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Note Title 3/22/2016

$\underline{\mathfrak{n}}$ - Cohomology

SETTING: $\mathfrak{g} \supseteq \underline{\mathfrak{q}} = \underline{\mathfrak{l}} + \underline{\mathfrak{u}} \supseteq \underline{\mathfrak{b}} =: \mathfrak{h} + \underline{\mathfrak{n}}$

$\mathfrak{q}$ called a Parabolic subalgebra

(π, V) a $\mathfrak{g}$-module

DEF: $\operatorname{Hom}_{\mathbb{C}}[\Lambda^i \underline{\mathfrak{u}}, V] \xrightarrow{d} \operatorname{Hom}_{\mathbb{C}}[\Lambda^{i+1} \underline{\mathfrak{u}}, V]$

$$df(x_0 \wedge \dots \wedge x_i) = \sum (-1)^j x_j \cdot f(x_0 \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_i)$$

satisfies $d^2 = 0$

$$H^i(\underline{\mathfrak{u}}, V) := \ker d_i / \operatorname{im} d_i$$

$\boxtimes$

<u>Remarks</u>

- This comes from de Rham cohomology; use left invariant forms $\boxtimes$
- $H^0(\underline{\mathfrak{u}}, V) = \{ f : \mathbb{C} \to V \mid df(x) = x \cdot f(1) = 0 \}$
  $\simeq V^{\underline{\mathfrak{u}}}$.
- see Jacobson, Cartan-Eilenberg for def's involving projectives / injectives

- H^i satisfies usual cohomology properties

$$0 \longrightarrow U \longrightarrow V \xrightarrow{\pi} W \longrightarrow 0$$

$$\Longrightarrow H^i(\underline{u}, U) \to H^i(\underline{u}, V) \to H^i(\underline{u}, W) \xrightarrow{\delta} H^{i+1}(\underline{u}, U) \to$$

THEOREM: $H^i(\underline{u}, V.)$ are $(\underline{\ell}, Z)$-modules.

δ is an (ℓ, Z)-homomorphism $\square$

"**Pf**": The "easy" part: action of $\underline{\ell}, Z$ commutes with d.

Harder Part: Commutation with δ

- $f : \Lambda^i \underline{u} \to W$, $df = 0$ gives rise to $\overline{\omega} \in H^i$
- $V \to W \to 0$ implies $\exists\, h \ni \pi h = f$.
- $dh : \Lambda^{i+1} \to V$ satisfies $d(dh) = 0$. Then

$\pi(dh) = d(\pi h) = df = 0$; so $dh : \Lambda^{i+1}\underline{u} \to U$.

Let $\delta\overline{\omega} := \overline{dh}$. Then $d(zh) = z\,dh \Rightarrow \delta(z\overline{\omega}) = z\delta\overline{\omega}$ $\boxtimes$

MAIN INTENTION:

$$\sum (-1)^i \operatorname{Ch} H^i(\underline{u}, V) = \sum (-1)^i \operatorname{Ch} \operatorname{Hom}_{\mathbb{C}}[\Lambda^i \underline{u}, V]$$

$$= \left(\sum (-1)^i \operatorname{Ch} \Lambda^i \underline{u}^*\right) \cdot \operatorname{Ch} V = \prod (1 - e^{-\alpha}) \cdot \operatorname{Ch} V$$

If we can compute $H^i(\underline{u}, V)$, we can compute $\mathrm{ch}\,V$.
This is what happens with irreducible $(\mathfrak{g}, K)$-modules $\square$

Finite Dimensional Rep's / Kostant's Theorem

(I) (π, V) has an infinitesimal character

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{u}^-) \otimes \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{u}^+)$$
$$= \mathcal{U}(\underline{\mathfrak{l}}) + [\quad \underline{\mathfrak{u}}^- \, \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\, \underline{\mathfrak{u}}^+ \quad]$$

as before, $\mathfrak{u}^- \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{l}} = \mathcal{U}(\mathfrak{g})^{\mathfrak{l}} \cap \mathcal{U}(\mathfrak{g})\mathfrak{u}^+$

$P_{\mathfrak{l}} : Z(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})^{\underline{\mathfrak{l}}}$

<u>THM</u> (Casselman-Osborne)

$$z \cdot \overline{\omega} = P_{\underline{\mathfrak{u}}}(z) \cdot \overline{\omega} \qquad \overline{\omega} \in H^i(\underline{\mathfrak{u}}, V)$$

$Z(\mathfrak{g})$

$\mathcal{U}(\underline{\mathfrak{l}})$ action

$\boxtimes$

<u>Pf:</u> Clear for $i = 0$.

assume true for i. $0 \to V \to I \to Q \to 0$

with I injective

$$\to H^i(\underline{u}, I) \to H^i(\underline{u}, Q) \xrightarrow{\delta} H^{i+1}(\underline{u}, V) \to \underbrace{H^{i+1}(u, I)}_{=0}$$

δ is <u>onto</u> and an $(\underline{l}, Z)$-homomorphism.

