Parametrization

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Paraboloid $z = x^2 + 4y^2$

The graph $z = F(x, y)$ can always be parameterized by

$$\Phi(u, v) = \langle u, v, F(u, v) \rangle.$$
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$$\Phi(u, v) = \langle u, v, F(u, v) \rangle .$$

Parameters $u$ and $v$ just different names for $x$ and $y$ resp.
Paraboloid $z = x^2 + 4y^2$

The graph $z = F(x, y)$ can always be parameterized by

$$\Phi(u, v) = \langle u, v, F(u, v) \rangle.$$

Use this idea if you can’t think of something better.
Paraboloid $z = x^2 + 4y^2$

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Paraboloid \( z = x^2 + 4y^2 \)

The graph \( z = F(x, y) \) can always be parameterized by

\[
\Phi(u, v) = \langle u, v, F(u, v) \rangle.
\]

Note the curves where \( u \) and \( v \) are constant are visible in the wireframe.
A trigonometric parametrization will often be better if you have to calculate a surface integral.
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Algebraically, we are rescaling the algebra behind polar coordinates where

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]

leads to \( r^2 = x^2 + y^2 \).
A trigonometric parametrization will often be better if you have to calculate a surface integral.

\[ \Phi(u, v) = \langle 2u \cos v, u \sin v, 4u^2 \rangle. \]

Here we want \( x^2 + 4y^2 \) to be simple. So

\[ x = 2r \cos \theta \]
\[ y = r \sin \theta \]

will do better.
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\[ y = r \sin \theta \]

will do better.

Plug \( x \) and \( y \) into \( z = x^2 + 4y^2 \) to get the z-component.
Graph parametrizations are often optimal for parabolic cylinders.
Parabolic Cylinder $z = x^2$

$\Phi(u, v) = \langle u, v, u^2 \rangle$
Parabolic Cylinder $z = x^2$

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Parabolic Cylinder $z = x^2$

\[ \Phi(u, v) = \langle u, v, u^2 \rangle \]

One of the parameters ($v$) is giving us the “extrusion” direction. The parameter $u$ is just being used to describe the curve $z = x^2$ in the zx plane.
Elliptic Cylinder $x^2 + 2z^2 = 6$

The trigonometric trick is often good for elliptic cylinders
Elliptic Cylinder $x^2 + 2z^2 = 6$

$$\Phi(u, v) = \langle \sqrt{3} \cdot \sqrt{2} \cos v, u, \sqrt{3} \sin v \rangle = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle$$
Elliptic Cylinder \( x^2 + 2z^2 = 6 \)

\[ \Phi(u, v) = < \sqrt{6} \cos v, u, \sqrt{3} \sin v > \]
Elliptic Cylinder $x^2 + 2z^2 = 6$

$$\Phi(u, v) = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle$$
Elliptic Cylinder $x^2 + 2z^2 = 6$

$\Phi(u, \nu) = \langle \sqrt{6} \cos \nu, u, \sqrt{3} \sin \nu \rangle$
Elliptic Cylinder $x^2 + 2z^2 = 6$

$$\Phi(u, v) = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle$$

What happened here is we started with the polar coordinate idea

\[
\begin{align*}
x & = r \cos \theta \\
z & = r \sin \theta
\end{align*}
\]

but noted that the algebra wasn’t right for $x^2 + 2z^2$ so shifted to

\[
\begin{align*}
x & = \sqrt{2} r \cos \theta \\
z & = r \sin \theta
\end{align*}
\]
Elliptic Cylinder $x^2 + 2z^2 = 6$

\[ \Phi(u, v) = \langle \sqrt{6} \cos v, u, \sqrt{3} \sin v \rangle \]

\[ x = \sqrt{2} r \cos \theta \]
\[ z = r \sin \theta \]

makes the left hand side work out to $2r^2$ which will be 6 when $r = \sqrt{3}$. 
Ellipsoid \( x^2 + 2y^2 + 3z^2 = 4 \)

A similar trick occurs for using spherical coordinate ideas in parameterizing ellipsoids.
Ellipsoid \( x^2 + 2y^2 + 3z^2 = 4 \)

A similar trick occurs for using spherical coordinate ideas in parameterizing ellipsoids.

\[
\Phi(u, v) = <2 \sin u \cos v, \sqrt{2} \sin u \sin v, \sqrt{\frac{4}{3}} \cos u>
\]
Ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

$\Phi(u, v) = < 2\sin u \cos v, \sqrt{2} \sin u \sin v, \sqrt{\frac{4}{3}} \cos u >$
You may have run into the hyperbolic functions

\[
\cosh x = \frac{e^x + e^{-x}}{2}
\]
\[
\sinh x = \frac{e^x - e^{-x}}{2}
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\]

Just as \(\cos^2 \theta + \sin^2 \theta = 1\) helps with ellipses, the hyperbolic version \(\cosh^2 \theta - \sinh^2 \theta = 1\) leads to the nicest hyperbola parameterizations.
Hyperbolic Cylinder $x^2 - z^2 = -4$

Just as $\cos^2 \theta + \sin^2 \theta = 1$ helps with ellipses, the hyperbolic version $\cosh^2 \theta - \sinh^2 \theta = 1$ leads to the nicest hyperbola parameterizations.

