Math 1710 Class 5

Random Variable Operations, Reverse Conditioning
Dr. Back

Sep. 7, 2009
Announcements

Please write prominently on your HW which Th Recitation you are *attending*.

Please bring two pennies (or other coins) to class on Wed.
Mean of an RV

The RV $X$ with probability distribution

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_n$</td>
<td>$p_n$</td>
</tr>
</tbody>
</table>

Mean:

$$\mu = EX = \sum p_i x_i.$$
Mean of an RV

The RV \( X \) with probability distribution

<table>
<thead>
<tr>
<th>( X )</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( p_2 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_n )</td>
<td>( p_n )</td>
</tr>
</tbody>
</table>

Mean:

\[ \mu = EX = \sum p_i x_i. \]

Variance:

\[ \text{Var}(X) = \sum p_i(x_i - \mu)^2 \]

Standard Deviation:

\[ \sigma_X = \sqrt{\text{Var}(X)}. \]
### Two Fair Dice

<table>
<thead>
<tr>
<th>$T$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{36}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2}{36}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3}{36}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{4}{36}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{5}{36}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{6}{36}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{5}{36}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{4}{36}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{3}{36}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{2}{36}$</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>
Two Fair Dice

\[ T = U_1 + U_2 \]

<table>
<thead>
<tr>
<th>( U )</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{6} )</td>
</tr>
</tbody>
</table>
Two Fair Dice

<table>
<thead>
<tr>
<th>U</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{6} )</td>
</tr>
</tbody>
</table>

\[ \mu = EU = 3.5 \]

\[ Var(U) = \frac{1}{6} (1 - 3.5)^2 + \frac{1}{6} (2 - 3.5)^2 + \frac{1}{6} (3 - 3.5)^2 \]

\[ + \frac{1}{6} (4 - 3.5)^2 + \frac{1}{6} (5 - 3.5)^2 + \frac{1}{6} (6 - 3.5)^2 \]

\[ = 2.917. \]

and \( \sigma = \sqrt{2.917} = 1.708. \)
Three Important Operations

- Addition of RV’s as in \( T = U_1 + U_2 \).
- Adding a constant: \( Y = X + c \).
- Multiplying by a constant: \( Y = cX \).
Thinking about $cX$ and $X + c$

A simple die game:
1, 2, or 3 gives $0; 4 or 5 gives $5; 6 gives $50.$
A simple die game:
1, 2, or 3 gives $0; 4 or 5 gives $5; 6 gives $50.

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>5</td>
<td>1/3</td>
</tr>
<tr>
<td>50</td>
<td>1/6</td>
</tr>
</tbody>
</table>
Thinking about $cX$ and $X + c$

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>50</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

$$\mu = \frac{5}{3} + \frac{25}{3} = 10; \sigma = 18.03.$$
Thinking about $cX$ and $X + c$

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>50</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

$\mu = \frac{5}{3} + \frac{25}{3} = 10$; $\sigma = 18.03$.

Doubling the stakes: $Y = 2X$

<table>
<thead>
<tr>
<th>$Y$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>100</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>
Thinking about $cX$ and $X + c$

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>50</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

$\mu = \frac{5}{3} + \frac{25}{3} = 10; \sigma = 18.03.$

Doubling the stakes: $Y = 2X$

<table>
<thead>
<tr>
<th>$Y$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>100</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

$\mu = 20; \sigma = 36.06.$
Thinking about $cX$ and $X + c$

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>50</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

$$\mu = \frac{5}{3} + \frac{25}{3} = 10; \quad \sigma = 18.03.$$  

Including a Payment of 12$ to Play the Game: $W = X - 12$

<table>
<thead>
<tr>
<th>$W$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-12$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$-7$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$38$</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>
Thinking about $cX$ and $X + c$

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>50</td>
<td>(\frac{1}{6})</td>
</tr>
</tbody>
</table>

\[\mu = \frac{5}{3} + \frac{25}{3} = 10; \quad \sigma = 18.03.\]

