t - Inference for Means
Dr. Back

Nov. 9, 2009
Suppose:

1. 33% of 75 Perdue chickens contaminated.
2. 45% of 75 Store Brand chickens contaminated.
3. 56% of 75 Tyson chickens contaminated.
Chicken Contamination

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Questions:

1. Purdue safer than Store Brand?
2. Tyson safer than Store Brand?
3. Tyson different in safety than Store Brand?
4. Confidence interval for difference in safety between Store Brand and Tyson?
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**Notation:** Let $p_1$ denote the proportion of Purdue which are contaminated and $p_2$ the proportion for Store Brand.
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- $H_0$: $p_1 = p_2$ (or $p_1 \geq p_2$)
- $H_a$: $p_1 < p_2$
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$$\hat{p}_{pooled} = \frac{.33 \cdot 75 + .45 \cdot 75}{75 + 75} = .39$$
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$$\hat{p}_{pooled} = \frac{.33 \cdot 75 + .45 \cdot 75}{75 + 75} = .39$$

$$SE_{pooled}(\hat{p}_1 - \hat{p}_2) = \sqrt{.39 \cdot .61 \left(\frac{1}{75} + \frac{1}{75}\right)} = .0796.$$
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$$z = \frac{\hat{p}_1 - \hat{p}_2}{SE_{pooled}} = \frac{-1.12}{.0796} = -1.51.$$
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$$z = \frac{\hat{p}_1 - \hat{p}_2}{SE_{pooled}} = \frac{-0.12}{0.0796} = -1.51.$$ 

P-value = tail probability = $P(Z < -1.51) = 0.0655$. 
At a level of $\alpha = 0.05$, we’d retain $H_0$. 
Purdue might not be safer.
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\[ \hat{p}_{pooled} = \frac{.45 \cdot 75 + .56 \cdot 75}{75 + 75} = .505 \]
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\[ \hat{p}_{pooled} = \frac{.45 \cdot 75 + .56 \cdot 75}{75 + 75} = .505 \]

\[ SE_{pooled}(\hat{p}_2 - \hat{p}_3) = \sqrt{.505 \cdot .495 \left( \frac{1}{75} + \frac{1}{75} \right)} = .0816. \]
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\[ z = \frac{\hat{p}_2 - \hat{p}_3}{SE_{\text{pooled}}} = \frac{-0.11}{0.0816} = -1.35. \]
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\[
z = \frac{\hat{p}_2 - \hat{p}_3}{SE_{pooled}} = \frac{-.11}{.0816} = -1.35.
\]

Which side provides as much or more support for \( H_a \) of \( p_3 < p_2 \)?
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Our statistic provides no support for $H_a$ so we immediately retain $H_0$.
It is a matter of convention whether we’d view the p-value as .5 or even larger.
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- $H_0$: $p_2 = p_3$
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Still $\hat{p}_{pooled} = .505$, $SE_{pooled}(\hat{p}_2 - \hat{p}_3) = .0816$, $z = -1.35$. 
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tail probability = $P(Z < -1.35) = 0.0885$.
P-value = $2(\text{tail probability}) = 2(0.0885) = 0.177$

At a level of $\alpha = 0.05$, we’d retain $H_0$. Tyson might not have a different level of safety than Store Brand.
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Confidence interval for difference in safety between Store Brand and Tyson?
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Confidence interval for difference in safety between Store Brand and Tyson?

\[
SE_{pooled} = \sqrt{\frac{.45 \cdot .55}{75} + \frac{.56 \cdot .44}{75 + 75}} = .0812
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Confidence interval for difference in safety between Store Brand and Tyson?

\[ SE_{pooled} = \sqrt{\frac{.45 \cdot .55}{75} + \frac{.56 \cdot .44}{75 + 75}} = .0812 \]

A 95% CI for \( p_2 - p_3 \) would be

\[ -.11 \pm 1.96 \cdot .0812 = -.11 \pm .159 = (-.269, .049) \]
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A 95% CI for \( p_2 - p_3 \) would be

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The fact that this CI contains 0 is another way of doing the last 2 HT’s.
A coffee vending machine dispenses coffee into a paper cup. You’re supposed to get 10 ounces of coffee, but the amount varies slightly from cup to cup. Here are the amounts measured in a random sample of 20 cups.
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Is there evidence that the machine is shortchanging customers?
Coffee Machine

Is there evidence that the machine is shortchanging customers?

A natural HT situation.
Is there evidence that the machine is shortchanging customers?

We’ll summarize the data by its mean $\bar{x} = 9.845$ and its standard deviation $s = .1986$. 
Is there evidence that the machine is shortchanging customers?

We’ll summarize the data by its mean $\bar{x} = 9.845$ and its standard deviation $s = .1986$.

**Notation:** Let $\mu$ denote the mean amount of coffee in a dispensed cup.
Is there evidence that the machine is shortchanging customers?

