FROM LAST TIME: (1) $L^2(G/\Gamma) = \bigoplus m_\pi \mathcal{H}_\pi$

with $m_\pi < \infty$

$$\operatorname{tr} \Pi(\varphi) = \sum m_\pi \operatorname{tr} \pi(\varphi)$$

(2) $$\operatorname{tr} \Pi(\varphi) = \sum_{\{\Gamma\}} \operatorname{vol}(G_\gamma/\Gamma_\gamma) \int_{G/G_\gamma} \varphi(x\gamma x^{-1})\, dx$$

PROBLEM(S) . convergence.

Example: $G = SL(2,\mathbb{R})$.

$$\gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} -e^t & 0 \\ 0 & -e^{-t} \end{pmatrix}, \begin{pmatrix} \pm 1 & \pm 1 \\ 0 & \pm 1 \end{pmatrix}$$

$t > 0$

elliptic hyperbolic unipotent

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

$$p_\lambda(g) = \det(\lambda I - g)$$

$$= \lambda^2 - (a+d)\lambda + 1$$

$\Delta = (a+d)^2 - 4$

< 0 elliptic

$= 0$ unipotent

> 0 hyperbolic

Want to express $\displaystyle\int_{G/G_\gamma} \varphi(x\gamma x^{-1})\,dx$

in terms of $\operatorname{tr}\pi$; both are distributions...

The fact that this is possible is H-C's Plancherel Formula.

For $SL(2)$ see Lang ...

Introduction to Arthur's Notes

3-8-3

Universal Enveloping Algebra

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) \Big/ \{x \otimes y - y \otimes x - [x,y]\}.$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\iota"'] & \mathcal{A} \\ \mathcal{U}(\mathfrak{g}) \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

Lie algebra homomorphism

$\tilde{\pi}$ algebra hom $1 \to 1$

$$\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$$

<u>PBW:</u> $X_1 \ldots X_n \in \mathfrak{g}$ an ordered basis. Then $\{X_1^{m_1} \cdots X_n^{m_n}\}$ is a basis for $\mathfrak{g}$

$\mathcal{U}(\mathfrak{g})$ has a filtration $\mathbb{C} = \mathcal{U}_0(\mathfrak{g}) \subset \mathcal{U}_1(\mathfrak{g}) = \mathfrak{g} \subset \mathcal{U}_2(\mathfrak{g}) \subset \cdots \subset \mathcal{U}_n(\mathfrak{g}) \subset$

$$\mathcal{U}_n \cdot \mathcal{U}_m \subset \mathcal{U}_{m+n}$$

$X \in \mathcal{U}_n$, $X \cdot u - u \cdot X \in \mathcal{U}_{n-1}$ $\forall X \in \mathfrak{g}$. $uv - vu \in \mathcal{U}_{m+n-1}$

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$$

$$\mathrm{ad}X(X_1 \cdots X_n) = X X_1 \cdots X_n - X_1 \cdots X_n \cdot X$$

$$= \sum X_1 \cdots [X, X_i] \cdots X_n$$

$Gr(\mathcal{U}) := \bigoplus \mathcal{U}_n / \mathcal{U}_{n-1}$ an abelian algebra

PBW $\iff Gr(\mathcal{U}) \simeq S(\mathfrak{g})$ as a graded algebra.

Inverse: $x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$

THM $\mathcal{U}(\mathfrak{g}) \simeq$ Algebra of left (right) invariant differential operators.

Example: $\mathfrak{g} = sl(2, \mathbb{C})$ generated by e, h, f

Ω $h^2 + 4ef \in S^2(\mathfrak{g})$

$$\operatorname{ad} h\, \Omega = 0$$

$$\operatorname{ad} e\, \Omega = [e,h]h + h[e,h] + 4[e,e]f + 4e[e,f]$$

$$= -2eh - 2eh + 4eh = 0.$$

$$\Lambda(\Omega) = \tfrac{1}{2} 2h^2 + 4 \cdot \tfrac{1}{2}(ef + fe) = h^2 + 2(ef + fe)$$

$$= h^2 + 2(fe + h + fe) = h^2 + 2h + 4fe$$

$$= h^2 - 2h + 4ef$$

We are interested in $\mathcal{U}(\mathfrak{g})^G$.

