Mathematics and Tensegrity

Group and representation theory make it possible to form a complete catalogue of "strut-cable" constructions with prescribed symmetries

Robert Connelly and Allen Back

In the autumn of 1948, while experimenting with ways to build flexible, modular towers, a young artist named Kenneth Snelson constructed a sort of sculpture that had never been seen before. As ethereal in appearance as a mobile, with no obvious weight-bearing elements, it nonetheless retained its shape and stability. "I was quite amazed at what I had done," Snelson recalled four decades later. The following summer he showed the sculpture to his mentor, the not-yet-famous inventor, artist and self-styled mathematician R. Buckminster Fuller. Before long, Fuller had adapted Snelson's invention as a centerpiece of his system of synergetics, even to the point of calling the new objects "my structures" and promoting them in his many inspirational, free-ranging lectures. In the process, he gave them the name by which they are known today, referring to their integrity under tension: *tensegrity.*

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Snelson's sculptures, in which rigid sticks or "compression members" (as an engineer might call them) are suspended in midair by almost invisible cables or very thin wires, can still be seen around the world. A remarkable, 60-foot-high sculpture, "Needle Tower," is displayed at the Hirshhorn Museum and Sculpture Garden in Washington, D.C. The idea has penetrated into low art as well. A number of baby toys employ the same principles as Snelson's original tensegrities. One could even argue that the first tensegrities were not made by human beings: A spider web can also be viewed as a tensegrity, albeit one with no rigid parts.

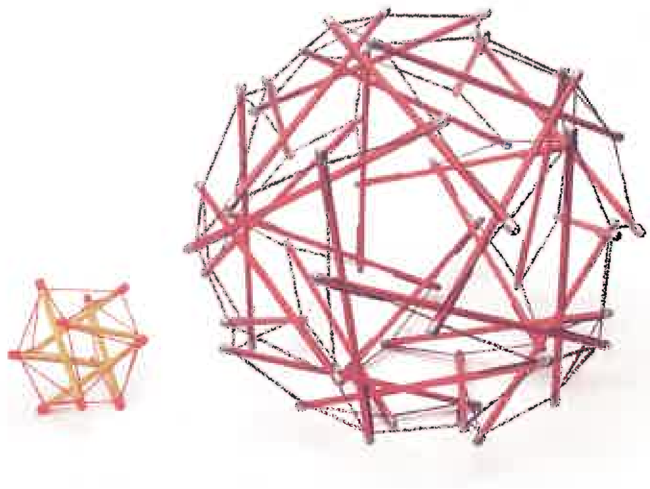
Although Fuller's geodesic domes and synergetics gained him worldwide renown, most of the mathematics that he used was already well established. However, his student Snelson's discovery posed genuinely new mathematical questions, which are far from being completely resolved: What is a tensegrity? Why is it stable? Can tensegrities be classified or listed?

Branko Grünbaum, a mathematician at the University of Washington in Seattle, was especially responsible for rekindling the interest of mathematicians in such questions, with a wonderful set of mimeographed notes written in the early 1970s, called "Lectures on Lost Mathematics." In 1980, one of us (Connelly) proved a conjecture of Grünbaum's that allows the systematic construction of stable planar tensegrities. But the wonder and beauty of Snelson's sculptures surely lies in their three-dimensional nature. One of the motivations of our recent work, therefore, was to find a proper three-dimensional generalization. The mathematical tools of group theory and representation theory, coupled with the powerful graphic and computational capabilities of computers, have now made it possible to draw up a complete catalogue of tensegrities with certain prescribed types of stability and symmetry, including some that have never been seen before.

What Is a Tensegrity?

Tensegrities have a purity and simplicity that lead very naturally to a mathematical description. Putting aside the physical details of the construction, every tensegrity can be modeled mathematically as a configuration of points, or vertices, satisfying simple distance constraints. Snelson's structures are held together with two types of design elements (engineers say members), which can be called *cables* and *struts.* The two elements play complementary roles: Cables keep vertices close together; struts hold them apart. Two vertices connected by a cable may be as close together as desired—they might even be on top of one another if the tensegrity collapsed—but they may never be farther apart than the length of the cable joining them. Similarly, two vertices joined by a strut may never be closer than the length of the strut, but may be arbitrarily far apart.

The last point may seem surprising at first, because in most real tensegrities the struts cannot get either longer or shorter. In fact, the term "bar" has been used to describe a design element of fixed length. However, we have found that, most of the time, bars can be replaced by struts without sacrificing stability. Moreover, the concept of struts can be applied to other problems, such as the packing of spherical balls. (In any such packing, the centers of the balls must keep a minimum distance but can be as far apart as desired.



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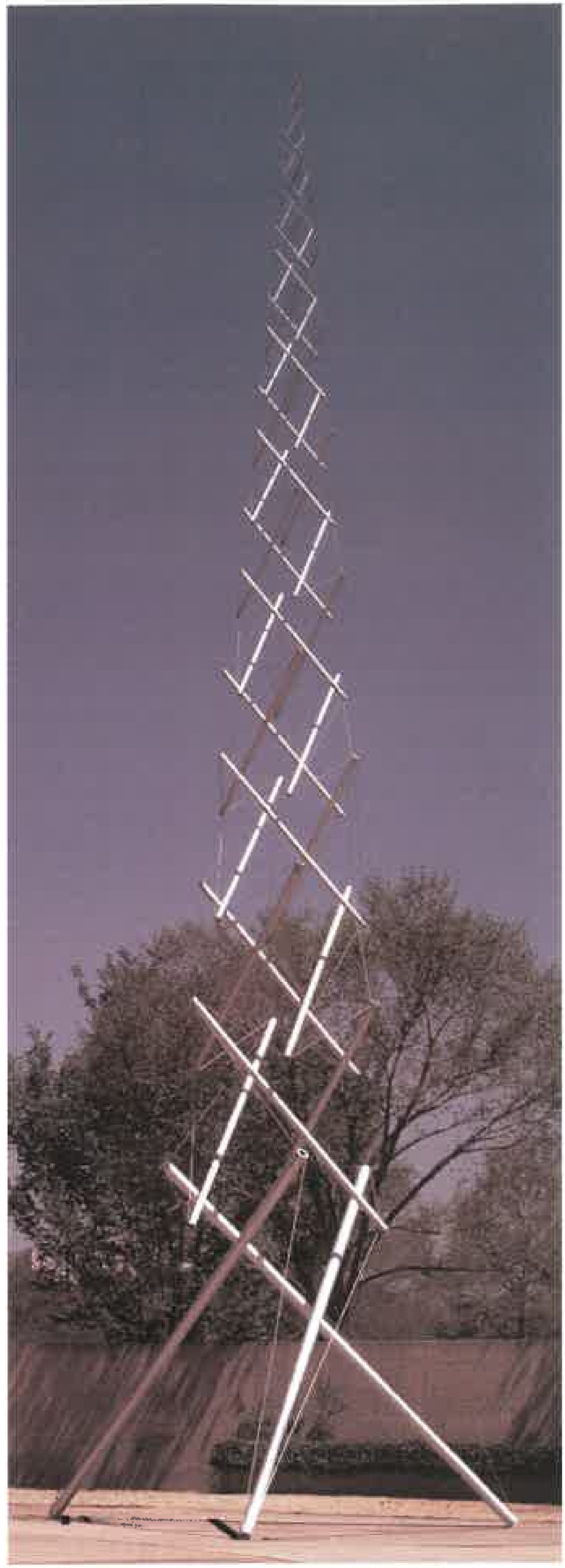


