<u>THM</u>: (π, X) a finitely generated $(\mathfrak{g}, K)$-module.
Then $X(\xi)$ is f.g. as a Z-module.

- We reduced the problem to $(S(\underline{s}), K)$-modules via Gr. (Recall $\mathfrak{g} = \underline{k} + \underline{s}$)

Need to show that

① $S(\underline{s})^K$ is finitely generated over $\mathrm{Res}_{\underline{s}}\, S(\mathfrak{g})^{\mathfrak{g}}$

② $\mathrm{gr}X$ is f.g. as a $S(\underline{s})^K$-module.

①: Recall $\mathfrak{a} \subseteq \underline{s}$ maximal abelian

<u>CLAIM</u> $P(\mathfrak{a})$ is integral over $\mathrm{Res}_{\mathfrak{a}}(S(\mathfrak{g})^{\mathfrak{g}})$.

$\Delta(\mathfrak{g}, \mathfrak{a})$ are the roots of $\mathfrak{a}$ in $\mathfrak{g}$.

$\mathfrak{a}$ acts semisimply (via ad)

$\mathfrak{g}_\beta := \{ x \in \mathfrak{g} : [h, x] = \beta(h)x \}$

- $\Delta(\mathfrak{g}, \mathfrak{a})$ spans $\mathfrak{a}^*$
- Every $\beta \in \Delta(\mathfrak{g}, \mathfrak{a})$ is algebraic/$\mathrm{Res}_{\mathfrak{a}}$:

$P_t(x) = \det(tI - \mathrm{ad}X)$ has invariant coeff's. For $x \in \mathfrak{a}$ get $\sum \beta^j p_j = 0$

② Consider $\mathrm{Hom}_{\mathbb{C}}[V_\xi, X]$. This is an $(S(\underline{\Delta}), K)$ module via

$k \cdot T := \pi(k) \circ T \circ \xi(k^{-1})$

$x \cdot T := \pi(x) \circ T$ (action is on X)

$X(\xi)$ is the image of $\mathrm{Hom}_K[V_\xi, X]$ under the map $(T, v) \longmapsto T(v)$.

Since (1) X is f.g. as an $S(\underline{\Delta})$-module

(2) $\dim V_\xi < \infty$

(3) $S(\underline{\Delta})$ is Noetherian,

$S(\underline{\Delta}) \cdot \mathrm{Hom}_K[V_\xi, X]$ is f.g. by say $T_1, \ldots, T_d \in \mathrm{Hom}_K[V_\xi, X]$.

So any $T \in \mathrm{Hom}_K[V_\xi, X]$ is

$$T = \sum p_i T_i$$

Integrate over K to get

$$T = \sum p_i^K T_i \qquad \boxtimes$$

Harish-Chandra Homomorphism

$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ ($\mathfrak{g}$ cx)

$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) + \mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$ (not a direct sum)

$P :=$ projection of $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ onto $\mathcal{U}(\mathfrak{h})$.

CLAIM: Any $u \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$ is also in $\mathcal{U}(\mathfrak{g})^{h} \cap \mathfrak{n}^-\mathcal{U}(\mathfrak{g})$.

Let $X_{\alpha_1} \cdots X_{\alpha_d}$ $X_{-\alpha_1} \cdots X_{-\alpha_d}$ be the root vectors. By PBW, consider

$$u = X_{-\alpha_1}^{-m_1} \cdots X_{-\alpha_d}^{-m_d} \cdot u_h \cdot X_{+\alpha_1}^{n_1} \cdots X_{+\alpha_d}^{n_d}$$

If $u \in \mathcal{U}(\mathfrak{g})^h \cap \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$ one of $n_j > 0$.

$$0 = [h, u] = \sum (-m_i \alpha_i(h) + n_j \alpha_j(h)) u.$$

& choose $h \ni \alpha_i(h) > 0$. $\forall \alpha > 0$.

It follows that some $m_i > 0$.

$\mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \subseteq \mathcal{U}(\mathfrak{h}) + \mathcal{U}(\mathfrak{g})\underline{\mathfrak{n}}^+$

P projection onto $\mathcal{U}(\mathfrak{h})$.

Example: $\mathfrak{g} = sl(2)$. $h^2 + 2(ef + fe) \longmapsto h^2 + 2h$

- P is an algebra homomorphism.

<u>Pf</u>: $u_1 = u_1^0 + u_1^+$ $\qquad u_2 = u_2^0 + u_2^+$

$u_1 \cdot u_2 = u_1^0 \cdot u_2^0 + u_1^+ u_2^0 + u_1^0 u_2^+ + u_1^+ u_2^+$

Commute u_1^+ and u_2^0 in the second term.

- $\mu : \mathcal{U}(\mathfrak{h}) \simeq S(\mathfrak{h}) \simeq P(\mathfrak{h}) \longrightarrow \mathcal{U}(\mathfrak{h}) \simeq \ldots$

defined by $u(\lambda) \longmapsto u(\lambda - \rho)$

where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$

<u>CLAIM</u>: $\gamma \simeq \mu \circ P : Z \xrightarrow{\sim} \mathcal{U}(\mathfrak{h})^W$

γ called the Harish-Chandra homomorphism; independent of choice $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$.

- We will only show that the image lands in $\mathcal{U}(\mathfrak{h})^W$.

<u>Example</u> $h^2 + 2(ef + fe) \longmapsto h^2 + 2h \mapsto (h-1)^2 + 2(h-1)$

$= h^2 - 2h + 1 + 2h - 2 = h^2 - 1$

<u>CONSEQUENCE</u>: $\hat{Z} \simeq \widehat{B(\mathfrak{h})}^W$.

