From Math 2220 Class 39

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Integration of a conservative vector field cartoon.

\[ \int \nabla F \cdot ds = F(\text{end}) - F(\text{start}) \]
Green’s Theorem cartoon.

\[ \iint \nabla \cdot \mathbf{F} \, dA = 
\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} \]
Stokes and Gauss

Why Green’s and Gauss’

Conservative Vector Fields

Systematic Method of Finding a Potential

Integral Theorem

Problems

Surface of Revolution Case

Graph Case

Surface Integrals

Stokes’ Theorem cartoon.

\[
\iint_S (\nabla \times \mathbf{v}) \cdot \hat{n} \, dS = \int_C \mathbf{v} \cdot d\mathbf{s}
\]
Both sides of Stokes involve integrals whose signs depend on the orientation, so to have a chance at being true, there needs to be some compatibility between the choices.

The rule is that, from the “positive” side of the surface, (i.e. the side chosen by the orientation), the positive direction of the curve has the inside of the surface to the left.

As with all orientations, this can be expressed in terms of the sign of some determinant. (Or in many cases in terms of the sign of some combination of dot and cross products.)
Problem: Let $S$ be the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Orient the hemisphere with an upward unit normal. Let $\vec{F}(x, y, z) = (y, -x, e^{z^2})$. Calculate the value of the surface integral

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS.$$
Stokes and Gauss

Gauss' Theorem field cartoon.

\[ \iiint \text{div}(\mathbf{F}) \, dV = \oiint (\mathbf{F} \cdot \mathbf{n}) \, ds \]
Stokes and Gauss

The surface integral side of Gauss depends on the orientation, so there needs to be a choice making the theorem true.

The rule is that the normal to the surface should point outward from the inside of the region.

(For the 2d analogue of Gauss (really an application of Green’s) \[
\int_C \vec{F} \cdot \hat{n} = \iint_{\text{inside}} (P_x + Q_y) \, dx \, dy
\]
we also use an outward normal, where here \( C \) must of course be a closed curve.)
Problem: Let \( \mathcal{W} \) be the solid cylinder \( x^2 + y^2 \leq 3 \) with \( 1 \leq z \leq 5 \). Let \( \vec{F}(x, y, z) = (x, y, z) \). Find the value of the surface integral

\[
\iint_{\partial \mathcal{W}} \vec{F} \cdot \hat{n} \, dS.
\]
Green’s theorem says that for simple closed (piecewise smooth) curve $C$ whose inside is a region $\mathcal{R}$, we have

$$\int_C P(x, y) \, dx + Q(x, y) \, dy = \iiint_\mathcal{R} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy$$

as long as the vector field $\vec{F}(x, y) = (P(x, y), Q(x, y))$ is $C^1$ on the set $\mathcal{R}$ and $C$ is given its usual “inside to the left” orientation.
Why Green’s and Gauss’

For a $y$-simple region

$$
\int\int_{\mathcal{R}} -P_y \, dy \, dx = \int_{\mathcal{C}=\partial \mathcal{R}} P \, dx.
$$

is fairly easily justified.
For an $x$-simple region

\[ \iint_{\mathcal{R}} Q_x \, dx \, dy = \int_{C=\partial \mathcal{R}} Q \, dy. \]
Why Green’s and Gauss’

So for a region that is both \( y \)-simple and \( x \)-simple we have Green:

\[
\int_{\partial \mathcal{R}} P(x, y) \, dx + Q(x, y) \, dy = \iint_{\mathcal{R}} Q_y - P_x \, dx \, dy.
\]
Intuitively, why the different signs? $+Q_x$ yet $-P_y$. And why this combination of $Q_x - P_y$?
Why Green’s and Gauss’

Intuitively, why the different signs? \( +Q_x \) yet \(-P_y\).

And why this combination of \( Q_x - P_y \)?

Think about the line integral around a small rectangle with sides \( \Delta x \) and \( \Delta y \).