Figure 1. Tensegrity, a concept invented by Kenneth Snelson for use in sculptures, describes a structure that retains its integrity under tension. Consisting of struts and cables, these three-dimensional assemblages may soar into the sky, float out across landscapes or describe more familiar geometric figures. Tensegrities appear in high art, low art and nature, as illustrated by Snelson's sculpture "Needle Tower" at the Hirshhorn Museum and Sculpture Garden in Washington, D.C., a spider web and a child's toys (Stik-Trix and Tensegritoy, *left and right*). All of these structures hold their shape because of internal tension, yet mathematical generalizations of their sometimes complex structure have been difficult to develop. The authors have used the mathematical tools of group theory and representation theory, combined with the graphic capabilities of computers, to develop a complete catalogue of tensegrities with certain prescribed types of stability and symmetry. (Photograph at right courtesy of Kenneth Snelson; upper left photograph courtesy of Design Science Toys Ltd.)

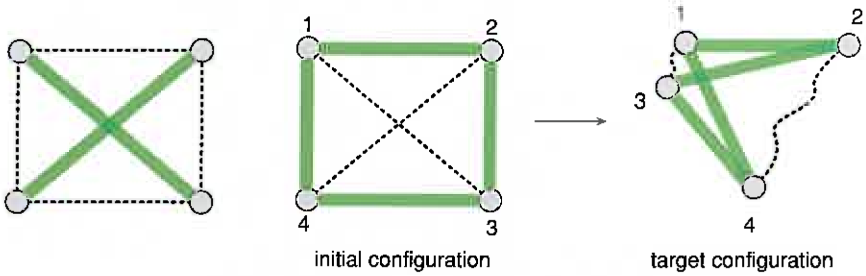


Figure 2. **Snelson's X tensegrity *(left)* and its inverse *(right)*, with struts and cables interchanged, demonstrate the concept of super stability. The X tensegrity is super stable: Any comparable configuration must either have shorter struts or longer cables. The inverse is not super stable, because it can be deformed by flipping along one digonal, so that the struts remain the same length while the other diagonal cable becomes shorter. Struts are shown in green; cables are dashed lines.**

Thus a packing can be considered to be a tensegrity with invisible struts.)

Some people have defined a tensegrity in such a way that no two struts share an end vertex, and each vertex is at the end of a strut. Again, for reasons of generality we do not adopt these conventions, but for many of the examples that are mentioned later, these properties happen to hold.

What is Stability?

As in the definition of a tensegrity itself, there are several different plausible notions of stability, each appropriate for certain circumstances: infinitesimal rigidity, static rigidity, first- and second-order rigidity, prestress stability and others. (See Connelly and Whiteley 1996 for a definition of these terms.)

We have chosen a very direct and strong definition, whose name was proposed by one of our undergraduate students, Alex Tsow. We can call two configurations "comparable" if they have the same number of vertices, connected by cables and struts in the same way. Tsow called a given tensegrity *super stable* if any comparable configuration of vertices either violates one of the distance constraints—one of the struts is too short, or one of the cables too long—or else is an identical copy of (in geometrical terms, congruent to) the original.

For example, Snelson's X tensegrity in Figure 2 is super stable. One elementary tensegrity that is not super stable is a hinge (two struts sharing one vertex): By opening or closing the hinge a little bit, one obtains a new configuration with the same strut lengths but a different shape from the original. More interestingly, if one reverses the roles of cables and struts in the X tensegrity, it fails to be super stable, even considering only configurations in the plane. It is rigid in the plane, in the sense that there is no continuous or gradual motion of the vertices that preserves the cable and strut constraints. However, it is not rigid in space; like a hinge, it can be flexed into new shapes that are not congruent to the original.

Unlike a rigid tensegrity, a super stable tensegrity must win against all the comparable configurations in any number of dimensions—including dimensions 4 and higher. Mathematicians are used to such spaces, as Pythagoras's formula for distance and Descartes's idea of coordinates make them as easy to work with as 2- and 3-dimensional space *(see Figure 3)*.

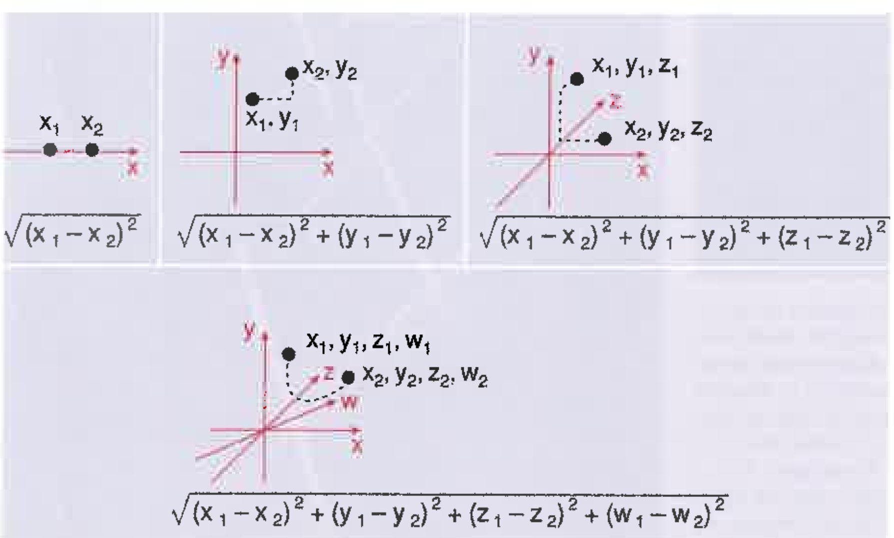


Figure 3. Points in *n*-dimensional Euclidean space can be identified with their coordinates in a Cartesian coordinate system. The number of coordinates is the dimension. Distance in *n*-dimensional space is calculated by analogy with the Pythagorean formula from plane geometry. A tensegrity can be defined in any dimension because its points and the distance constraints governing the struts and cables can all be stated in terms of coordinates. Here examples are shown for 1- through 4-dimensional space.

Spider Webs and Stability

To prove that a tensegrity is stable in such a strong sense, we often invoke a concept borrowed from physics—the idea of potential energy. When a structure is deformed, physically it adsorbs or gives up energy. However, mathematicians need not be constrained to physically realistic energy functions, but may invent convenient fictional energy functions to facilitate the verification of super stability.