<u>COROLLARY</u>: (1) V is $Z(\mathfrak{g})$-finite $\Longrightarrow$ $H^i(\underline{u}, V)$ is $Z(\mathfrak{l})$ finite

(need $Z(\mathfrak{l})$ finite over $Z(\mathfrak{g})$; uses
$Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W$)

(2) $\lambda \in \mathfrak{h}^*$ is the (generalized) inf'l character of V

$\Longrightarrow$ $w(\lambda) - \rho(\underline{u})$ is the (gen.) inf'l char. of H^i

$\rho(\underline{u}) = 1/2 \sum \alpha$ with $\alpha \in \Delta(\underline{u}, \mathfrak{h})$

"Pf of (2)": $T_\varsigma \circ P_\mathfrak{g} : Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$

$T_{\varsigma(\mathfrak{l})} \circ P_\mathfrak{l} : Z(\underline{\mathfrak{l}}) \to \mathcal{U}(\mathfrak{h})$

$$P_\mathfrak{g} = P_\mathfrak{l} \circ P_\mathfrak{u} \quad , \quad T_\varsigma = T_{\varsigma(\mathfrak{u})} \circ T_{\varsigma(\mathfrak{l})}$$

$$\gamma_\mathfrak{g} = T_\varsigma \circ P_\mathfrak{g} = T_{\varsigma(\mathfrak{u})} \circ T_{\varsigma(\mathfrak{l})} \circ P_\mathfrak{l} \circ P_\mathfrak{u} = T_{\varsigma(\mathfrak{u})} \circ \gamma_\mathfrak{l} \circ \underline{P_\mathfrak{u}}$$

<u>GENERALIZED INF'L CHARACTER</u>

$$\boxed{(z - \chi(z))^n v = 0 \quad \forall v \in V \text{ (some } n\text{.)}}$$

Say $I \subseteq Z$ of finite codimension annihilates V

Then $\gamma_{\underline{\ell}}(P_{\underline{u}}(I))$ annihilates H^i. But

$(\gamma_{\ell} \circ P_{\underline{u}})(I) = T_{-\rho(\underline{u})}(\gamma_{\mathfrak{g}}(I)) \subset T_{-\rho(u)}(\mathcal{U}(\mathfrak{h})^W) \subseteq \mathcal{U}(\mathfrak{h})^{W_\ell}$

finite codimension

Let μ occur in $H^i(\underline{u}, V)$. Say $\mu \neq w\lambda - \rho(u)$ for any $w \in W$. There is $p \in S(\mathfrak{h})^W \ni$ $p(\mu + \rho(\underline{u})) \neq 0$ but $p(\lambda) = p(w\lambda) = 0 \ \forall w$. Let $z \in Z$ be such that $p = \gamma(z)$. Then $z^n = 0$ on V; may as well assume $z = 0$ on V. $P_{\underline{u}}(z)$ annihilates $H^i(\underline{u}, V)$.

$(\gamma_{\mathfrak{l}} \circ P_{\underline{u}})(z)(\mu) = 0$; but

$$0 = \left[T_{-\rho(\underline{u})} \circ \gamma_{\mathfrak{g}}(z) \right](\mu) = \gamma_{\mathfrak{g}}(z)(\mu + \rho(\underline{u}))$$
$$= P(\mu + \rho(\underline{u})) \neq 0 \quad \boxtimes$$

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want to illustrate this for $F = F_\Lambda$ f. dim'l

$$0 \longrightarrow M(\Lambda + \rho) \longrightarrow F(\Lambda) \longrightarrow 0$$

- Inf'l Character of $F(\Lambda)$ is $\Lambda + \rho$.
- $\underline{\mathfrak{l}}$ - modules in $H^i(\underline{u}, F)$ have inf'l character $w(\Lambda + \rho) - \rho(u)$

NOTATION: $W = W(\mathfrak{g},\mathfrak{h})$, $\Delta^+(\mathfrak{g},\mathfrak{h}) = \Delta^+$

$w \in W$ $\quad \Delta_+(w) := \{\alpha \in \Delta^+ ;\ w^{-1}\alpha \notin \Delta^+\}$

$$\ell(w) = |\Delta_+(w)|$$

$$W^1 = \{w \in W : \Delta^+(w) \subseteq \Delta(\underline{u},\mathfrak{h})\}$$

$$= \{w \in W : \mu \in \mathfrak{h}^* \text{ dominant for } \Delta^+ \Rightarrow w\mu \text{ dominant for } \Delta^+(\underline{\ell},\mathfrak{h})\}$$