If you assume (or justify) that evaluating the vector field in the middle of each edge gives a good approximation in the line integral, then \( Q_x - P_y \) emerges quite naturally.
If you can cut a region into two pieces where we know Green’s holds on each piece (e.g. a ring shaped region), then Green also holds for the entire region. (Because the line integrals over the “cuts” show up twice with opposite signs (think “inside to the left”) and cancel.)
So in the end, Green’s theorem holds for regions whose boundaries include several closed curves (multiply-connected regions) as long as we orient each boundary curve according to the inside to the left rule.
For what might be called a $z$-simple region $\mathcal{W}$,

$$g(x, y) \leq z \leq h(x, y) \text{ with } (x, y) \in D \subset \mathbb{R}^2$$

a very similar argument shows that

$$\iint_{\partial \mathcal{W}} (0, 0, R) \cdot \hat{n} \ dS = \iiint_{\mathcal{W}} R_z \ dz \ dx \ dy$$

as Gauss says about the $z$-part of any $C^1$ vector field.
As with Green, the other components can be handled similarly for appropriately shaped elementary regions. If you can cut a region into two pieces where we know Gauss holds on each piece (e.g. a doughnut shaped region), then Gauss also holds for the entire region.
While the proof of Stokes’ shares many elements with the proofs of Green’s and Gauss’, the best “classical style” proof of Stokes’ involves using a parametrization to reduce Stokes’ to Green’s in the parameterizing (i.e. $uv$ plane)
The differential forms point of view introduced in section 8.5 makes all these theorems one theorem, usually called Stokes’, and the proof becomes a combination of more advanced linear algebra constructions (differential forms) together with the one variable Fundamental Theorem of Calculus.

Time permitting, we’ll talk a bit about this on Monday.
A vector field $\vec{F}(x, y, z)$ which can be written as

$$\vec{F} = \nabla f$$

is called conservative. We already know

$$\int_C \vec{F} \cdot d\vec{s} = 0$$

for any *closed* curve.
The origin of the term is physics (I think) where in the case of $\vec{F}$ a force, it does no work (and so saps/adds no energy) as a particle traverses the closed curve.
Conservative Vector Fields

A vector field \( \vec{F}(x, y, z) \) which can be written as

\[
\vec{F} = \nabla f
\]

is called conservative. We already know

\[
\int_C \vec{F} \cdot d\vec{s} = 0
\]

for any closed curve.

In physics, the convention is to choose \( \phi \) so that

\[
\vec{F} = -\nabla \phi
\]

and \( \phi \) is referred to as the potential energy.
Conservation of energy (in e.g. mechanics) becomes a theorem in multivariable calculus combining the definition of a flow line with the computation of a line integral.

Newton’s 2nd law ($\vec{F} = m\vec{a}$ and other versions) is also key . . . .
The concept of **voltage** arises here too; it is just a potential energy per unit charge.
Conservative Vector Fields

The theorem (vector identity) \( \text{curl}(\nabla f) = 0 \) means the

\[ \text{curl}(\vec{F}) = 0 \]

is a necessary condition for the existence of a function \( f \) satisfying

\[ \nabla f = \vec{F}. \]
Conservative Vector Fields

It turns out that for vector fields defined on e.g. all of $\mathbb{R}^2$ or $\mathbb{R}^3$, the converse of the theorem $\text{curl}(\nabla f) = 0$ is true. (For $\mathbb{R}^2$, we’re thinking of the scalar curl.) In other words, in such a case, if $\text{curl}(\vec{F}) = 0$, (for a $C^1$ vector field), there is guaranteed to be a function $f(x, y, z)$ such that

$$\nabla f = \vec{F}.$$
While this hold for vector fields with domains $R^2$, $R^3$, or more generally any “simply connected” region, the example $d\theta$ below shows this converse does not hold in general.
Conservative Vector Fields

Time permitting, we’ll talk about simple connectivity next week.
Finding a potential by inspection is fine when you can, but it is not systematic. I often ask on a final exams for this.
Problem: Use a systematic method to find a function $f(x, y, z)$ for which

$$\nabla f = (2xy, x^2 + z^2, 2yz + 1).$$
**Systematic Method of Finding a Potential**