A good starting point for understanding these functions is the tensegrity that was constructed long before Kenneth Snelson—the spider web. A spider web differs from the tensegrities discussed so far in two respects. First, it has some "pinned" vertices, fixed in space or in the plane; any comparable configuration must have vertices in exactly the same positions. Second, a spider web has only cables and no struts *(see Figure 4)*.

The energy functions considered for spider webs are motivated by, but not identical to, the physical potential energy for an ideal spring. The English physicist Robert Hooke (1635–1703) found that the force needed to displace a spring was proportional to the displacement from its rest position. (He wrote this empirical observation, later known as Hooke's law, as an anagram: "ceiinosssttuu." The unscrambled anagram—"Ut tensio, sic vis"—translates from Latin as, "As the extension, so is the force.") Although Hooke did not phrase the law in terms of energy, it implies that the energy in a spring is proportional to the square of the distance it is stretched or compressed.

In a spider-web tensegrity, the energy function for each cable is simply proportional to the total length squared—as if the cable were a spring with a resting length of zero. It remains to determine what the constants of proportionality for each cable should be or (if we think of the cables as springs) how "strong" the springs are. For a given configuration, the goal is to choose these fictional strengths in such a way that the configuration represents a unique minimum for the corresponding energy function, the sum of the energy functions of all the cables. Then any comparable spider web that does not increase any cable length must have the same energy or smaller, because each cable contributes the same, or less, to the total. But since the given configuration is supposed to represent a unique minimum energy, the two configurations must be identical. Consequently, such a spider web (one that minimizes some energy function) is super stable. This is called the *principle of least work*, in honor of the similar principle that is used in structural engineering.

How does one recognize when a spider web has an energy function that is precisely minimized by the given configuration? One answer is the equilibrium of stresses—another concept borrowed from engineering. Again, it helps to imagine the cables that meet at any given vertex as springs, each one tugging in a different direction. Remember that this force can be made "stronger" or "weaker" by a proportionality constant, which we will call the *stress* in the cable; thus the strength of each cable's tug is equal to its length times its stress. If the stresses are picked in just the right way, so that the tug-of-war among all the cables leading to a given vertex is a draw, then the stresses at that vertex are in *equilibrium*. If that happens at every unpinned vertex, then the whole spider web is said to be in equilibrium.

These stresses are simply numbers that are assigned to cables; they need not have anything to do with the size of the cable or even its physical or mechanical characteristics. However, it is important that all the stresses are positive: If some are zero, then the given configuration may not be a unique minimum of the energy function, and the principle of least work will not apply.

For tensegrities in general, equilibrium of the stresses is not enough to guarantee that the configuration has the minimum energy. However, for the special case of a spider web with no unstressed cables, it is enough. The reason is that the energy function, composed as it is of quadratic polynomials with positive coefficients, has a property called *convexity*. No matter how you move the (unpinned) vertices around, starting from a critical equilibrium configuration, the energy function will increase. A convex function, like a parabola, can only have one minimum point. If a spider web satisfies the equilibrium condition, it has to be the one. Thus the principle of least work applies to show that the spider web is super stable.

Note that this method does not just provide a "local" result only valid for small (or even bounded) perturbations.

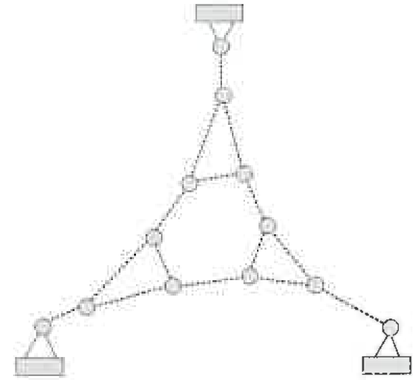


Figure 4. Mathematical spider web consists only of cables, with no struts. Three vertices are pinned *(shaded rectangles)* and cannot move relative to the background. In order for the web to be super stable, the three cables coming from the pinned vertices must determine lines that go through a single point.

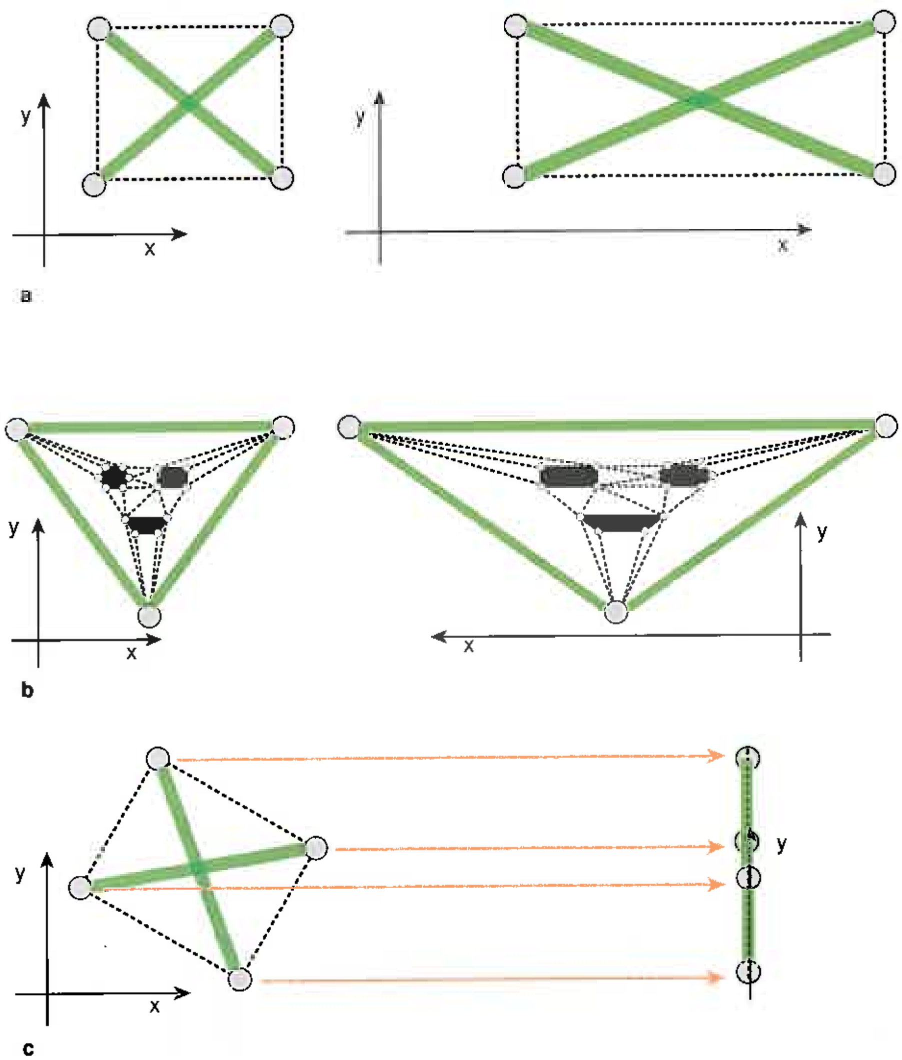


Figure 5. Affine transformations include stretches *(a)*, flips *(b)* and projections *(c)*. Note that a projection may cause many struts and cables to overlap. All of these transformations have the property that an equilibrium stress for the original tensegrity also serves as an equilibrium stress for the transformed tensegrity.