$$\Phi(u, v) = \langle 2 \sinh v, u, 2 \cosh v \rangle$$
Hyperbolic Cylinder $x^2 - z^2 = -4$

$$\Phi(u, v) = \langle 2 \sinh v, u, 2 \cosh v \rangle$$
Saddle $z = x^2 - y^2$

The hyperbolic trick also works with saddles.
Saddle $z = x^2 - y^2$

\[ \Phi(u, v) = \langle u \cosh v, u \sinh v, u^2 \rangle \]
Saddle $z = x^2 - y^2$

$\Phi(u, v) = \langle u \cosh v, u \sinh v, u^2 \rangle$
Hyperboloid of 1 Sheet \( x^2 + y^2 - z^2 = 1 \)

The spherical coordinate idea for ellipsoids with \( \sin \phi \) replaced by \( \cosh u \) works well here.
Hyperboloid of 1 Sheet \( x^2 + y^2 - z^2 = 1 \)

\[
\Phi(u, v) = \langle \cosh u \cos v, \cosh u \sin v, \sinh u \rangle
\]
Hyperboloid of 1 Sheet $x^2 + y^2 - z^2 = 1$
Hyperboloid of 2-Sheets $x^2 + y^2 - z^2 = -1$

$\Phi(u, v) = \langle \sinh u \cos v, \sinh u \sin v, \cosh u \rangle$
Hyperboloid of 2-Sheets $x^2 + y^2 - z^2 = -1$
Top Part of Cone \( z^2 = x^2 + y^2 \)

So \( z = \sqrt{x^2 + y^2}. \)
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The polar coordinate idea leads to

$$
\Phi(u, v) = \langle u \cos v, u \sin v, u \rangle
$$
Top Part of Cone $z^2 = x^2 + y^2$

So $z = \sqrt{x^2 + y^2}$.

The polar coordinate idea leads to

$$\Phi(u, v) = < u \cos v, u \sin v, u >$$
Mercator Parametrization of the Sphere

For $0 \leq v \leq \infty$, $0 \leq u \leq 2\pi$

$\Phi(u, v) = (\text{sech}(v) \cos u, \text{sech}(v) \sin u, \tanh(v))$.

(Note $\tanh^2(v) + \text{sech}^2(v) = 1$)
Surface Integrals

Picture of $\vec{T}_u, \vec{T}_v$ for a Lat/Long Param. of the Sphere.
Surface Integrals

Basic Parametrization Picture
Surface Integrals

Parametrization \( \Phi(u, v) = (x(u, v), y(u, v), z(u, v)) \)

Tangents \( T_u = (x_u, y_u, z_u) \quad T_v = (x_v, y_v, z_v) \)

Area Element \( dS = \| \vec{T}_u \times \vec{T}_v \| \, du \, dv \)

Normal \( \vec{N} = \vec{T}_u \times \vec{T}_v \)

Unit normal \( \hat{n} = \pm \frac{\vec{T}_u \times \vec{T}_v}{\| \vec{T}_u \times \vec{T}_v \|} \)

(Choosing the \( \pm \) sign corresponds to an orientation of the surface.)
Surface Integrals

Two Kinds of Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$:

$$
\int \int_S f(x, y, z) \ dS
$$

Surface Integral of a vector field $\vec{F}(x, y, z)$:

$$
\int \int_S \vec{F}(x, y, z) \cdot \hat{n} \ dS.
$$
Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$ calculated by

$$\int\int_S f(x, y, z) \, dS = \int\int_D f(\Phi(u, v)) \| \vec{T}_u \times \vec{T}_v \| \, du \, dv$$

where $D$ is the domain of the parametrization $\Phi$.

Surface Integral of a vector field $\vec{F}(x, y, z)$ calculated by

$$\int\int_S \vec{F}(x, y, z) \cdot \hat{n} \, dS = \pm \int\int_D \vec{F}(\Phi(u, v)) \cdot \left( \frac{\vec{T}_u \times \vec{T}_v}{\| \vec{T}_u \times \vec{T}_v \|} \right) \| \vec{T}_u \times \vec{T}_v \| \, du \, dv$$

where $D$ is the domain of the parametrization $\Phi$. 
Surface Integrals

3d Flux Picture
Surface Integrals

The preceding picture can be used to argue that if \( \vec{F}(x, y, z) \) is the velocity vector field, e.g. of a fluid of density \( \rho(x, y, z) \), then the surface integral

\[
\iint_S \rho \vec{F} \cdot \hat{n} \ dS
\]

(with associated Riemann Sum

\[
\sum \rho(x_i^*, y_j^*, z_k^*) \vec{F}(x_i^*, y_j^*, z_k^*) \cdot \hat{n}(x_i^*, y_j^*, z_k^*) \Delta S_{ijk}
\]

depicted in the preceding picture represents the rate at which material (e.g. grams per second) crosses the surface.
From this point of view the orientation of a surface simple tells us which side is accumulating mass, in the case where the value of the integral is positive.
There’s an analogous 2d Riemann sum and interp of

$$\int_C \vec{F} \cdot \hat{n} \, ds.$$
Surface Integrals

At level of linear approx, parallelogram on right consists of fluid that crossed $C$ between times $t$ and $t+\Delta t$. 

$\Delta s$ parallel to $V$ 

Crossing $C$ at time $t$ 

$\Delta t$
Problem: Calculate

\[ \int \int_S \vec{F}(x, y, z) \cdot \hat{n} \, dS \]

for the vector field \( \vec{F}(x, y, z) = (x, y, z) \) and \( S \) the part of the paraboloid \( z = 1 - x^2 - y^2 \) above the \( xy \)-plane. Choose the positive orientation of the paraboloid to be the one with normal pointing downward.
Problem: Calculate the surface area of the above paraboloid.