Including a Payment of 12\$ to Play the Game: $W = X - 12$

<table>
<thead>
<tr>
<th>$W$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>−12</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>−7</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>38</td>
<td>(\frac{1}{6})</td>
</tr>
</tbody>
</table>

\[\mu = -2; \quad \sigma = 18.03.\]
Means are very intuitive!

\[ \mu = \sum p_i x_i \]

\[ \mu_X + \mu_Y = \mu_X + \mu_Y \]

\[ \mu_{X+c} = \mu_X + c \]

\[ \mu_{cX} = c \mu_X \]

(Handy: \( \sum p_i = 1. \))
Means are very intuitive!

\[ \mu = \sum p_i x_i \]

\[ \mu X + Y = \mu X + \mu Y \]

\[ \mu X + c = \mu X + c \]

\[ \mu cX = c \mu X \]

(Handy: \( \sum p_i = 1. \))
Means are very intuitive!

\[ \mu = \Sigma p_i x_i \]

\[ \mu_{X+Y} = \mu_X + \mu_Y \]

\[ \mu_{X+c} = \mu_X + c \]

\[ \mu_{cX} = c \mu_X \]

(Handy: \( \Sigma p_i = 1 \).)
Announcements

Last Time
Operations on RV’s
Behavior of $\mu, \sigma$ under operations.

Bernoulli and Binomial RV’s

Word Problems

Why the formulas for $\mu, \sigma$ of $X + Y$?

Hard Expectation Problem

Reverse Conditioning

Std. Dev. for $cX$ and $X + c$ are pretty natural.

$$\text{Var}(X) = \Sigma p_i (x_i - \mu)^2$$

$$\text{Var}(X + c) = \text{Var}(X).$$

$$\sigma_{X+c} = \sigma_X$$

$$\text{Var}(cX) = c^2 \text{Var}(X).$$

$$\sigma_{cX} = |c| \sigma_X. \text{ So } \sigma_X = \sigma_{-X}.$$
Std. Dev. for $cX$ and $X + c$ are pretty natural.

\[
\text{Var}(X) = \sum p_i (x_i - \mu)^2
\]

\[
\text{Var}(X + c) = \text{Var}(X).
\]

\[
\sigma_{X+c} = \sigma_X
\]

\[
\text{Var}(cX) = c^2 \text{Var}(X).
\]

\[
\sigma_{cX} = |c| \sigma_X. \text{ So } \sigma_X = \sigma_{-X}.
\]
Std. Dev. for $cX$ and $X + c$ are pretty natural.

$$\text{Var}(X) = \sum p_i(x_i - \mu)^2$$

$$\text{Var}(X + c) = \text{Var}(X).$$

$$\sigma_{X+c} = \sigma_X$$

$$\text{Var}(cX) = c^2\text{Var}(X).$$

$$\sigma_{cX} = |c|\sigma_X. \text{ So } \sigma_X = \sigma_{-X}.$$
Std. Dev. for $cX$ and $X + c$ are pretty natural.

$$\text{Var}(X) = \sum p_i(x_i - \mu)^2$$

$$\text{Var}(X + c) = \text{Var}(X).$$

$$\sigma_{X+c} = \sigma_X$$

$$\text{Var}(cX) = c^2 \text{Var}(X).$$

$$\sigma_{cX} = |c| \sigma_X. \textbf{ So } \sigma_X = \sigma_{-X}.$$
Variance of a Sum of *Independent* RV’s

There is no general formula for $\text{Var}(X + Y)$ in terms of $\text{Var}(X)$ and $\text{Var}(Y)$. 
Variance of a Sum of *Independent* RV’s

There is no general formula for $\text{Var}(X + Y)$ in terms of $\text{Var}(X)$ and $\text{Var}(Y)$.