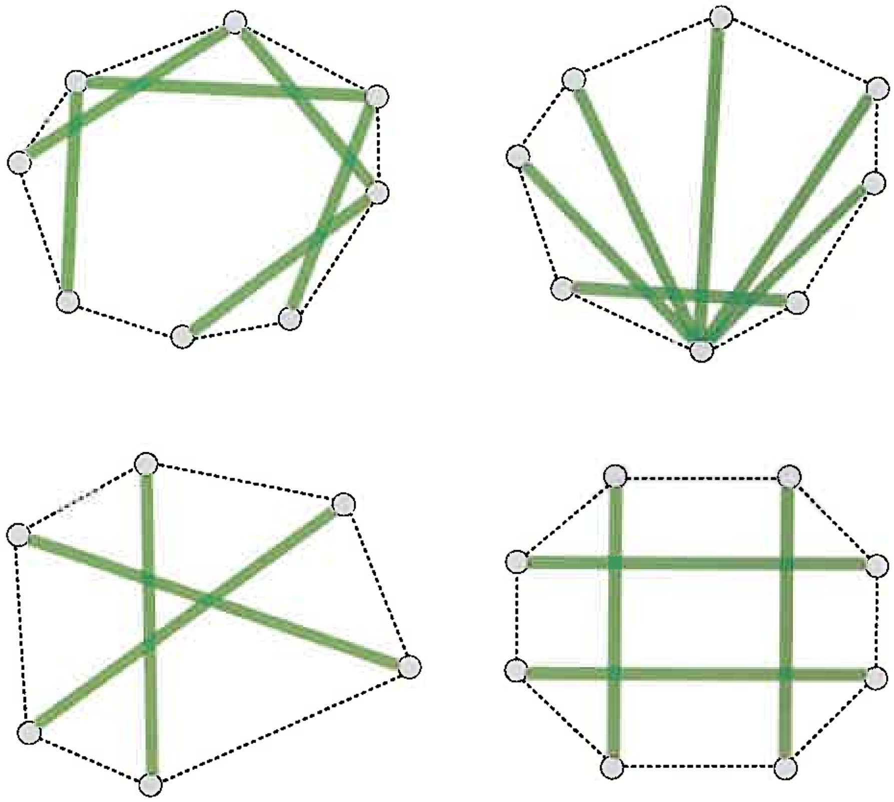
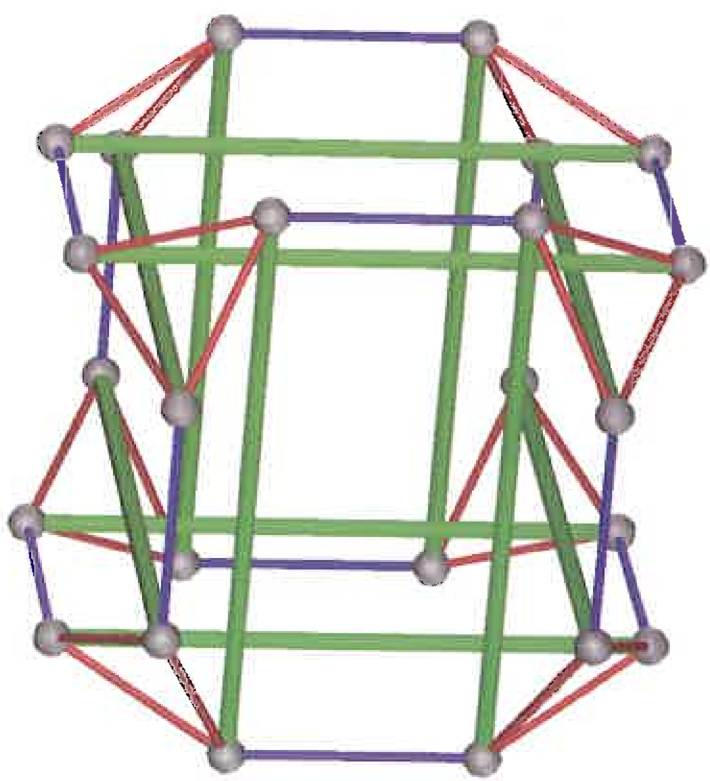


Figure 6. Super stable planar tensegrities can be generated by a 1980 theorem of Connelly. If the cables form a strictly convex polygon, if the struts are internal diagonals, and if there is a positive stress for each cable, a negative stress for each strut and equilibrium at each vertex, then the tensegrity is super stable. The stability is not always intuitively obvious. A configuration inspired by a rigidity theorem proved by French mathematician Augustin Louis Cauchy in 1813 is shown at top left. An example of a class suggested by the Branko Grünbaum of the University of Washington is shown at top right. A configuration that is super stable only if the vertices lie on an ellipse appears at bottom left.



It works for any other configuration one can conceive in any higher-dimensional Euclidean space. Spiders cannot have their webs ruined even by flies in higher dimensions.

Stability of 2-Dimensional Tensegrities

The quadratic functions that were considered for the spider web tensegrities work very nicely, but if there are struts as well as cables, the situation gets more complicated. Because a strut is, in a certain sense, the opposite of a cable, it is mathematically natural—although it does not model a physical potential energy function directly—to define its energy function similarly but with a negative proportionality constant. This is as if the rest position is when the strut has infinite length! The total energy is, as before, the sum of the energies in each of the cables and struts.

Incidentally, although the energy function just described is not physically realistic, it is not completely divorced from engineering reality. If one analyzes the local static properties of a structure, the quadratic energy described here is one of two terms that enter into a description of the structure under sufficiently small perturbations. When the stability can be detected by such a quadratic approximation, the structure is called *prestress stable* in the engineering literature.

If we consider only small perturbations of the physical energy of a prestress stable structure, the second term (which we have not described) only adds to the stability of the structure. Indeed, for a super stable tensegrity, as long as there is no catastrophic buckling of the struts or breaking of the cables, increasing the stress tends to stabilize the tensegrity. This is not necessarily the case for a tensegrity that is only prestress stable.

Figure 7. Super stable three-dimensional tensegrities can be generated from the action of a symmetry group on one strut and two cables. In this example, which has the symmetry group of half the symmetries of a cube, each strut *(green)* can be superimposed on each other one by a rotation or a reflection. Similarly, each red cable can be superimposed on each other red cable, and each blue cable on each other blue cable. The whole tensegrity can be seen as being made up of six identical "stretchers" joined together at their ends with the red cables. If the predetermined ratio of the stress in the blue cables to the stress in the red cables is increased, the blue cables shorten, and the configuration approaches that of the baby-toy tensegrity.