**Problem:** Use a systematic method to find a function \( f(x, y, z) \) for which

\[
\nabla f = (y^2 ze^{xyz} + \frac{1}{y}, (1 + xyz)e^{xyz} - \frac{x}{y^2}, -\cos^2(xyz))e^z + xy^2 e^{xyz} - e^z \sin^2(xyz)).
\]
Systematic Method of Finding a Potential

**Problem:** Use a systematic method to find a function $f(x, y, z)$ for which

$$\nabla f = (2xz, 2y, x^2 + 6e^z).$$
\[ \int_C -y \, dx + x \, dy \]

for \( C \) the unit circle traversed counterclockwise.
\[ \int_C x \, dx + y \, dy \]

for \( C \) the unit circle traversed counterclockwise.
\[ \int_C \frac{-y \, dx + x \, dy}{x^2 + y^2} \]
is still nonzero for \( C \) a circle of radius \( R \) centered at the origin traversed counterclockwise.

This is remarkable since

\[ \nabla \tan^{-1} \frac{y}{x} = \frac{1}{x^2 + y^2} (-y, x) \].
Why does this not contradict \( \int_C \nabla f \cdot d\vec{s} = 0 \) for a closed curve (i.e. start=end) \( C \)?
Problem: Let

\[ \vec{F}(x, y, z) = (y^2 + z^2, x^2 + z^2, x^2). \]

Find

\[ \int_C \vec{F} \cdot d\vec{s} \]

where \( C \) is the boundary of the plane \( x + 2y + 2z = 2 \) intersected with the first octant, oriented counterclockwise from above.
Problem: Find the flux of the vector field

\[ \vec{F}(x, y, z) = (xy, yz, xz) \]

through the boundary of the unit cube

\[ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1 \]

where the boundary of the cube has its usual outward normal.
Problem: Find \[ \int \int_S \vec{F} \cdot \hat{n} \, dS \]

for \( \vec{F}(x, y, z) = (0, yz, z^2) \) and \( S \) the portion of the cylinder \( y^2 + z^2 = 1 \) with \( 0 \leq x \leq 1, \ z \geq 0 \), and the positive orientation chosen to be a radial outward (from the axis of the cylinder) normal.
\[ \frac{1}{2} \int_C \mathbf{\times} \, dy - y \, dx \quad \text{for } C \text{ the boundary of the ellipse} \]

\[ \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \]

oriented counterclockwise.
Integral Theorem Problems

Let

\[ \vec{F} = \frac{1}{x^2 + y^2} (-y, x). \]

If \( C_1 \) and \( C_2 \) are two simple closed curves enclosing the origin (and oriented with the usual inside to the left), can you say whether one of \( \int_{C_1} \vec{F} \cdot d\vec{s} \) and \( \int_{C_2} \vec{F} \cdot d\vec{s} \) is bigger than the other?
Surface of Revolution Case

This is not worth memorizing!

If one rotates about the $z$-axis the path (curve) $z = f(x)$ in the $xz$-plane for $0 \leq a \leq x \leq b$, one obtains a surface of revolution with a parametrization

$$\Phi(u, v) = (u \cos v, u \sin v, f(u))$$

and $dS =$?
Surface of Revolution Case

\[ dS = u \sqrt{1 + (f'(u))^2} \, du \, dv. \]
Graph Case

This is not worth memorizing!
For the graph parametrization of $z = f(x, y)$,

$$
\Phi(u, v) = (u, v, f(u, v))
$$

and $dS =$?
Graph Case

\[ dS = \sqrt{1 + f_u^2 + f_v^2} \, du \, dv. \]
For such a graph, the normal to the surface at a point 
\( (x, y, f(x, y)) \) (this is the level set \( z - f(x, y) = 0 \)) is
\[
(-f_x, -f_y, 1)
\]

so we can see that
\[
\cos \gamma = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}
\]
determines the angle \( \gamma \) of the normal with the \( z \)-axis. And so at the point \( (u, v, f(u, v)) \) on a graph,
\[
dS = \frac{1}{\cos \gamma} \, du \, dv.
\]
(Note that \( du \, dv \) is essentially the same as \( dx \, dy \) here.)
Surface Integrals

