Highest Weight Modules

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ semisimple, $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{b} + \mathfrak{n}$

(π, V) a rep'n of $\mathfrak{g}$

Highest Weight Module $V = \bigoplus V_\mu \quad \mu \in \mathfrak{h}^*$

V_μ eigenspace for $\mathfrak{h}$ & $\exists\, v_\Lambda \in V_\Lambda$, $\mathfrak{n}^+ v_\Lambda = 0$

& v_Λ generates V.

Standard (Verma) Module $M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_{\lambda-\rho}$

$\mathfrak{g}$ acts by left-multiplication on

PROPERTIES: (1) $M(\lambda) \simeq \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho}$

$$M(\lambda) = \bigoplus M(\lambda)_\mu \qquad \mu = \lambda - \rho - \sum_{\substack{\alpha \in \Delta(\mathfrak{n}) \\ m_i \in \mathbb{N}}} m_i \alpha_i$$

$$\dim M(\lambda)_\mu = \#\{(m_1, \ldots, m_d) : \mu = \lambda - \rho - \sum m_i \alpha_i\}$$

(2) Character of $M(\lambda) := \sum \dim M(\lambda)_\mu \cdot e^{\mu} = \dfrac{e^{\lambda-\rho}}{\prod_{\alpha>0}(1 - e^{-\alpha})}$

Converges absolutely if you plug in $H \in \mathfrak{h} \ni \alpha(H) > 0 \ \forall \alpha$

(3) Z acts by scalars. $\chi(z) = \gamma(z)(\lambda) = P(z)(\lambda - \rho)$

(A) If (π, V) is highest weight Λ, $M(\Lambda+\rho) \longrightarrow V \longrightarrow 0$

$M(\lambda)$ has a unique irreducible Quotient.

So any irreducible h.w. module is a quotient of a Verma module.

$$L(\Lambda) \simeq L(\Lambda') \iff \Lambda + \rho = \Lambda' + \rho$$

(5) $M(\lambda)$ has finite composition series

assume $\lambda \in W \cdot \rho$. Reparametrize $Z(w) := M_{-w\rho-\rho}$.

$L(w)$ irreducible quotient

$$\mathrm{Ch}\, Z(w) = \sum_{y \in W} M(y,w) L(y); \quad \mathrm{Ch}\, L(y) = \sum m(y,w)\, \mathrm{Ch}\, Z(y)$$

(6) If $(\lambda, \check{\alpha}) \in \mathbb{N}$ for some $\alpha \in \Pi$, then

$$M(s_\alpha \lambda) \hookrightarrow M(\lambda).$$

CONSEQUENCE: $\gamma(z)(s_\alpha \lambda) = \gamma(z)(\lambda) \quad \forall \lambda \in \Lambda^+$

$\Lambda^+ := \{ \lambda \in \mathfrak{h}^* : (\lambda, \check{\alpha}) \geq 0 \;\; \forall \alpha \in \Delta(\underline{n}) \}$.

$\gamma(z) \in \mathcal{U}(\mathfrak{h})^W$ because W is generated by s_α, $\alpha \in \Pi$

PROOF: Enough to show that $X_{-\alpha}^{(\lambda, \check{\alpha})} \otimes v_{\lambda - \rho}$

is annihilated by $\underline{n}^+$. This is then a h.w.,

and we get a map $M(s_\alpha \lambda) \longrightarrow M(\lambda)$;
injection because n^- acts freely.

$\beta \in \Pi \quad \beta \neq \alpha \implies \beta - \alpha$ is NOT a root.

$$[X_\beta, X_{-\alpha}] = 0 \implies X_\beta X_{-\alpha}^{(\lambda,\check\alpha)} = X_{-\alpha}^{(\lambda,\check\alpha)} X_\beta$$

$\beta = \alpha$: $\quad [X_\alpha, X_{-\alpha}] = h_{\check\alpha} \ni [h_{\check\alpha}, X_{\pm\alpha}] = \pm 2 X_{\pm\alpha}$.

$$h_{\check\alpha} X_{-\alpha}^m = X_{-\alpha}^m h_{\check\alpha} - 2m X_{-\alpha}$$

$$X_\alpha X_{-\alpha}^m = X_{-\alpha}^m X_\alpha + \sum_{k=1}^{m-1} X_{-\alpha}^{k-1} h_{\check\alpha} X_\alpha^{m-k} =$$

$$= X_{-\alpha}^m X_\alpha + X_{-\alpha}^{m-1} \sum_{k=1}^{m-1} \left(h_{\check\alpha} - 2(m-k) \right) =$$

$$= X_{-\alpha}^m X_\alpha + X_{-\alpha}^{m-1} \left[(m-1) h_{\check\alpha} - m(m-1) \right]$$

$$= X_{-\alpha}^m X_\alpha + (m-1) X_{-\alpha}^{m-1} \left[h_{\check\alpha} - m \right]$$