A second problem when we move from spider webs to tensegrities is the lack of pinned vertices, which technically rules out the whole idea of a "unique" energy minimum. Since nothing is pinned, the whole configuration can be rigidly moved about, and the energy will remain the same. Not only that, but there also can be massive distortions to the configuration—certain kinds of rescaling transformations as well as projections of a tensegrity down to its "shadow" in a lower number of dimensions—that do not alter the energy. The equilibrium condition is preserved by such maps, which are called *affine linear transformations (see Figure 5).*

As before, call the coefficient of the energy function for each cable or strut the stress. As with the spider web tensegrities, the tensegrity is said to be in equilibrium with that collection of stresses if the stresses balance at each vertex. For example, Snelson's X tensegrity is in equilibrium when all four cables have stress 1, and the two struts have stress –1.

As in the case of spider webs, if a tensegrity has an energy function that is minimized for a certain configuration, then the configuration is in equilibrium. But the converse is no longer true: Even when the configuration is at equilibrium for a given set of stresses, the energy may not be at a minimum. For example, simply reverse the roles of cables and struts for Snelson's X tensegrity, and reverse the signs of the stresses as well. The new tensegrity will still be in equilibrium, but the energy will be at a maximum instead of a minimum, and the tensegrity will not be super stable.

Thus, to prove that a given tensegrity is super stable, there are three tasks we must complete. First, we must show that the energy function is at a minimum—not just at an equilibrium. Second, we must show that the only affine transformations that do not violate any cable and strut constraints are actually congruences (that is, no stretching or shrinking is allowed). Finally, we must show that the given tensegrity cannot be the "shadow" of any higher-dimensional tensegrity that is also in equilibrium. (Maria Terrell of Cornell University has called the latter property the *universality* of the tensegrity.) Under these conditions, the principle of least work implies that the given tensegrity is super stable.

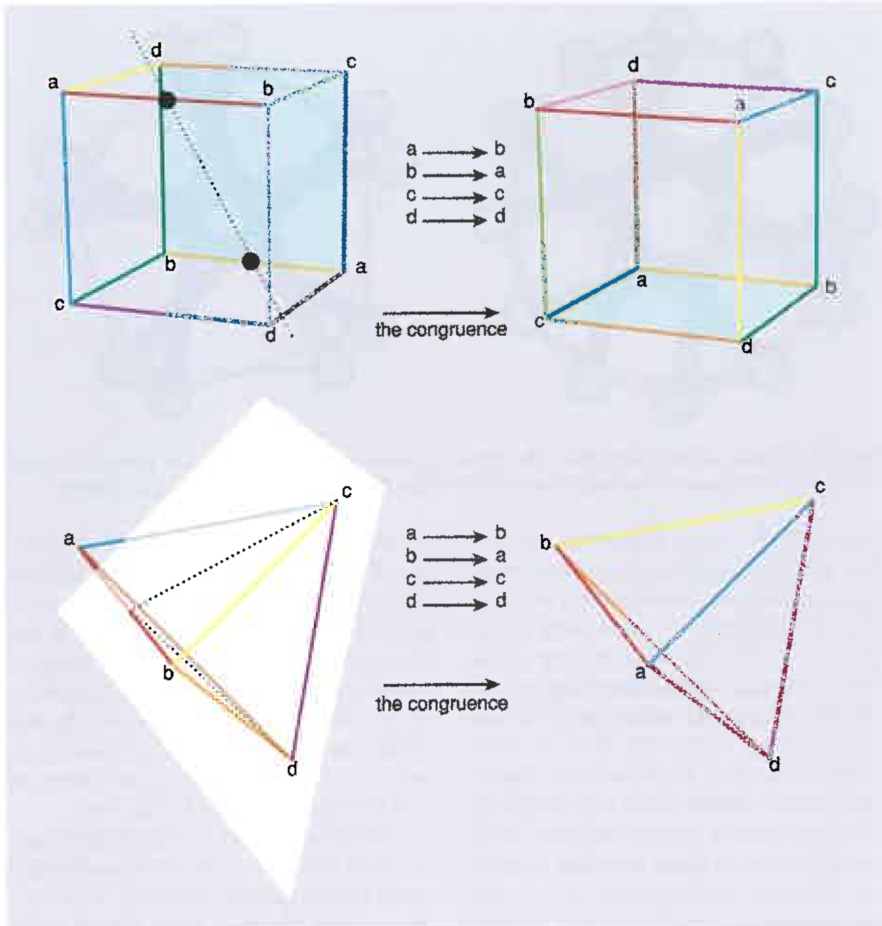


Figure 8. Group of permutations of four letters acts on a cube and a tetrahedron in two different ways. On the cube *(top)*, the permutation that switches vertices labeled (a) with those labeled (b) while leaving (c) and (d) alone represents a rotation. On the tetrahedron *(bottom)*, the same permutation represents a reflection. Thus the group of permutations of four letters (called S_4) is seen to have two distinct 3-dimensional representations.

This approach has made it possible to identify a large number of super stable tensegrities. For example, take any convex polygon in the plane, where the edges are cables and some collection of the internal diagonals are struts. (Here the word "convex" is applied in a different context from before. A polygon is *convex* if the line segment connecting any two of its vertices is contained entirely in the interior.) One of us (Connelly) proved in 1980 that, if an equilibrium collection of stresses can be found, positive on the external edge cables and negative on the internal diagonal struts, then each of the three conditions holds; thus, any such tensegrity is super stable. Therefore, for this class of convex planar tensegrities as well as for spider webs, equilibrium implies stability *(see Figure 6).*

The question remains as to what the proper generalization is for 3-dimensional space. A natural choice for a configuration is the collection of vertices of a 3-dimensional polytope (polyhedron), and perhaps the edges of the polytope should provide the cables of the tensegrity. But it is not clear just how to identify precisely a satisfyingly general class of super stable tensegrities. One idea is to specialize somewhat and look at tensegrities that have a great deal of symmetry. This can be used to short-circuit the unpleasant parts of the calculations.

Symmetric Tensegrities

Some of the most appealing tensegrities made by Snelson and later by others are highly symmetric, exhibiting a subgroup of the symmetries of the cube or the regular dodecahedron. To be efficient about the analysis, and—even more important—to provide a framework for the classification of such structures, it is convenient to use the theory of representations of finite groups.