Picture of $\vec{T}_u, \vec{T}_v$ for a Lat/Long Param. of the Sphere.
Surface Integrals

Basic Parametrization Picture
Surface Integrals

Parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$

Tangents $T_u = (x_u, y_u, z_u)$ $T_v = (x_v, y_v, z_v)$

Area Element $dS = \| \mathbf{T}_u \times \mathbf{T}_v \| \, du \, dv$

Normal $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$

Unit normal $\mathbf{\hat{n}} = \pm \frac{\mathbf{T}_u \times \mathbf{T}_v}{\| \mathbf{T}_u \times \mathbf{T}_v \|}$

(Choosing the $\pm$ sign corresponds to an orientation of the surface.)
Surface Integrals

Two Kinds of Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$:

$$\iint_{S} f(x, y, z) \, dS$$

Surface Integral of a vector field $\vec{F}(x, y, z)$:

$$\iint_{S} \vec{F}(x, y, z) \cdot \hat{n} \, dS.$$
Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$ calculated by

$$\int \int_S f(x, y, z) \, dS = \int \int_D f(\Phi(u, v)) \| \vec{T}_u \times \vec{T}_v \| \, du \, dv$$

where $D$ is the domain of the parametrization $\Phi$.

Surface Integral of a vector field $\vec{F}(x, y, z)$ calculated by

$$\int \int_S \vec{F}(x, y, z) \cdot \hat{n} \, dS$$

$$= \pm \int \int_D \vec{F}(\Phi(u, v)) \cdot \left( \frac{\vec{T}_u \times \vec{T}_v}{\| \vec{T}_u \times \vec{T}_v \|} \right) \| \vec{T}_u \times \vec{T}_v \| \, du \, dv$$

where $D$ is the domain of the parametrization $\Phi$. 
Surface Integrals
Surface Integrals

The preceding picture can be used to argue that if $\vec{F}(x, y, z)$ is the velocity vector field, e.g. of a fluid of density $\rho(x, y, z)$, then the surface integral

$$\int \int_S \rho \vec{F} \cdot \hat{n} \, dS$$

(with associated Riemann Sum

$$\sum \rho(x_i^*, y_j^*, z_k^*) \vec{F}(x_i^*, y_j^*, z_k^*) \cdot \hat{n}(x_i^*, y_j^*, z_k^*) \Delta S_{ijk}$$)

represents the rate at which material (e.g. grams per second) crosses the surface.
Surface Integrals

From this point of view the orientation of a surface simple tells us which side is accumulating mass, in the case where the value of the integral is positive.
There’s an analogous 2d Riemann sum and interp of
\[
\int_C \vec{F} \cdot \hat{n} \, ds.
\]
Surface Integrals

At level of linear approx, parallelogram on right consists of fluid that crossed $C$ between times $t$ and $t+\Delta t$. 

$\Delta l \cdot \hat{\Delta t} = (\nabla \cdot \mathbf{V}) \Delta t$

Parallelogram to left has height $1/2 \cos \theta \Delta t$. 

Crossing $C$ at time $t$
Problem: Calculate

$$\iint_S \vec{F}(x, y, z) \cdot \hat{n} \, dS$$

for the vector field $\vec{F}(x, y, z) = (x, y, z)$ and $S$ the part of the paraboloid $z = 1 - x^2 - y^2$ above the $xy$-plane. Choose the positive orientation of the paraboloid to be the one with normal pointing downward.
Problem: Calculate the surface area of the above paraboloid.
Paraboloid $z = x^2 + 4y^2$

The graph $z = F(x, y)$ can always be parameterized by

$$\Phi(u, v) = \langle u, v, F(u, v) \rangle.$$
Paraboloid \( z = x^2 + 4y^2 \)

The graph \( z = F(x, y) \) can always be parameterized by

\[
\Phi(u, v) = \langle u, v, F(u, v) \rangle.
\]

Parameters \( u \) and \( v \) just different names for \( x \) and \( y \) resp.
The graph $z = F(x, y)$ can always be parameterized by
\[ \Phi(u, v) = \langle u, v, F(u, v) \rangle. \]