Applied to $v_{\lambda-\rho}$ and $m = (\lambda, \check\alpha)$, we get

$$(\lambda, \check\alpha) + 1 - (\rho, \check\alpha) - (\lambda, \check\alpha) = 0 \quad \boxtimes$$

<u>alternate:</u> (π, V) satisfies

- $\mathcal{Z}$ acts by scalars
- $V^{\mathfrak{n}} \neq (0)$
- V irreducible

<u>PROP:</u> (π, V) is h.w. as in the previous Definition

"Sketch of Proof". <u>If</u> $0 \neq v \in V^{\mathfrak{n}^+}$, then $\dim \mathcal{U}(\mathfrak{h})v < \infty$

Follows from $S(\mathfrak{h})$ is finite over $S(\mathfrak{h})^W$

$\underline{n}$ - cohomology

$\mathfrak{g} \supseteq \mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supset \mathfrak{b} = \mathfrak{h} + \underline{n}$

$\mathfrak{q}$ called parabolic subalgebra.

(π, V) a $\mathfrak{g}$-module

$\underline{\text{DEF}}$: $\operatorname{Hom}_{\mathbb{C}}[\Lambda^i \mathfrak{u}, V] \xrightarrow{d} \operatorname{Hom}_{\mathbb{C}}[\Lambda^{i+1} \mathfrak{u}, V]$

$$df(x_0 \wedge \cdots \wedge x_i) := \sum (-1)^j (x_j \cdot f)(x_0 \wedge \cdots \hat{x}_j \cdots x_i).$$

Then $d^2 = 0$ and we define

$$H^i(\underline{u}, V) := \ker d_i / \operatorname{im} d_{i-1}$$

$$H^0(\underline{u}, V) = \{ f : \mathbb{C} \to V \mid df(x) := \pi(x) f(1) = 0 \}$$
$$= V^{\underline{u}}$$

Can define these groups via injective/projective resolutions; see Jacobson, Cartan-Eilenberg

T. $H^i(\mathfrak{u}, V)$ has usual functorial properties.

$0 \to U \xrightarrow{j} V \xrightarrow{\pi} W \to 0 \implies$

$$H^i(\underline{u}, U) \to H^i(\underline{u}, V) \to H^i(\underline{u}, W) \xrightarrow{\delta} H^{i+1}(\underline{u}, U) \to$$

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Z and $\underline{l}$ act on $H^i(\underline{u}, V)$:

Check that their action on V commutes with d.

MORE INVOLVED: δ is a Z, ℓ - homomorphism

$\omega \in H^i(\Lambda^i \underline{u}, W)$ comes from $f : \Lambda^i \underline{u} \longrightarrow W$, $df = 0$.

Since $V \to W \to 0$ $\exists\, h : \Lambda^i \underline{u} \longrightarrow V$ $f = \pi h$

Then $dh : \Lambda^{i+1} \underline{u} \longrightarrow V$ maps to $0 \in W$, so takes

values in U. $d(dh) = 0$, so define $\delta \bar{\omega} = \overline{dh}$

$d(zh) = z \cdot dh \implies \delta(z\omega) = z\,\delta(\omega)$.

Now consider $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\underline{u}^-) \otimes \mathcal{U}(\ell) \otimes \mathcal{U}(u^+)$

$= \mathcal{U}(\ell) \oplus (\underline{u}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\, \underline{u}^+)$

As before, $Z(\mathfrak{g}) \hookrightarrow \mathcal{U}(\mathfrak{g})^{\underline{\ell}}$

and $\bar{u}\, \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})^{\ell} = \mathcal{U}(\mathfrak{g}) u \cap \mathcal{U}(\mathfrak{g})^{\ell}$

and we can project $P_\ell : Z(\mathfrak{g}) \longrightarrow \mathcal{U}(\underline{\ell})^{\ell}$

with kernel $(\mathcal{U}(\mathfrak{g}) u)^{\ell}$.

<u>THM</u> $z \cdot \omega = \gamma(z)\,\omega$ (Casselman-Osborne)

$$\underset{\substack{\uparrow\\ Z(\mathfrak{g})\text{-action}}}{z\cdot\omega} = \underset{\substack{\uparrow\\ \mathcal{U}(\mathfrak{l})\text{-action}}}{\gamma(z)\cdot\omega}$$

"Proof" $H^0(\underline{u}, V) = V^{\mathfrak{u}}$. The statement is clear for $i=0$. For $i>0$ do an induction

$$0 \to V \to I \to Q \to 0$$ with I injective.

