HARISH-CHANDRA HOMOMORPHISM

1. $\mathfrak{g} = \underline{n}^+ + \mathfrak{h} + n^-$

$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) + n^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g}) n$

$\underset{\sim}{q} : \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \longrightarrow \mathcal{U}(\mathfrak{h})$ projection via $(n^+ \mathcal{U}(\mathfrak{g}))^{\mathfrak{h}} \simeq (\mathcal{U}(\mathfrak{g}) n)^{\mathfrak{h}}$

Lemma: $q : \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \longrightarrow \mathcal{U}(\mathfrak{h})$ is an algebra homomorphism

Proof: $u = \underbrace{h + N u^+ + u^- N^-}$ if $u \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$.

From PBW

$u_1 \cdot u_2 = h_1 h_2 + N_1^+ (-) + (-) N_2^-$

2. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ $\qquad \alpha > 0$ means $X_\alpha \in \underline{n}^+$

There is a map

$\mu : S(\mathfrak{h}) \longrightarrow S(\mathfrak{h}) \qquad \mu(h) := h - \rho(h)$

NOT HOMOGENEOUS.

THM: $\gamma = \mu \circ q : \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{h})^{W}$ is an isomorphism.

Proof:

Let $\lambda \in \mathfrak{h}^*$. We define a module $M(\lambda)$,

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \mathbb{C}_{\lambda-\rho}$$

where $\mathfrak{n}_+$ acts on $\mathbb{C}_{\lambda-\rho}$ by 0.

By PBW, $M(\lambda) \simeq \mathcal{U}(\mathfrak{n}^-) \underset{\mathbb{C}}{\otimes} \mathbb{C}_{\lambda-\rho}$

It has a unique irreducible quotient $L(\lambda)$.

Let $v_{\lambda-\rho}$ be the "highest weight." If $z \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$,

$$z \cdot v_{\lambda-\rho} = q(z) \cdot v_{\lambda-\rho} = q(z)(\lambda-\rho)\, v_{\lambda-\rho}$$

It follows that z acts by $\gamma(z)(\lambda)$ on $M(\lambda)$.

FACT: assume λ is dominant. If $\lambda' = s_\alpha \lambda$ where $s_\alpha \in W$ is the reflection $\leftrightarrow \alpha$, then $M(\lambda') \hookrightarrow M(\lambda)$.

This follows from an $sl(2)$-calculation.

So z must act by the same scalar:

$$\gamma(z)(s_\alpha \lambda) = \gamma(z)(\lambda).$$

More general, if $(\lambda, \alpha) > 0$ $\quad s_\alpha \lambda = \lambda - (\lambda, \check{\alpha})\alpha$

and $M(s_\alpha \lambda) \hookrightarrow M(\lambda)$.

The fact that the map is an isomorphism follows from using $gr: U(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{g})$ and the fact that it gives $U(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{g})^{\mathfrak{g}}$.

Example $\mathfrak{g} = sl(2, \mathbb{C})$ $\quad e, h, f.$

$\rho(h) = 1 \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha(h) = 2, \check{\alpha}(h) = 2$

$M(\lambda) = \bigoplus f^n \otimes v_{(\lambda-1)}$

$h \cdot (f^n \otimes v_{\lambda-1}) = (\lambda - 1 - 2n) \cdot (f^n \otimes v_{\lambda-1})$

$f \cdot (f^n \otimes v_{\lambda-1}) = f^{n+1} \otimes v_{\lambda-1}$

$e \cdot (f^n \otimes v_{\lambda-1}) = \sum_{k=1}^{n-1} f^k h f^{n-1-k} \otimes v_{\lambda-1}$

$= \sum_{k=0}^{n-1} (\lambda - 1 - 2(n-1-k)) f^{n-1} \otimes v_{\lambda-1} = [(\lambda-1)(n) - n(n-1)] f^{n-1} \otimes v_{\lambda-1}$

$= (n \quad)(\lambda \quad - n) f^{n-1} \otimes v_{\lambda-1}$

If $\lambda = N \in \mathbb{N}$ then $e \cdot (f^N \otimes v_{\lambda-1}) = 0$, and

$M(-N \quad) \subset M(N)$ and $s_\alpha \lambda = -\lambda$ in this case