</table>
**G. N = 12 (16)**

| 1. 0 | -2, -6, -8, -10 |
| 2. 4 | 0, -6           |
| 3. 2 | -1, -5, -6, -8  |
| 4. 5 | 5, 4, -4, -5    |

**H. N = 14 (16)**

| 1. 0 | -1, -3, -4, -6, -8, -10 |
| 2. 3 | 0, -4, -5, -7           |
| 3. 5 | 2, 1, -2, -5            |
| 4. 7 | 7, 5, 4, 3, 1, 0, -1, -3 |
Appendix G. (Proposition XII.4; \( v = 2^{1} (2^{1} + 2) \))

1. \( l > 3 \), \( N_{0} = 0, 2, 2^{1} - 4, \text{ or } 2^{1} - 2 \)

V-duality applied to (1)-(4) in appendix F (part I)
and \( r, v \)-duality applied to (5)-(8) give essentially all equations needed.

II. \( l = 2 \)

V-duality applied to appendix F (part II) and the following three equations provide essentially all that are needed:

G. \( N = 4 \) (16)

\[ \phi(16k+6) = \overline{F}^{16k+4} + \overline{F}^{16k+3} + \overline{F}^{16k+2} + \overline{F}^{16k} + \overline{F}^{16k-1} + \overline{F}^{16k-2} \]

F. \( N = 10 \) (16)

\[ \phi(16k+15) = \overline{F}^{16k+15} + \overline{F}^{16k+14} + \overline{F}^{16k+7} + \overline{F}^{16k+6} \]

G. \( N = 12 \) (16)

\[ \phi(16k+14) = \overline{F}^{16k+14} + \overline{F}^{16k+13} + \overline{F}^{16k+12} + \overline{F}^{16k+11} + \overline{F}^{16k+10} + \overline{F}^{16k+8} + \overline{F}^{16k+7} + \overline{F}^{16k+5} \]