But if $X$ and $Y$ are independent RV’s:

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$
Variance of a Sum of \textit{Independent} RV’s

There is no general formula for \( \text{Var}(X + Y) \) in terms of \( \text{Var}(X) \) and \( \text{Var}(Y) \).

But if \( X \) and \( Y \) are independent RV’s:

\[
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)
\]

Very important in Statistics!
Variance of a Sum of *Independent* RV’s

There is no general formula for $\text{Var}(X + Y)$ in terms of $\text{Var}(X)$ and $\text{Var}(Y)$.

But if $X$ and $Y$ are independent RV's:

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

Very important in Statistics!

Without independence, this often fails; e.g.

$$\text{Var}(X + X) = \text{Var}(2X) = 2^2\text{Var}(X) = 4\text{Var}(X)$$

$$\neq \text{Var}(X) + \text{Var}(X) = 2\text{Var}(X)$$
X is the number of heads in one toss of a coin, 
$p$ being the probability of a head.
X is the number of heads in *one* toss of a coin, $p$ being the probability of a head.

<table>
<thead>
<tr>
<th>X</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q = 1 - p$</td>
</tr>
<tr>
<td>1</td>
<td>$p$</td>
</tr>
</tbody>
</table>
X is the number of heads in *one* toss of a coin, $p$ being the probability of a head.

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q = 1 - p$</td>
</tr>
<tr>
<td>1</td>
<td>$p$</td>
</tr>
</tbody>
</table>

$\mu$, $\sigma$?
X is the number of heads in *one* toss of a coin, $p$ being the probability of a head.

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q = 1 - p$</td>
</tr>
<tr>
<td>1</td>
<td>$p$</td>
</tr>
</tbody>
</table>

$\mu = q(0) + p(1) = p$. 
X is the number of heads in one toss of a coin, $p$ being the probability of a head.

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q = 1 - p$</td>
</tr>
<tr>
<td>1</td>
<td>$p$</td>
</tr>
</tbody>
</table>

$\mu = q(0) + p(1) = p$.  
$\text{Var}(X)$?
X is the number of heads in *one* toss of a coin, $p$ being the probability of a head.

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q = 1 - p$</td>
</tr>
<tr>
<td>1</td>
<td>$p$</td>
</tr>
</tbody>
</table>

$\mu = q(0) + p(1) = p$.

$\text{Var}(X) = q(0 - p)^2 + p(1 - p)^2$. 
X is the number of heads in \emph{one} toss of a coin, 
\( p \) being the probability of a head.

<table>
<thead>
<tr>
<th>( X )</th>
<th>\text{probability}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( q = 1 - p )</td>
</tr>
<tr>
<td>1</td>
<td>( p )</td>
</tr>
</tbody>
</table>

\[ \mu = q(0) + p(1) = p. \]
\[ \text{Var}(X) = q(0 - p)^2 + p(1 - p)^2 \]
\[ = qp^2 + pq^2 = pq(p + q) = pq. \]
X is the number of heads in *one* toss of a coin, 
*p* being the probability of a head.

<table>
<thead>
<tr>
<th>X</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>q = 1 − p</td>
</tr>
<tr>
<td>1</td>
<td>p</td>
</tr>
</tbody>
</table>

\[ \mu = q(0) + p(1) = p. \]

\[ \text{Var}(X) = q(0 - p)^2 + p(1 - p)^2 = qp^2 + p(q)^2 = pq(p + q) = pq. \]

\[ \sigma = \sqrt{pq}. \]
Y is the number of heads in $n$ tosses of a coin, $p$ being the probability of a head.
Y is the number of heads in \( n \) tosses of a coin, 
\( p \) being the probability of a head.

\[ Y = X_1 + X_2 + \ldots + X_n \]

were \( X_i \) is a Bernoulli(p) RV describing the \( i \)’th toss.
Y a Binomial(n,p) RV

Y is the number of heads in \( n \) tosses of a coin, \( p \) being the probability of a head.