The theory of representations of finite groups, developed at the turn of the past

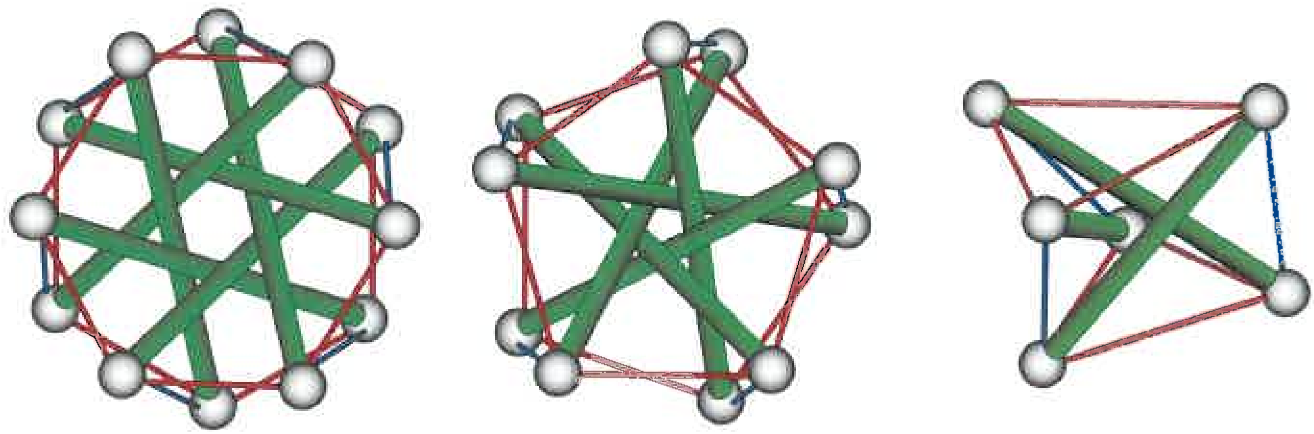


Figure 9. Super stable tensegrities like these are generated by the dihedral groups, in joint work of Connelly and Maria Terrell of Cornell University. This class includes Snelson's "octet truss," one of his earliest designs *(right)*.

century by F. Georg Frobenius and Isai Shur, was originally motivated by a problem in algebra, but soon found a wide variety of applications, especially to the physics of the then new theory of the atom. A typical success story begins with a fairly complicated mathematical model of a structure with some form of (geometric) symmetry. Representation theory allows one to break down the complicated model into a predetermined, small number of much more tractable models, each of which can be treated more or less independently. Fan Chung and Shlomo Sternberg (1993) gave a very nice example of such an application: the analysis of the infrared spectra of the buckyball, a molecule with 60 carbon atoms that has the symmetry of a regular dodecahedron. In our situation, the representation theory is called on in almost the same way, except that the underlying mathematical objects are different.

Suppose that a tensegrity can be rotated about some line in such a way that the rotated tensegrity is indistinguishable from the original: each vertex superimposed on a vertex, each cable superimposed on a cable and each strut superimposed on a strut. All the motions, or *congruences*, that superimpose the tensegrity on itself in this fashion form a mathematical structure called the *group of symmetries* of the tensegrity. (It may include some reflections and other congruences, and always includes the "identity," a Zen-like motion that simply leaves everything untouched.)

If a tensegrity has enough symmetries that any vertex can be superimposed onto any other by a congruence, then the group of symmetries is said to be transitive on the vertices. Similarly, any pair of vertices (or any pair of cables or any pair of struts) is said to be in the same *orbit* if at least one symmetry of the tensegrity superimposes one of the pair onto the other. In Figure 7, each cable or strut in a given orbit has the same color. The group of symmetries of these tensegrities is transitive on the struts (because there is only one orbit, colored green) but not quite transitive on the cables (because there are two orbits, colored red and blue).

With this in mind, it is especially easy to check whether there is an equilibrium stress for the cables and struts. If there is any energy function at all that is minimized by the given configuration, then there is one that has the same symmetry as the tensegrity itself. Thus the same stress can be assigned to each cable or strut in any given orbit. Similarly, the balance of stresses only needs to be checked at one vertex, because all vertices are alike under the group of symmetries. For almost all the examples considered here, it turns out to be easy to check that the only affine motions that preserve the cable and strut constraints are congruences. Hence only one difficulty remains before we can apply the least work principle: We have to make sure that the energy is a minimum. That is where group representations enter the picture.

Group Representations

So far, groups have appeared in only one guise: the group of symmetries of a tensegrity. But groups can also be defined in the abstract, without reference to any particular physical object. From this point of view, a group is simply a set whose elements can be "multiplied" and that obeys certain rules, such as the existence of an identity element. In the case just presented, the elements of the group were motions that superimpose a tensegrity on itself, and "multiplication" of two congruences is defined by first performing one motion and then the other.

Group theory is something like the ancient Oriental game of go. There are only a few simple rules, and they are easy to learn. But their consequences can take many years of intense study to master. Like certain move sequences (or "joseki") in go, certain groups occur often enough to have their own names. The simplest, and perhaps most ubiquitous, is a group with two elements called Z_2. Its elements can be thought of as the numbers +1 and –1, with the operation of multiplication. (Note that a product of either +1 or –1 with either +1 or –1 again gives +1 or –1.) Or they can be thought of as the words "even" and "odd," with the operation of addition. Or they can be thought of as the identity motion and reflection in a mirror.

Of the many guises an abstract group can assume, among the most convenient ones are sets of linear functions. Rotations and reflections are examples of linear functions. If the elements of the abstract group are thought of as actors and the linear functions as roles, then the playbill, which assigns certain actors to certain roles, is called a *representation* of the group. The abstract group itself can be used to do group calculations and effectively provide a common point of reference. A representation, on the other hand, may have more structure, and may give deep and subtle information about the abstract group.

An essential concept in dealing with representations is that of *equivalence*. Consider, for example, the group of symmetries of the word MOM. There are two ways to superimpose this word on itself: Either leave it alone (the identity motion) or reflect it about a vertical line through the center of the O. This symmetry group, therefore, is a representation of the abstract group Z_2. Likewise, if we turned the word MOM on its side, its symmetry group would still be a representation of Z_2. Although the actual reflection involved is different—it is now a reflection through a horizontal line—the essential symmetry of the word has not been changed by turning it on its side. A mathematician would say that these two representations of Z_2 are equivalent.

Compare this with the group of symmetries of the word MOW. Again, there are two ways to superimpose the word on itself: the identity and a 180-degree rotation about the center of the O. This symmetry group is also a representation of Z_2, but somehow it feels different. The difference is not in the group itself, but in the geometry—the linear function involved is a rotation, not a reflection. Thus a mathematician would say that this representation of Z_2 is not equivalent to the previous one.

The symmetries of 3-dimensional figures involve representations of some more interesting groups than Z_2. The group of rotations of the cube and the group of all symmetries of the regular tetrahedron are both representations of the group of permutations of four letters (denoted S_4). These two representaions of S_4 turn out to be inequivalent, as can be seen in Figure 8.