Assume G connected $\mathcal{U}(\mathfrak{g})^G \simeq \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$.

More General Case: $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ Cartan decomposition

coming from $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$.

Interested in $S(\mathfrak{s})^K \simeq S(\mathfrak{s})^{\mathfrak{k}}$ $(K = K^\circ)$

$\mathfrak{g}_0$ cx $\mathfrak{g}_0 = \mathfrak{u}_0 + i\mathfrak{u}_0$ and $(i\mathfrak{u}_0)_{\mathbb{C}} \simeq \mathfrak{g}$ etc.

STEP 1 $S(\mathfrak{s}) \simeq P(\mathfrak{s})$ via $(\cdot,\cdot)$ a positive definite K-invariant form.

Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a maximal abelian subalgebra.

STEP 2: $K \times \mathfrak{a} \xrightarrow{\sim} \mathfrak{s}$ $\quad (k, h) \longmapsto \mathrm{Ad}\,k(h)$.

Let $W := N_K(\mathfrak{a}) / C_K(\mathfrak{a})$ a finite group which acts on $\mathfrak{a}$.

$$P(\mathfrak{s})^K \xrightarrow{\quad Res \quad} P(\mathfrak{a})^W$$

THM: Res is an isomorphism.

Example: $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathfrak{so}(n) + \mathfrak{s}$
$= \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) + \mathfrak{s}$

where $\mathfrak{s} :=$ symmetric matrices (trace 0).

From Linear Algebra:

any symmetric matrix is diagonalizable by a (special) orthogonal matrix

So $\ddot{Res}$ is an injection by STEP2.

For $sl(n, \mathbb{R})$ we can finish the proof as follows:

$$X \in \mathfrak{g} \longmapsto \det(tI - X) = \sum_{i=0}^{n} t^i P_i(X) = t^n + \cdots + P_0(X)$$

The P_i are clearly invariant under K.

$P(\mathfrak{a})^K$ is symmetric polynomials on $\mathbb{R}^n \left(\mathbb{R}^{n-1} \cdots\right)$

This is a polynomial algebra with basis

$$x_1 + \cdots + x_n, \ \sum x_i x_j, \ \sum x_i x_j x_k, \ \text{etc.}$$

$$\sum x_i^2, \ \sum x_i^3 \quad \text{etc.}$$

If $X \sim \operatorname{diag}(a_1, \ldots, a_n)$ then $\det(tI - X)\big|_{\mathfrak{a}} = \prod (t - a_m)$

$P_i(X) = \pm \sum a_{m_1} \cdots a_{m_i}$,

So the map is onto

This does not quite work for an arbitrarily semisimple $\mathfrak{g}$; even a classical one

Example $\mathfrak{g} = so(4, \mathbb{C})$ $X = \left[\begin{array}{c|c} A & B \\ \hline C & -{}^{t}A \end{array}\right]$

with B, C skew symmetric. The bilinear form comes from $Q(x_1\ x_2\ x_3\ x_4) = x_1 x_2 + x_3 x_4$

X diagonalizes to $\mathrm{diag}(\alpha_1\ \alpha_2\ -\alpha_1\ -\alpha_2)$.

$\det(tI - X) = t^4 - (\alpha_1^2 + \alpha_2^2)t^2 + \alpha_1^2\alpha_2^2$

But $\alpha_1\alpha_2$ is an invariant.

W = permutations & even sign changes of (x_1, x_2).

Fix:

- Can add $\det(tI - \pi(X)) = \sum_i P_{i,\pi}(X) t^i$ the polynomials in $P(\mathfrak{s})^K$.
- Or go through the proof in the notes (from Helgason)

3-8-8

A theorem of Chevalley: (Warner) ch 2

Finite Reflection Groups

DEF: V a vector space $r: V \to V$ is a reflection if

(1) $r^2 = Id$

(2) r leaves a hyperplane fixed

$W \subset Aut(V)$ is a finite reflection gp if it is finite & generated by reflections.

THM: If $W \curvearrowright V$ is a finite reflection gp,

$P(V)^W$ is a polynomial algebra generated by $n = \dim V$ lin. indep. homogeneous polynomials.

$P(\mathfrak{a})^W$ is of this type.