\[
Y = X_1 + X_2 + \ldots + X_n
\]

were \( X_i \) is a Bernoulli(p) RV describing the \( i \)'th toss. The \( X_i \) are independent copies of \( X \) with

\[
\mu_X = p, \quad \text{Var}(X) = pq.
\]
Y is the number of heads in $n$ tosses of a coin, $p$ being the probability of a head.

$$Y = X_1 + X_2 + \ldots X_n$$

were $X_i$ is a Bernoulli($p$) RV describing the $i$'th toss. So $\mu_Y = p + p + \ldots + p = np$. 
Y is the number of heads in $n$ tosses of a coin, $p$ being the probability of a head.

$$Y = X_1 + X_2 + \ldots X_n$$

were $X_i$ is a Bernoulli(p) RV describing the $i$'th toss.

So $\mu_Y = p + p + \ldots + p = np$.

And $\text{Var}(Y) = pq + pq + \ldots + pq = npq$. 
Y is the number of heads in $n$ tosses of a coin, $p$ being the probability of a head.

$$Y = X_1 + X_2 + \ldots X_n$$

where $X_i$ is a Bernoulli($p$) RV describing the $i$'th toss.

So $\mu_Y = p + p + \ldots + p = np$. 
And $\text{Var}(Y) = pq + pq + \ldots + pq = npq$. 
Making $\sigma_Y = \sqrt{npq}$. 
Joan tosses a coin 10,000 times and sees 5,300 heads.
Announcements

Last Time
Operations on RV's
Behavior of $\mu, \sigma$ under operations.
Bernoulli and Binomial RV's

Word Problems

Why the formulas for $\mu, \sigma$ of $X + Y$?

Hard Expectation Problem
Reverse Conditioning

---

Application

Joan tosses a coin 10,000 times and sees 5,300 heads. Note Binomial(10,000,.5) has $\mu = 5,000, \sigma = 50$. We'll see later that models based on Binomial RV's almost always fall within 3 std. dev. of the mean. 5300 is 6 std dev from the mean. Joan concludes her coin is probably not fair.
Announcements

Last Time
Operations on RV's
Behavior of \( \mu, \sigma \) under operations.
Bernoulli and Binomial RV's

Word Problems

Why the formulas for \( \mu, \sigma \) of \( X + Y \)?

Hard Expectation Problem
Reverse Conditioning

Application

Joan tosses a coin 10,000 times and sees 5,300 heads. Note Binomial(10,000,.5) has \( \mu = 5,000, \sigma = 50 \).
We’ll see later that models based on Binomial RV's almost always fall within 3 std. dev. of the mean.
Joan tosses a coin 10,000 times and sees 5,300 heads. Note Binomial(10,000,.5) has $\mu = 5,000, \sigma = 50$. We’ll see later that models based on Binomial RV’s almost always fall within 3 std. dev. of the mean. 5300 is 6 std dev from the mean.
Joan tosses a coin 10,000 times and sees 5,300 heads. Note Binomial(10,000,.5) has $\mu = 5,000$, $\sigma = 50$. We’ll see later that models based on Binomial RV’s almost always fall within 3 std. dev. of the mean. 5300 is 6 std dev from the mean. Joan concludes her coin is probably not fair.
Suppose a 2-step process:

**Mechanical Checkup:** $50 per hour, 90 min average, 15 min standard deviation.

**Appearance Prep:** $6 per hour, 60 min average, 5 min standard deviation.

Find $\mu$ and $\sigma$ for:

- Total Expense.
- Difference in Expense of Phases.
Let $M$ be an RV describing the time for the mechanical part.
Let $M$ be an RV describing the time for the mechanical part. Let $A$ be an RV describing the time for the appearance part.
Solution:

Let $M$ be an RV describing the time for the mechanical part. Let $A$ be an RV describing the time for the appearance part. Let $T$ be an RV describing the total expense.
Solution:

Let $M$ be an RV describing the time for the mechanical part. Let $A$ be an RV describing the time for the appearance part. Let $T$ be an RV describing the total expense. We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>
Solution:

Let \( M \) be an RV describing the time for the mechanical part. Let \( A \) be an RV describing the time for the appearance part. Let \( T \) be an RV describing the total expense. We’re given (in hours):

\[
\begin{array}{c|cc}
 & \mu & \sigma \\
\hline
M & 3 & 1/2 \\
A & 1 & 1/4 \\
\end{array}
\]

\[ T = 50M + 6A. \]
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>3</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>2</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$T = 50M + 6A$. 
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>( \mu )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>3</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( A )</td>
<td>2</td>
<td>( \frac{1}{12} )</td>
</tr>
</tbody>
</table>

\[ T = 50M + 6A. \]
\[ \mu_T = 50\mu_M + 6\mu_A. \]
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$T = 50M + 6A$.

$\mu_T = 50\mu_M + 6\mu_A$.

$\mu_T = 50\left(\frac{3}{2}\right) + 6(1) = 75 + 6 = 81$. 
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$T = 50M + 6A$.
Assuming independence of $M$ and $A$,
$\text{Var}(T) = 50^2 \text{Var}(M) + 6^2 \text{Var}(A)$. 
Solution Continued:

We’re given (in hours):

\[
\begin{array}{c|cc}
 & \mu & \sigma \\
 M & 3 & 1 \\
 A & 1 & 1/12 \\
\end{array}
\]

\[ T = 50M + 6A. \]

Assuming independence of \( M \) and \( A \),
\[ \text{Var}(T) = 50^2 \text{Var}(M) + 6^2 \text{Var}(A). \]

\[ \text{Var}(T) = 50^2 \left(\frac{1}{4}\right)^2 + 6^2 \left(\frac{1}{12}\right)^2. \]
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$T = 50M + 6A$.

Assuming independence of $M$ and $A$,

$\text{Var}(T) = 50^2 \text{Var}(M) + 6^2 \text{Var}(A)$.

$\text{Var}(T) = 50^2 \left(\frac{1}{4}\right)^2 + 6^2 \left(\frac{1}{12}\right)^2$.

$\text{Var}(T) = \left(\frac{25}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{626}{4} = 156.5$. 
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>( \mu )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( 1 )</td>
<td>( \frac{1}{12} )</td>
</tr>
</tbody>
</table>

\( T = 50M + 6A. \)

Assuming independence of \( M \) and \( A \),
\[
\text{Var}(T) = 50^2 \text{Var}(M) + 6^2 \text{Var}(A).
\]
\[
\text{Var}(T) = 50^2 \left(\frac{1}{4}\right)^2 + 6^2 \left(\frac{1}{12}\right)^2.
\]
\[
\text{Var}(T) = \left(\frac{25}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{626}{4} = 156.5.
\]
\[
\sigma_T = \sqrt{156.5} = 12.51.
\]
Solution Continued:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>( \mu )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( 1 )</td>
<td>( \frac{1}{12} )</td>
</tr>
</tbody>
</table>

\( T = 50M + 6A \).

Assuming independence of \( M \) and \( A \),

\[
\text{Var}(T) = 50^2 \text{Var}(M) + 6^2 \text{Var}(A).
\]

\[
\text{Var}(T) = 50^2 \left( \frac{1}{4} \right)^2 + 6^2 \left( \frac{1}{12} \right)^2.
\]

\[
\text{Var}(T) = \left( \frac{25}{2} \right)^2 + \left( \frac{1}{2} \right)^2 = \frac{626}{4} = 156.5.
\]

\[
\sigma_T = \sqrt{156.5} = 12.51.
\]

Is the independence of \( M \) and \( A \) natural to assume?
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A$. 
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>3</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>2</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A.$

$\mu_D = 50\mu_M - 6\mu_A.$

$D = 50 \times 3 - 6 \times 2.$

$D = 150 - 12.$

$D = 138.$
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A.$

$\mu_D = 50\mu_M - 6\mu_A.$

$\mu_D = 50\left(\frac{3}{2}\right) - 6(1) = 75 - 6 = 69.$
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>3</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A$.