So $H^i(\underline{u}, I) = 0 \quad i>0$

$$\to H^i(\underline{u}, I) \to H^i(\underline{u}, Q) \xrightarrow{\delta} H^{i+1}(\underline{u}, V) \to \underbrace{H^{i+1}(\underline{u}, I)}_{=0}$$

I is $\mathcal{U}(\mathfrak{g})$-injective $\Longrightarrow$ I is $\mathcal{U}(\underline{u})$-injective.

$\omega \in H^{i+1}(\underline{u}, V)$ equals $\delta\tilde{\omega} = \omega$, $\omega \in H^i(\underline{u}, Q)$.

$$z\cdot\omega = z\,\delta\tilde{\omega} = \delta(z\tilde{\omega}) = \delta(\gamma(z)\tilde{\omega}) = \gamma(z)\delta(\tilde{\omega}) \quad \text{QED.}$$

COROLLARY: (1) V is $Z(\mathfrak{g})$ finite $\Longrightarrow H^i(\mathfrak{u}, V)$ is $Z(\mathfrak{l})$-finite

(2) If $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ are the generalized inf'l char's of V, then

$w\lambda_i - \rho(\mathfrak{u})$ are the generalized inf'l characters of $H^i(\underline{u}, V)$

CONSEQUENCE

a Method for Computing Ch V:

Compute $H^i(\mathfrak{u}, V)$ & use $\sum (-1)^i H^i(\underline{u}, V) = \left(\sum (-1)^i \Lambda^i \underline{u}\right) ch(V)$.

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<u>Proof:</u> $T_{\rho} \circ P_{\mathfrak{g}}^{\gamma_{\mathfrak{g}}} : Z(\mathfrak{g}) \to U(\mathfrak{h})$, $T_{\rho(\mathfrak{l})} \circ P_{\mathfrak{l}}^{\gamma_{\mathfrak{l}}} : Z(\mathfrak{l}) \to U(\mathfrak{h})$

are the two projections, $P_{\mathfrak{u}} : Z(\mathfrak{g}) \to U(\mathfrak{l})^{\mathfrak{l}} \simeq Z(\mathfrak{l})$

$\mathfrak{n} = \underline{\mathfrak{n}}^{\mathfrak{l}} + \mathfrak{u}$. $\quad \underline{\mathfrak{n}}^{\mathfrak{l}}$ positive roots $\Delta(\mathfrak{l} \cap \mathfrak{n}, \mathfrak{h})$

$P_{\mathfrak{g}} = P_{\mathfrak{l}} \circ P_{\mathfrak{u}} \qquad \rho = \rho(\mathfrak{l}) + \rho(\mathfrak{u})$

$T_{\rho} \circ P_{\mathfrak{g}} = T_{\rho(\mathfrak{u})} \circ T_{\rho(\mathfrak{l})} \circ P_{\mathfrak{l}} \circ P_{\underline{\mathfrak{u}}} = T_{\rho(\mathfrak{u})} \circ \gamma_{\mathfrak{l}} \circ P_{\underline{\mathfrak{u}}}$

<u>Generalized Inf'l Character</u> means

annihilated by an ideal $I \subseteq Z(\mathfrak{g})$ of finite codimension

$(z - \chi(z))^n \cdot v = 0 \quad \forall z \in Z(\mathfrak{g}),\ v \in V$.

$\gamma_{\mathfrak{l}}(P_{\mathfrak{u}}(I)) = T_{-\rho(\mathfrak{u})}(\gamma_{\mathfrak{g}}(I)) \subseteq T_{-\rho(\mathfrak{u})} \cdot (U(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})})$

So $H^i(\underline{\mathfrak{u}}, V)$ is annilated by $P_{\mathfrak{u}}(I)$, & this has finite codimension in $U(\mathfrak{h})^{W}$.

Let μ occur in $H^i(\underline{\mathfrak{u}}, V)$. Want $\mu = w\lambda_i - \rho(\mathfrak{u})$.

Suppose not, $\exists\ p \in S(\mathfrak{h}) \ni p \in S(\mathfrak{h})^W$ &

$p(\mu + \rho(\mathfrak{u})) \neq 0, \quad p(\lambda_i) = 0\ \forall i$

Let $z \in Z \ni \gamma(z) = p$. Then z^n is 0 on V.

May as well assume z is 0 on V.

But $P_{\underline{u}}(z)$ annihilates $H^1(\underline{u}, V)$. Thus

$$(\gamma_\ell \circ \gamma(z))(\mu) = 0$$

So $0 = \left[T_{-\rho(u)} \circ \gamma_{\mathfrak{g}}(z)\right](\mu) = \gamma_{\mathfrak{g}}(z)(\mu + \rho(u))$

$= p(\mu + \rho(u))$ a contradiction.