Use this idea if you can’t think of something better.
Paraboloid $z = x^2 + 4y^2$

The graph $z = F(x, y)$ can always be parameterized by

$$
\Phi(u, v) = \langle u, v, F(u, v) \rangle .
$$
Paraboloid \[ z = x^2 + 4y^2 \]

The graph \( z = F(x, y) \) can always be parameterized by
\[
\Phi(u, v) = \langle u, v, F(u, v) \rangle .
\]

Note the curves where \( u \) and \( v \) are constant are visible in the wireframe.
Paraboloid \( z = x^2 + 4y^2 \)

A trigonometric parametrization will often be better if you have to calculate a surface integral.
Paraboloid $z = x^2 + 4y^2$

A trigonometric parametrization will often be better if you have to calculate a surface integral.

$$\Phi(u, v) = \langle 2u \cos v, u \sin v, 4u^2 \rangle.$$
A trigonometric parametrization will often be better if you have to calculate a surface integral.

\[ \Phi(u, v) = \langle 2u \cos v, u \sin v, 4u^2 \rangle. \]
Paraboloid $z = x^2 + 4y^2$

A trigonometric parametrization will often be better if you have to calculate a surface integral.

$$\Phi(u, v) = \langle 2u \cos v, u \sin v, 4u^2 \rangle.$$  

Algebraically, we are rescaling the algebra behind polar coordinates where

$$x = r \cos \theta$$
$$y = r \sin \theta$$

leads to $r^2 = x^2 + y^2$. 
Paraboloid $z = x^2 + 4y^2$

A trigonometric parametrization will often be better if you have to calculate a surface integral.

$$\Phi(u, v) = \langle 2u \cos v, u \sin v, 4u^2 \rangle .$$

Here we want $x^2 + 4y^2$ to be simple. So

$$x = 2r \cos \theta$$
$$y = r \sin \theta$$

will do better.
Paraboloid \( z = x^2 + 4y^2 \)

A trigonometric parametrization will often be better if you have to calculate a surface integral.

\[ \Phi(u, v) = \langle 2u \cos v, u \sin v, 4u^2 \rangle. \]

Here we want \( x^2 + 4y^2 \) to be simple. So

\[ x = 2r \cos \theta \]
\[ y = r \sin \theta \]

will do better.

Plug \( x \) and \( y \) into \( z = x^2 + 4y^2 \) to get the \( z \)-component.
Parabolic Cylinder $z = x^2$

Graph parametrizations are often optimal for parabolic cylinders.
Parabolic Cylinder $z = x^2$

$\Phi(u, v) = \langle u, v, u^2 \rangle$
Parabolic Cylinder $z = x^2$

$\Phi(u, v) = \langle u, v, u^2 \rangle$
Parabolic Cylinder $z = x^2$

$\Phi(u, v) = \langle u, v, u^2 \rangle$

One of the parameters ($v$) is giving us the “extrusion” direction. The parameter $u$ is just being used to describe the curve $z = x^2$ in the zx plane.
Elliptic Cylinder $x^2 + 2z^2 = 6$

The trigonometric trick is often good for elliptic cylinders
Elliptic Cylinder $x^2 + 2z^2 = 6$

$$\Phi(u, v) = \langle \sqrt{3} \cdot \sqrt{2} \cos v, u, \sqrt{3} \sin v \rangle = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle$$
Elliptic Cylinder $x^2 + 2z^2 = 6$

\[ \Phi(u, v) = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle \]
Elliptic Cylinder $x^2 + 2z^2 = 6$

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Elliptic Cylinder $x^2 + 2z^2 = 6$

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Elliptic Cylinder $x^2 + 2z^2 = 6$

$$\Phi(u, v) = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle$$

What happened here is we started with the polar coordinate idea

$$x = r \cos \theta$$
$$z = r \sin \theta$$

but noted that the algebra wasn’t right for $x^2 + 2z^2$ so shifted to

$$x = \sqrt{2} r \cos \theta$$
$$z = r \sin \theta$$
Elliptic Cylinder $x^2 + 2z^2 = 6$