<u>THEOREM</u>: $A \subset B$ f.g. algebras $/ \mathbb{C}$. If B is integral over A, then

(i) each $\varphi : A \to \mathbb{C}$ extends to $\Phi : B \to \mathbb{C}$

(ii) if each extension is unique, $Fr(B) = Fr(A)$

REFERENCE HELGASON 1962, Chapter X.5
actually any graduate algebra text.

- $\widehat{P(\mathfrak{h})}^{W} \simeq \mathfrak{h}^{*}/W$

Highest Weight Modules

$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. $\qquad \mathfrak{g} = \mathfrak{n}^{-} + \mathfrak{h} + \mathfrak{n}$

$\pi : \mathfrak{g} \longrightarrow End(V)$

$V_\mu := \{ v \in V : \pi(h)v = \mu(h)v \}$ Weight Space

(1) $\pi(\mathfrak{g}^{\alpha}) V_\mu \subseteq V_{\mu+\alpha}$

(2) $V_\mu \cap V_\gamma = (0)$ unless $\mu = \gamma$

$\bigoplus V_\mu \subset V$ is $\mathfrak{g}$-stable

DEF: (π, V) is called a highest weight module if

(1) $V = \bigoplus V_\mu$

(2) V generated by a $v_\Lambda \in V_\Lambda$ annihilated by $\mathfrak{n}$

STANDARD MODULE:

$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$ (we will see later why $\lambda - \rho$)

(1) $M(\lambda) := \mathcal{U}(\mathfrak{n}^{-}) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho}$

So $M(\lambda)$ is free as an $\mathfrak{n}^-$-module.

any $v \in M(\lambda)$ satisfies $\dim \mathcal{U}(\mathfrak{b}) v < \infty$

$M(\lambda) = \bigoplus M(\lambda)_\mu$ & $\dim M(\lambda)_\mu < \infty$

$$\dim M(\lambda)_\mu = \#\text{ of ways } \mu = \lambda - \sum_{\substack{\alpha > 0 \\ n_\alpha \in \mathbb{N}}} n_\alpha \alpha$$

(2) If V is h.w. $\lambda - \rho$, $\exists!\ \Psi: M(\lambda) \longrightarrow V \longrightarrow 0$

(3) $M(\lambda)$ has an infinitesimal character

$z = z_{\mathfrak{h}} + \mathcal{U}\mathfrak{n}^+ \qquad z \cdot v_{\lambda-\rho} = z_{\mathfrak{h}}(\lambda - \rho)$

& $z \cdot (u \otimes 1\!\!1_{\lambda-\rho}) = uz \otimes 1\!\!1_{\lambda-\rho}$

(4) $M(\lambda)$ has a unique irreducible quotient

$N(\lambda) :=$ sum of all submodules which do not contain $v_{\lambda-\rho}$.

$V \subseteq M \qquad V = \bigoplus V_\mu = \bigoplus V \cap M_\mu$

$L(\lambda) :=$ unique irreducible quotient

(5) If $(\lambda, \check{\alpha}) \in \mathbb{N}$, for α simple, then $M(s_\alpha \lambda) \hookrightarrow M(\lambda)$.

This proves that $z \longmapsto P(z)(\lambda - \rho) \in S(\mathfrak{h})^W$

because it implies $\gamma(z)(\lambda) = \gamma(z)(s_\alpha \lambda)$

for any $\lambda \ni (\lambda, \check{\alpha}) \in \mathbb{N} \quad \alpha \in \Delta(\mathfrak{n}^+, \mathfrak{h})$.

It is enough to show that $X_{-\alpha}^{(\lambda,\check{\alpha})} \otimes v_{\lambda-\rho}$ is annihilated by n^+.

Enough to do it for X_β with $\beta \in \Pi$ simple

$\beta \neq \alpha \Rightarrow \beta - \alpha$ not a root. So $[X_\beta, X_{-\alpha}] = 0$.

$\beta = \alpha$ $\quad X_\alpha, h_{\check{\alpha}}, X_{-\alpha}$ with $[X_\alpha, X_{-\alpha}] = h_{\check{\alpha}}$ and $(\check{\alpha}, \alpha) = 2$.

$$h_{\check{\alpha}} \cdot X_{-\alpha}^m \otimes v_{\lambda-\rho} = [(\lambda-\rho)(h_{\check{\alpha}}) - 2m]\, X_{-\alpha}^m \otimes v_{\lambda-\rho}$$

$$X_\alpha \cdot X_{-\alpha}^m = \sum_{k=1}^{m-1} X_{-\alpha}^k\, h_{\check{\alpha}}\, X_{-\alpha}^{m-1-k} + X_{-\alpha}^m \cdot X_\alpha$$

$$= X_{-\alpha}^{m-1} \cdot \sum_{k=1}^{m-1} \left(h_{\check{\alpha}} - 2(m-1-k) \right) + X_{-\alpha}^m X_\alpha$$

$$X_{-\alpha}^{m-1} \cdot \left[m\, h_{\check{\alpha}} - m(m-1) \right] + X_{-\alpha}^m \cdot X_\alpha$$

applied to $1 \otimes v_{\lambda-\rho}$:

$$m\left[(\lambda-\rho, \check{\alpha}) - m + 1\right] 1 \otimes v_{\lambda-\rho} = m\left[(\lambda, \check{\alpha}) - m\right] 1 \otimes v_{\lambda-\rho}$$

If $m = (\lambda, \check{\alpha})$ get 0.