THM: $$H^m(\underline{u},F) = \bigoplus_{\substack{w \in W^1 \\ \ell(w)=m}} F_{\underline{\ell}}\left[w(\Lambda+\rho)-\rho\right]$$

APPLICATION $\mathrm{ch}\, F_\Lambda = \dfrac{\sum (-1)^{\ell(w)} e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+} (1-e^{-\alpha})}$

LEMMA 1: (a) $\alpha \in \Delta(\mathfrak{l}) \Longrightarrow \langle \alpha, \rho(\mathfrak{u}) \rangle = 0$

(b) $\alpha \in \Delta(\underline{\mathfrak{u}}) \Longrightarrow \langle \alpha, \rho(\mathfrak{u}) \rangle > 0$

"Pf": If $\alpha \in \Pi \subset \Delta^+$ is simple, $\langle \rho, \check{\alpha} \rangle = 1$.

Comes from $s_\alpha(\Delta^+) = \Delta^+ \setminus \{\alpha\} \cup \{-\alpha\}$

$s_\alpha \rho = \rho - (\rho, \check{\alpha})\alpha$.

$\alpha \in \Pi_{\mathfrak{l}} \implies s_\alpha(\Delta(\underline{u})) = \Delta(\underline{u})$.

$$\alpha \in \Delta(\underline{u}) \implies \alpha = \sum_{\beta_i \in \Delta(u)} m_i \beta_i + \sum_{\beta_j \in \Delta(\mathfrak{l})} n_j \beta_j$$

with $m_i, n_j \geq 0$. So enough to do for $\alpha \in \Delta(u)$ simple.

<u>LEMMA 2</u> $\mu \in F_\Lambda \implies \langle \mu+\rho, \mu+\rho \rangle \leq \langle \Lambda+\rho, \Lambda+\rho \rangle$

with "$=$" $\iff \mu = \Lambda$

<u>"Pf"</u>: $\mu = \Lambda - \sum_{\beta \in \Delta^+} n_\beta \beta$, $n_\beta \geq 0$

$$\langle \mu+\rho, \mu+\rho\rangle = \langle \mu,\mu\rangle + 2(\mu,\rho) + (\rho,\rho) \leq$$
$$\langle \Lambda,\Lambda\rangle + 2\left(\Lambda - \sum m_\beta \beta, \rho\right) + (\rho,\rho) \leq (\Lambda+\rho, \Lambda+\rho)$$

LEMMA 3 $\{\alpha_i\}_{i=1\dots m} \subset \Delta(\underline{n})$ any subset.

$$\left\langle w(\Lambda+\rho)-\rho+\sum\alpha_i,\ w(\Lambda+\rho)-\rho+\sum\alpha_i\right\rangle \geq (\Lambda,\Lambda).$$

"=" $\iff \{\alpha_i\} = \Delta^+(w)$.

"<u>pf</u>" ① $\rho - w\rho = \sum_{\beta\in\Delta^+(w)} \beta$

② $w(\Lambda+\rho)-\rho = w\Lambda - \sum\beta + \sum\alpha_i$

$A = \Delta^+(w) \cap \{\alpha_i\}$, $B = \{\beta : -\beta \in \Delta^+ \setminus A\}$, $C = \{\alpha_i\} \setminus A$

$\Delta^+(w) \setminus A$

$$w(\Lambda+\rho) - \rho = w\Lambda + \sum_{\beta \in C \cup B} \beta \quad \boxtimes$$

<u>COROLLARY</u>: $\left[F_\ell\left[w(\Lambda+\rho) - \rho(\cancel{\underline{u}})\right] : \operatorname{Hom}_{\mathbb{C}}\left[\Lambda^m \underline{u}, F\right]\right] = 1$ (red: ρ)

in degree $m = \ell(w)$.

<u>Pf</u>: $\mu \in \operatorname{Hom}_{\mathbb{C}}[\Lambda^m \underline{u}, F]$ is of the form $\mu = \gamma - \sum \alpha_i$ with γ a weight of F.

<u>NEED</u>: $\langle \gamma, \gamma \rangle \leq \langle \Lambda, \Lambda \rangle$ with "=" $\Longleftrightarrow \gamma = w\Lambda$

$\boxtimes$ THE THEOREM FOLLOWS FROM THIS $\boxtimes$

$\gamma = \Lambda - \sum m_\beta \beta$ with $m_\beta > 0$, $\beta \in \Delta^+$.

$F(\Lambda)$ f. dim'l, may assume γ dominant

$[\gamma : F(\Lambda)] = [w\gamma : F(\Lambda)] \quad \forall\, w \in W$

$$\langle \gamma, \gamma \rangle = \langle \Lambda - \sum m_\beta \beta, \Lambda - \sum m_\beta \beta \rangle = \langle \Lambda, \Lambda \rangle - \underbrace{\langle \sum m_\beta \beta, \Lambda + \gamma \rangle}_{\geq 0}$$