$\text{Var}(D) = 50^2 \text{Var}(M) + (-6)^2 \text{Var}(A)$. 
The Difference $D$ in Expense of Phases:

We’re given (in hours):

\[
\begin{array}{ccc}
  M & \mu & \sigma \\
3 & 3 & \frac{1}{2} \\
2 & 1 & \frac{1}{4} \\
A & 1 & \frac{1}{12} \\
\end{array}
\]

\[D = 50M - 6A.\]

\[\text{Var}(D) = 50^2 \text{Var}(M) + (-6)^2 \text{Var}(A).\]

\[\text{Var}(D) = 50^2 \left(\frac{1}{4}\right)^2 + 6^2 \left(\frac{1}{12}\right)^2.\]
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A$.

Var($D$) = $50^2$Var($M$) + $(-6)^2$Var($A$).

Var($D$) = $50^2(\frac{1}{4})^2 + 6^2(\frac{1}{12})^2$.

Var($D$) = $(\frac{25}{2})^2 + (\frac{1}{2})^2 = \frac{626}{4} = 156.5$. 
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A.$

$\text{Var}(D) = 50^2 \text{Var}(M) + (-6)^2 \text{Var}(A).$

$\text{Var}(D) = 50^2 (\frac{1}{4})^2 + 6^2 (\frac{1}{12})^2.$

$\text{Var}(D) = (\frac{25}{2})^2 + (\frac{1}{2})^2 = \frac{625}{4} = 156.5.$

$\sigma_D = \sqrt{156.5} = 12.51.$
The Difference $D$ in Expense of Phases:

We’re given (in hours):

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$D = 50M - 6A$.

$\text{Var}(D) = 50^2 \text{Var}(M) + (-6)^2 \text{Var}(A)$.

$\text{Var}(D) = 50^2 \left(\frac{1}{4}\right)^2 + 6^2 \left(\frac{1}{12}\right)^2$.

$\text{Var}(D) = \left(\frac{25}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{626}{4} = 156.5$.

$\sigma_D = \sqrt{156.5} = 12.51$.

Same std. dev. for $D$ and $T$!
### Two Independent RV’s

<table>
<thead>
<tr>
<th>X</th>
<th>probability</th>
<th>Y</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$p_1$</td>
<td>$y_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$p_2$</td>
<td>$y_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_n$</td>
<td>$p_n$</td>
<td>$y_m$</td>
<td>$q_m$</td>
</tr>
</tbody>
</table>

Independence means $P\{X = x_1 \cap Y = y_1\} = p_1 q_1$.

In this case $X + Y$ would be $x_1 + y_1$. 

Reasoning, derivation, analysis...
Two Independent RV’s

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
<th>$Y$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$p_1$</td>
<td>$y_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$p_2$</td>
<td>$y_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_n$</td>
<td>$p_n$</td>
<td>$y_m$</td>
<td>$q_m$</td>
</tr>
</tbody>
</table>

Independence means $P\{X = x_1 \land Y = y_1\} = p_1 q_1$. 
Two Independent RV’s

<table>
<thead>
<tr>
<th>$X$</th>
<th>probability</th>
<th>$Y$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$p_1$</td>
<td>$y_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$p_2$</td>
<td>$y_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$x_n$</td>
<td>$p_n$</td>
<td>$y_m$</td>
<td>$q_m$</td>
</tr>
</tbody>
</table>

Independence means $P\{X = x_1 \& Y = y_1\} = p_1 q_1$. In this case $X + Y$ would be $x_1 + y_1$. 
Variance

So the probability model for $X + Y$ is
So the probability model for $X + Y$ is *roughly*

<table>
<thead>
<tr>
<th>$X + Y$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 + y_1$</td>
<td>$p_1 q_1$</td>
</tr>
<tr>
<td>$x_1 + y_2$</td>
<td>$p_1 q_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_n + y_m$</td>
<td>$p_n q_m$</td>
</tr>
</tbody>
</table>
Variance