The representations of a group can be combined in a simple but important way. For example, think of two representations of a group, one as motions of 3-space and the other as motions of 2-space. Now think of a 5-dimensional space, where the first three coordinates correspond to the 3-space, and the next two coordinates correspond to the 2-space. Create a new representation of the group by letting the first representation act on the first three coordinates and the second representation act on the next two coordinates. Then the new representation is called the *sum* of the two representations.

The process sometimes works in reverse, too: Given a representation, it may be possible to split it up as a sum of smaller-dimensional representations. For example, each of the two elements in the symmetry group of the word MOM superimposes a horizontal line through the center of the O onto itself. Likewise, each one superimposes a vertical line onto itself. This representation, therefore, is reducible to the sum of two 1-dimensional representations. But the situation is different for the two 3-dimensional representations of the group S_4. Although a line or plane may be superimposed on itself by an individual rotation, there is no line or plane that is superimposed on itself by *all* the symmetries of the cube or the tetrahedron. Thus these representations are called *irreducible*.

One of the major insights of Frobenius and Shur was that any representation can be decomposed in an essentially unique way into irreducible ones. Hence the irreducible representations are the building blocks of the theory, much like prime numbers in number theory.

Now we can define a representation that is very closely related to the energy function for tensegrities. Any group, such as the group S_4, has a representation as a permutation of its own elements. (If we return to the actor-and-role metaphor, this is like every actor playing himself.) Since S_4 has 24 elements, imagine a space with 24 coordinates, with each coordinate axis labeled by one element of the group. Each element of S_4 corresponds to a permutation of the axes, and each one of these extends in a natural way to a congruence of the 24-dimensional Euclidean space. These congruences form the *regular representation of* S_4.

Frobenius proved that the regular representation of any group "contains" all of the irreducible ones, each repeated a number of times equal to its dimension. For example, for S_4 the irreducible representations are the two 3-dimensional ones illustrated above, as well as a 2-dimensional one, a nontrivial 1-dimensional representation, and the trivial 1-dimensional representation (where all elements of the group are represented as the identity). So in the regular representation, the two 3-dimensional representations are repeated three times, accounting for nine dimensions each; the 2-dimensional representation is repeated twice, accounting

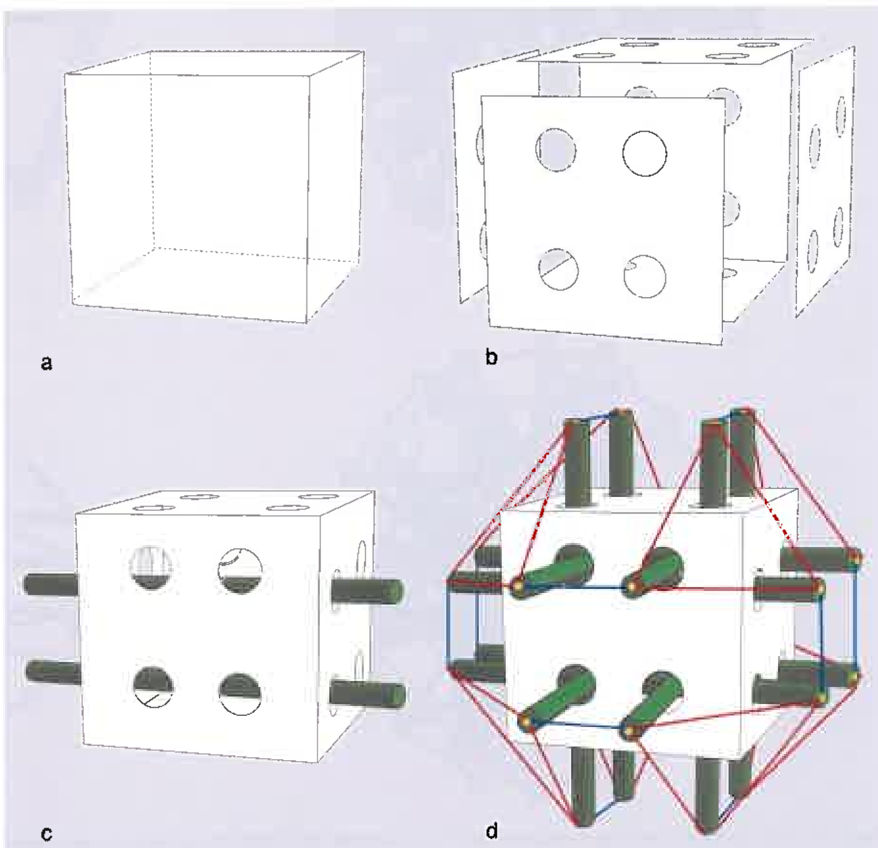


Figure 10. How to build your own tensegrity: Start with a cardboard model of the corresponding regular polyhedron *(a)*. Cut out the sides and use a paper punch to place holes at the proper points in each side *(b)*. It works best if the size of the holes just lets the sticks pass through. Here, however, the holes are shown larger for better viewing. Locations need not be precisely correct. Tape the edges back together again and pierce the polytope with the dowels according to the way they look in the final tensegrity *(c)*. (Compare Figure 7.) Insert rubber bands in the appropriate pattern through the notches in the ends of the dowel struts *(d)*. The figure shows pins at the ends of the dowel struts, which also works. Cut away the cardboard and let the whole structure come to equilibrium. Replace the rubber bands with string or cord. (The rubber bands may deteriorate within a few weeks.)