We’ll summarize the data by its mean \( \bar{x} = 9.845 \) and its standard deviation \( s = .1986 \).

**Notation:** Let \( \mu \) denote the mean amount of coffee in a dispensed cup.

**Hypotheses:**
- \( H_0: \mu = 10 \) (or \( \mu \geq 10 \))
- \( H_a: \mu < 10 \)
Is there evidence that the machine is shortchanging customers?

Recall by the CLT that the sampling distribution of $\bar{x}$ is

$$N(\mu, \frac{\sigma}{\sqrt{n}})$$

when $n$ is large.
Is there evidence that the machine is shortchanging customers?

\[ N(\mu, \frac{\sigma}{\sqrt{n}}) \]
Coffee Machine

Is there evidence that the machine is shortchanging customers?

As usual with HT’s, we are interested in whether the observed statistic of $\bar{x} = 9.845$ is reasonably consistent with the sampling distribution assuming $H_0$ is true.
Is there evidence that the machine is shortchanging customers?

\[ N(10, \frac{\sigma}{\sqrt{n}}) \]
Is there evidence that the machine is shortchanging customers?

\[ N(10, \frac{\sigma}{\sqrt{n}}) \text{ with } \bar{x} \]
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We know \( n = 20 \), but the major catch is not knowing \( \sigma \). What is the obvious approximation?
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We know \( n = 20 \), but the major catch is not knowing \( \sigma \).

What is the obvious approximation?

**Answer:** Use \( s = .1986 \) instead of \( \sigma \).
Is there evidence that the machine is shortchanging customers?

We know $n = 20$, but the major catch is not knowing $\sigma$. What is the obvious approximation?

Answer: Use $s = .1986$ instead of $\sigma$. 

$$N(10, \frac{s}{\sqrt{n}})$$
Coffee Machine

Is there evidence that the machine is shortchanging customers?

Since the standard error is

\[ SE(\bar{x}) = \frac{s}{\sqrt{n}} = \frac{.1986}{\sqrt{20}} = .0444 \]

\[ N(10, .0444) \text{ with } \bar{x} = 9.845. \]
Is there evidence that the machine is shortchanging customers?

If $s = \sigma$, we’d look at a $Z$-statistic

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{9.845 - 10}{\frac{0.1986}{\sqrt{20}}}$$

where we’ve written $H_0$ more abstractly as $\mu = \mu_0$, $\mu_0$ being the hypothesized value, 10 in this case.
Coffee Machine

Is there evidence that the machine is shortchanging customers?

Because $s$ will not exactly match $\sigma$, we actually get a bit of extra error here. This is compensated for by viewing

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{9.845 - 10}{\frac{.1986}{\sqrt{20}}} = -\frac{.155}{.0444} = -3.49.$$ 

as a $t$-Statistic.
Is there evidence that the machine is shortchanging customers?

Because $s$ will not exactly match $\sigma$, we actually get a bit of extra error here. This is compensated for by viewing

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{9.845 - 10}{\frac{.1986}{\sqrt{20}}} = \frac{-155}{.0444} = -3.49.$$

as a $t$-Statistic.

Since the error in approximating $\sigma$ by $s$ varies with the sample size, there is a different $t$-distribution for each sample size. These are labeled by the “degrees of freedom” which for a 1-sample t-test is:

$$df = n - 1.$$
Coffee Machine

Is there evidence that the machine is shortchanging customers?

These are all critical values \( t^* \).
For example \( P(t > 1.328) = .10 \) for the t distribution with 19 df.
Coffee Machine

Is there evidence that the machine is shortchanging customers?

Our $t$-statistic of -3.49 is more extreme than any on the $df=19$ row of the table.

The picture shows what the critical value $t^* = 2.861$ for a tail prob. of 0.005 means.
Is there evidence that the machine is shortchanging customers?

So by symmetry

\[ P(T < -2.861) = 0.005 \text{ as well.} \]
Is there evidence that the machine is shortchanging customers?

So our tail probability and p-value are both less than .005 and we reject the null. The machine does appear to be shortchanging.
Sps. our t-statistic had been 2.00 with the same 1-sided hypotheses

- $H_0$: $\mu = 10$ (or $\mu \geq 10$)
- $H_a$: $\mu < 10$

What P-value would we report?
Sps. our t-statistic had been 2.00 with the same 1-sided hypotheses

- $H_0: \mu = 10 \ (or \ \mu \geq 10)$
- $H_a: \mu < 10$

What P-value would we report?

Answer: A tail probability and P-value of between .025 and .05.
Sps. instead our t-statistic had been 2.00 with 2-sided hypotheses

- $H_0$: $\mu = 10$
- $H_a$: $\mu \neq 10$

What P-value would we report?
Sps. instead our t-statistic had been 2.00 with 2-sided hypotheses

- \( H_0: \mu = 10 \)
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What P-value would we report?

**Answer:** Our tail probability is still between .025 and .05 but our P-value is now between .05 and .10.