So the probability model for $X + Y$ is *roughly*

<table>
<thead>
<tr>
<th>$X + Y$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 + y_1$</td>
<td>$p_1 q_1$</td>
</tr>
<tr>
<td>$x_1 + y_2$</td>
<td>$p_1 q_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_n + y_m$</td>
<td>$p_n q_m$</td>
</tr>
</tbody>
</table>

(Why *roughly*: Some value $c$ of $X + Y$ may show up several times in the above table and then more precisely $p_i q_j$ is $P\{X = x_i \& Y = y_j\}$. And $P(\{X + Y = c\})$ would be the sum of several entries. But computing $\mu, \sigma$ won’t be affected by not combining such rows.)
Variance

Assuming we know \( E(X+Y) = EX + EY \), then \( \text{Var}(X + Y) \)

\[
= \sum_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
= \sum_{i,j} p_i q_j (x_i - \mu_X)^2 + \sum_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
\quad + 2 \sum_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
= (\sum_j q_j) \text{Var}(X) + (\sum_i p_i) \text{Var}(Y) \\
\quad + 2 (\sum_i p_i (x_i - \mu_X)) (\sum_j q_j (y_j - \mu_Y)) \\
= 1 \cdot \text{Var}(X) + 1 \cdot \text{Var}(Y) + 2 \cdot 0 \cdot 0 \\
= \text{Var}(X) + \text{Var}(Y).
\]
Variance

Assuming we know \( E(X+Y) = EX + EY \), then \( Var(X + Y) \)

\[
= \sum_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
= \sum_{i,j} p_i q_j (x_i - \mu_X)^2 + \sum_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
+ 2 \sum_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
= (\sum_j q_j) \Var(X) + (\sum_i p_i) \Var(Y) \\
+ 2 (\sum_i p_i (x_i - \mu_X)) (\sum_j q_j (y_j - \mu_Y)) \\
= 1 \cdot \Var(X) + 1 \cdot \Var(Y) + 2 \cdot 0 \cdot 0 \\
= \Var(X) + \Var(Y).
\]

where we used \((a + b)^2 = a^2 + 2ab + b^2\),
Variance

Assuming we know \( E(X+Y)=EX+EY \), then \( Var(X+Y) \)

\[
\begin{align*}
= & \sum_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
= & \sum_{i,j} p_i q_j (x_i - \mu_X)^2 + \sum_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
& + 2 \sum_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
= & (\sum q_j) Var(X) + (\sum p_i) Var(Y) \\
& + 2 (\sum p_i (x_i - \mu_X)) (\sum q_j (y_j - \mu_Y)) \\
= & 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0 \\
= & Var(X) + Var(Y)
\end{align*}
\]

\[\sum p_i = 1,\]
Variance

Assuming we know $E(X+Y)=EX+EY$, then $Var(X + Y)$

$$= \Sigma_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2$$
$$= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2$$
$$+ 2 \Sigma_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y)$$
$$= (\Sigma q_j) Var(X) + (\Sigma p_i) Var(Y)$$
$$+ 2 (\Sigma p_i (x_i - \mu_X)) (\Sigma q_j (y_j - \mu_Y))$$
$$= 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0$$
$$= Var(X) + Var(Y).$$

and $\Sigma p_i (x_i - \mu_X) = 0.$
Proof of Expectation of Mean Formula

\[ E(X + Y) \]

\[ = \sum_{i,j} (x_i + y_j)P(X = x_i \text{ and } Y = y_j) \]
\[ = \sum_{i,j} x_i P(X = x_i \mid Y = y_j)P(Y = y_j) \]
\[ + \sum_{i,j} y_j P(Y = y_j \mid X = x_i)P(X = x_i) \]
\[ = \sum_{i} x_i (\sum_{j} P(Y = y_j \mid X = x_i)P(Y = y_j)) \]
\[ + \sum_{j} y_j (\sum_{i} P(Y = y_j \mid X = x_i)P(X = x_i)) \]
\[ = \sum_{i} x_i P(X = x_i) \]
\[ + \sum_{j} y_j P(Y = y_j) \]
\[ = E(X) + E(Y). \]