$\Phi(u, v) = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle$

$x = \sqrt{2}r \cos \theta$
$z = r \sin \theta$

makes the left hand side work out to $2r^2$ which will be 6 when $r = \sqrt{3}$. 
Ellipsoid \[ x^2 + 2y^2 + 3z^2 = 4 \]

A similar trick occurs for using spherical coordinate ideas in parameterizing ellipsoids.
Ellipsoid \( x^2 + 2y^2 + 3z^2 = 4 \)

A similar trick occurs for using spherical coordinate ideas in parameterizing ellipsoids.

\[
\Phi(u, v) = \langle 2 \sin u \cos v, \sqrt{2} \sin u \sin v, \sqrt{\frac{4}{3}} \cos u \rangle
\]
Ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

$$\Phi(u, v) = \langle 2 \sin u \cos v, \sqrt{2} \sin u \sin v, \sqrt{\frac{4}{3}} \cos u \rangle$$
You may have run into the hyperbolic functions

\[
\cosh x = \frac{e^x + e^{-x}}{2}
\]

\[
\sinh x = \frac{e^x - e^{-x}}{2}
\]
You may have run into the hyperbolic functions

\[
\begin{align*}
cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2}
\end{align*}
\]

Just as \(\cos^2 \theta + \sin^2 \theta = 1\) helps with ellipses, the hyperbolic version \(\cosh^2 \theta - \sinh^2 \theta = 1\) leads to the nicest hyperbola parameterizations.
Hyperbolic Cylinder $x^2 - z^2 = -4$

Just as $\cos^2 \theta + \sin^2 \theta = 1$ helps with ellipses, the hyperbolic version $\cosh^2 \theta - \sinh^2 \theta = 1$ leads to the nicest hyperbola parameterizations.

$$\Phi(u, v) = \langle 2 \sinh v, u, 2 \cosh v \rangle$$
Hyperbolic Cylinder $x^2 - z^2 = -4$

\[ \Phi(u, v) = \langle 2 \sinh v, u, 2 \cosh v \rangle \]
The hyperbolic trick also works with saddles
Saddle \( z = x^2 - y^2 \)

\[
\Phi(u, v) = \langle u \cosh v, u \sinh v, u^2 \rangle
\]
Saddle $z = x^2 - y^2$

$\Phi(u, v) = < u \cosh v, u \sinh v, u^2 >$

Graph Case
The spherical coordinate idea for ellipsoids with $\sin \phi$ replaced by $\cosh u$ works well here.
Hyperboloid of 1 Sheet $x^2 + y^2 - z^2 = 1$

\[
\Phi(u, v) = \left< \cosh u \cos v, \cosh u \sin v, \sinh u \right>
\]
Hyperboloid of 1 Sheet $x^2 + y^2 - z^2 = 1$
Hyperboloid of 2-Sheets $x^2 + y^2 - z^2 = -1$

\[ \Phi(u, v) = \langle \sinh u \cos v, \sinh u \sin v, \cosh u \rangle \]
Hyperboloid of 2-Sheets \( x^2 + y^2 - z^2 = -1 \)
Top Part of Cone $z^2 = x^2 + y^2$

So $z = \sqrt{x^2 + y^2}$. 
Top Part of Cone $z^2 = x^2 + y^2$

So $z = \sqrt{x^2 + y^2}$.

The polar coordinate idea leads to

$$\Phi(u, v) = \langle u \cos v, u \sin v, u \rangle$$
Top Part of Cone $z^2 = x^2 + y^2$

So $z = \sqrt{x^2 + y^2}$.

The polar coordinate idea leads to

$$\Phi(u, v) = \langle u \cos v, u \sin v, u \rangle$$
Mercator Parametrization of the Sphere

For $0 \leq v \leq \infty$, $0 \leq u \leq 2\pi$

$\Phi(u, v) = (sech(v) \cos u, sech(v) \sin u, \tanh(v))$. 

(Note $tanh^2(v) + sech^2(v) = 1$)