Admissible Modules

(π, X) a $(\mathfrak{g}, K)$-module; finitely generated as a $\mathcal{U}(\mathfrak{g})$-module.

X_0 f. dim'l K-module, $\mathcal{U}(\mathfrak{g}) X_0 = X$.

Define $X_n := \mathcal{U}_n \cdot X_0$.

$\mathfrak{g} \cdot X_n = X_{n+1}$

$gr(X) := \bigoplus X_n / X_{n-1}$ is a $gr\,\mathcal{U}(\mathfrak{g}) \simeq S(\mathfrak{g})$ graded module. In fact $\mathfrak{k}$ acts by 0; so this is an $S(\mathfrak{g}/\mathfrak{k}) \simeq S(\mathfrak{s})$-module.

- But K still acts, as K-modules,

$$X \simeq gr(X).$$

$$X = \bigoplus_{\xi \in \hat{K}} X(\xi)$$

- $\mathcal{U}(\mathfrak{g})^K$ preserves each $X(\xi)$.
- $Z := \mathcal{U}(\mathfrak{g})^G \hookrightarrow \mathcal{U}(\mathfrak{g})^K$ therefore acts on each $X(\xi)$.

3-8-10

<u>Thm</u> If (π, X) is finitely generated, then $X(\xi)$ is finitely generated as a $\mathcal{Z}$-module.

<u>Corollary.</u> If (π, X) is finitely generated and locally $\mathcal{Z}$-finite, then it is admissible.

<u>Pf:</u> By the theorem, $X(\xi)$ is finitely generated as a $\mathcal{Z}$-module; say by $e_1 \ldots e_n$. But $\dim \mathcal{Z}e_i < \infty$.

<u>Pf of theorem:</u>

$gr(\mathcal{Z})$ maps to $S(\mathfrak{g})^{\mathfrak{g}} \simeq P(\mathfrak{g})^{\mathfrak{g}}$. Since $\mathfrak{k}$ acts trivially, $z \cdot v = \mathrm{Res}_{\mathfrak{s}}\, z \cdot v$.

<u>CLAIM:</u> $P(\mathfrak{s})^{K}$ is finitely generated as a $\mathrm{Res}_{\mathfrak{s}}\, P(\mathfrak{g})^{G}$ module.

<u>Using this:</u> $X(\xi) =$ image of $\mathrm{Hom}_{\mathbb{C}}[V_\xi, X] \otimes V_\xi$ under $T \otimes v \longmapsto T(v)$

- $\mathrm{Hom}_{\mathbb{C}}[V_\xi, X]$ is a K-module and an $S(\mathfrak{s})$-module

$\because \dim V_\xi < \infty$ 3-8-11

$S(\mathfrak{p})$ acts on X, so $\mathrm{Hom}_{\mathfrak{k}}[V_\xi, X]$ is f.g.

Then $\mathrm{Hom}_K[V_\xi, X]$ maps onto $X(\xi)$, and because $S(\mathfrak{p})$ is Noetherian, $S(\mathfrak{p})\cdot \mathrm{Hom}_K[V_\xi, X]$ is f.g.

$\exists\, T_1, \dots, T_d \ni$ they generate $\nearrow$.

If $T \in \mathrm{Hom}_K[V_\xi, X]$ $\exists\, p_1, \dots, p_d \ni T = \sum p_i T_i$

Then $T = \sum \left(\int \mathrm{Ad}\,k \cdot p_i \, dk \right) \cdot T_i$ & so $T_1, \dots, T_d$ generate $\mathrm{Hom}_K[V_\xi, X]$.

<u>Proof of the CLAIM:</u>

$$X \in \mathfrak{g} \longmapsto \det(tI + X) = \sum t^j q_j(X) \qquad \to S(\mathfrak{g})^{\mathfrak{g}}$$

Let $\beta \in \Delta(\mathfrak{g}, \mathfrak{a})$. Then $\sum \beta^j q_j|_{\mathfrak{a}} = 0$

Thus $P(\mathfrak{a})$ is f.g. under $\mathrm{Res}\, S(\mathfrak{g})^{\mathfrak{g}}$.

To finish the proof, need to compare

$$X_n \subset X_{n+1} \cdots$$

with $gr(X)$. Omitted.