Unsurprising, but not obvious looking.
Proof of Expectation of Mean Formula

\[ E(X + Y) \]

\[ = \sum_{i,j} (x_i + y_j)P(X = x_i \text{ and } Y = y_j) \]
\[ = \sum_{i,j} x_i P(X = x_i | Y = y_j)P(Y = y_j) \]
\[ + \sum_{i,j} y_j P(Y = y_j | X = x_i)P(X = x_i) \]
\[ = \sum_i x_i \left( \sum_j P(Y = y_j | X = x_i)P(Y = y_j) \right) \]
\[ + \sum_j y_j \left( \sum_i P(Y = y_j | X = x_i)P(X = x_i) \right) \]
\[ = \sum_i x_i P(X = x_i) \]
\[ + \sum_j y_j P(Y = y_j) \]
\[ = E(X) + E(Y). \]

Unsurprising, but not obvious looking.

Did not require \( X \) and \( Y \) to be independent!
7 Balls Randomly into 5 boxes

What is the probability that the first box remains empty?

What is the expected number of empty boxes?
7 Balls Randomly into 5 boxes

What is the probability that the first box remains empty?

What is the expected number of empty boxes?

The 1st question is a great hint for easily doing the 2nd!
Suppose

- 1 in 800 pregnant women affected.
- $\frac{8}{9}$ of affected women identified by the test
- $\frac{1}{4}$ of unaffected women show up as false positives.
Suppose

- 1 in 800 pregnant women affected.
- \( \frac{8}{9} \) of affected women identified by the test
- \( \frac{1}{4} \) of unaffected women show up as false positives.

P(Down syndrome if test is positive)?
D: foetus affected by Downs.
Ps: positive test result reports Downs.
Ng: not Ps. (Negative)
With Probabilities

$$P(D) = \frac{1}{800}$$

1/800

D

not D

799/800
With the Conditional Probabilities

\[ P(Ps|D) = \frac{8}{9}, \quad P(Ps|\text{not } D) = \frac{1}{4} \]
With the Conditional Probabilities

\[
\begin{array}{c}
1/800 \\
799/800 \\
1/9 \\
1/4 \\
8/9 \\
3/4
\end{array}
\]

\[
\begin{array}{c}
Ps \\
Ng
\end{array}
\]

\[
\begin{array}{c}
D \\
\text{not D}
\end{array}
\]
With the Conditional Probabilities

Math 1710
Class 5
V3

Announcements

Last Time
Operations on RV's
Behavior of $\mu, \sigma$ under operations.
Bernoulli and Binomial RV's
Word Problems
Why the formulas for $\mu, \sigma$ of $X + Y$?

Hard Expectation Problem
Reverse Conditioning

\[
\begin{align*}
1/800 & \quad \text{D} \\
799/800 & \quad \text{not D}
\end{align*}
\]

\[
\begin{align*}
1/900 & \quad Ps \\
8/9 & \\
1/9 & \quad Ng \\
1/4 & \quad Ps \\
3/4 & \quad Ng \\
1/7200 & \\
799/3200 & \\
2397/3200 & \end{align*}
\]
With the Conditional Probabilities

\[
P(D|Ps) = \frac{P(D \text{ and } Ps)}{P(Ps)} = \frac{1}{900} + \frac{799}{3200} \approx 0.0044.
\]
With the Conditional Probabilities

\[ P(D|Ps) = \frac{P(D \text{ and } Ps)}{P(Ps)} = \frac{\frac{1}{900}}{\frac{1}{900} + \frac{799}{3200}} \sim .0044. \]

Not very helpful!