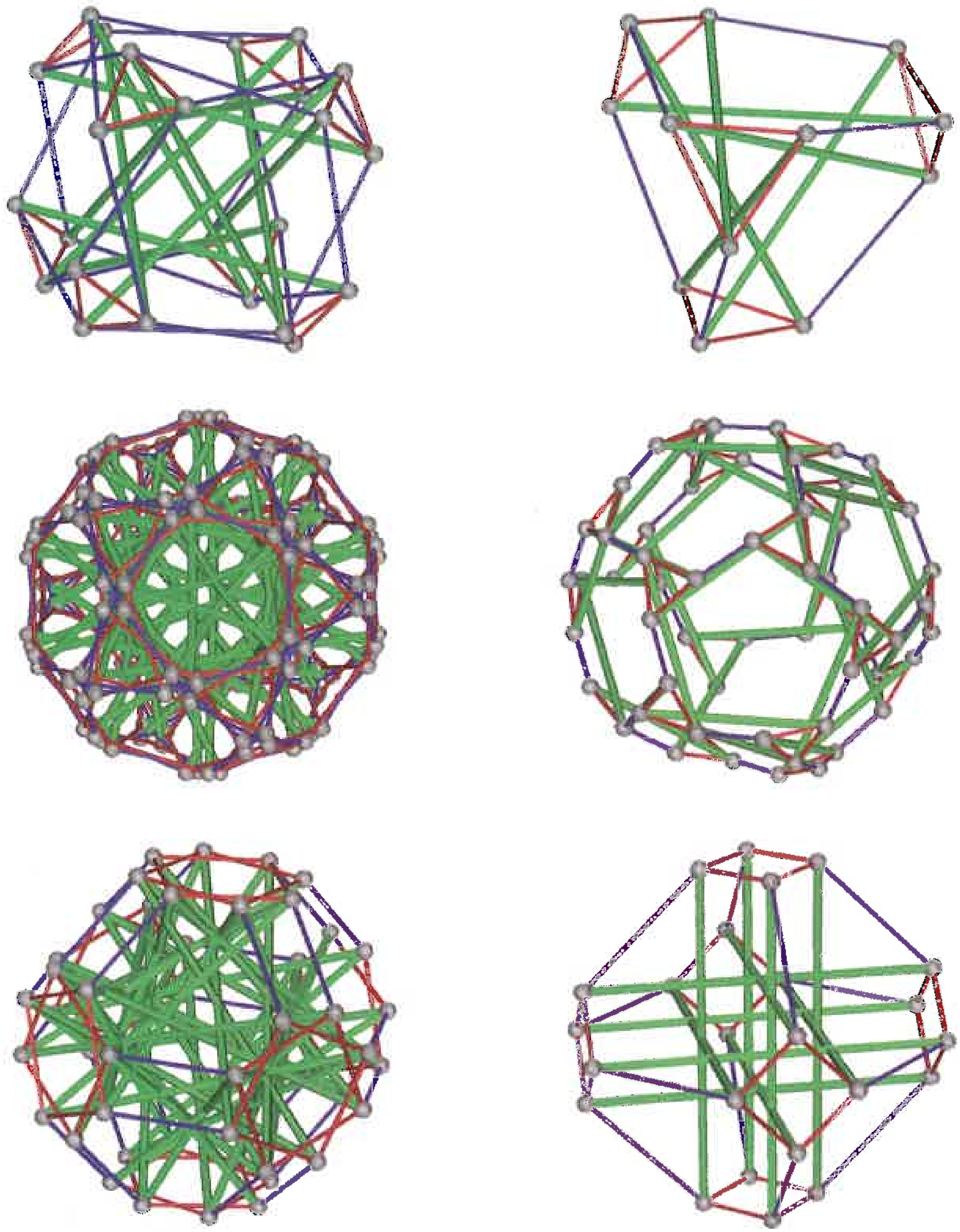


Figure 11. The authors' catalogue of super stable symmetric tensegrities includes dozens to hundreds of examples with the symmetry of each regular polyhedron. Here, one tensegrity representing each type of symmetry is portrayed. Tensegrities in the right hand column, from the top down, come from representations of the group of even permuations on four letters, the group of even permutations on five letters, and the group of all permutations of four letters. These representations turn out to be the rotations of the regular tetrahedron, the regular dodecahedron, and the cube respectively. The tensegrities in the left hand column come from the direct sum of the group with two elements and the group generating the right hand column.

for four dimensions; and the nontrivial and trivial 1-dimensional representations account for one more dimension each. In total, we get 9 + 9 + 4 + 1 + 1 = 24 dimensions accounted for.

Suppose that we have a tensegrity whose group of symmetries acts transitively on the vertices. Its energy function, which was defined using the stresses (motivated from the equilibrium condition), corresponds to the regular representation of the symmetry group. This was not at all obvious to us at first: Only after doing several special cases did we discover that Frobenius's theorem was made to order for our problem. Using that theorem, the energy function can be written as a sum of energy functions that correspond to each of the irreducible representations. So the calculations can be done for the much smaller-dimensional irreducible configurations. For example, calculating a minimum energy configuration for the group S_4, without the decomposition coming from representation theory, would involve solving equations in at least 24 variables. But by using the irreducible representations, it takes only two calculations with three variables, one calculation involving two variables and one with one variable.

Now consider the problem a bit differently. Start with the stresses, and then ask if one of the representations has an equilibrium configuration for that collection of stresses. This is like looking for the smile of the Cheshire cat (the stress) before we find the cat itself (the tensegrity). If all the stresses are positive, then it is clear that all the terms in the definition of the energy function are positive or zero. If the cable graph is connected, this means that the only representation that gives an equilibrium configuration will be the trivial one—in which all the vertices are on top of each other. Next, choose one of the orbits that you eventually want to be a strut. Decrease the stress coefficient for that strut, even allowing it to be negative. Keep decreasing that coefficient until the total energy function itself just starts to have values on the borderline of being negative (that is, zero). Then at least one of the nontrivial irreducible representations has a configuration that is in equilibrium with respect to that stress. Call these equilibrium configurations, resulting from irreducible representations, critical configurations. Usually, but not always, it turns out that there is only one representation that has a critical configuration. If so, this is the desired configuration and the corresponding tensegrity.

One unsettling feature of this process is that there do not seem to be any assurances beforehand as to which representation will be the one that provides the crucial stress. If one of the "winners" happens to be a 3-dimensional representation (as it has been in many of the cases we have tried), then we get a super stable tensegrity that we can see, rather than just inferring its existence in some higher dimension. But we do not know of any general theory that would predict the outcome without doing the calculation.

Not many abstract groups can occur as finite groups of symmetries in 3-space. They include two infinite families: the cyclic groups on n elements (which can be identified with the rotations of a regular n-sided polygon) and the dihedral group with $2n$ elements (the group of all symmetries, including reflections, of a regular n-sided polygon). There are six other possibilities: S_4; the "alternating groups" A_4 and A_5 with 12 and 60 elements respectively; and the "direct sum" of each of these with the group Z_2, which doubles the number of elements. Each of these six can be represented as a group or subgroup of symmetries of a regular polyhedron: For example, the group of rotations of a regular dodecahedron is a representation of A_5.

For any given finite group, our method allows us to compile efficiently a complete catalogue of the symmetric tensegrities with two orbits of cables, one orbit of struts and one orbit of vertices. The most intriguing—and most recently discovered—ones correspond to the six groups mentioned in the last paragraph. Because the struts can be connected to the cables in hundreds of different ways, maintaining the symmetry, the complete catalogue has well over a hundred different tensegrities. For Figure 11 we have chosen one representative to illustrate each of the six possible types of symmetry. The complete catalogue can be viewed on our World Wide Web page at http://mathlab.cit.cornell.edu/visualization/tenseg/tenseg.html.

In Figure 10 we show you how to design your own tensegrities. Although the number of configuration types is finite, there is still plenty of room for artistic experimentation: The lengths of the struts can be chosen more or less at will (provided they are all the same length), and the locations and lengths of the other members will change accordingly. Making the tensegrity initially with rubber bands helps the tensegrity "find" a super stable configuration, and the design can then be made permanent by replacing the rubber bands with string.

For readers who would like to build virtual tensegrities, we recommend a program called STRUCK, by Gerald de Jong and Karl Erickson. This program can be accessed on the World Wide Web at http://wolfenet.com/~setebos/springspace.html.

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