4_12_2016

Note Title 4/10/2016

PRINCIPAL SERIES

- $\underline{g} = \underline{k} + \underline{p}$ Cartan decomposition $\to \Theta$
- $\underline{g} = \underline{k} + \underline{\alpha}_0 + \underline{n}_0$ Iwasawa decomposition

Parabolic Subgroup / Algebra

$\underline{m}_0 := C_{\underline{k}}(\alpha_0) \subseteq \underline{k}$.

$P_0 = \underline{m}_0 + \alpha_0 + \underline{n}_0$ minimal parabolic subalgebra

$P_0 = M_0 A_0 N_0$ normalizer of P_0 in G minimal parabolic subgroup

(M_0 is compact)

$P = MAN \supseteq P_0$ called parabolic subgp.

$p = \mathcal{L}(P) = \underline{m} + \underline{a} + \underline{n}$

$\underline{m} \supseteq \underline{m}_0$, $\underline{\alpha} \subseteq \underline{a}_0$ and $\underline{n} \subseteq n_0$

$\underline{m} + \underline{\alpha} = C_g(\underline{\alpha})$ in fact $\alpha = Z_g(\underline{m} + \underline{\alpha})$.

$\Delta = \Delta(g / \underline{m}_0 + \underline{\alpha}_0, \alpha_0)$ nonzero roots of α_0 in g

$\tilde{\Delta} = \{\alpha \in \Delta \mid \frac{1}{2}\alpha \notin \Delta\}$ reduced roots

$M' = N_K(A)$, $W \simeq M'/M$ Weyl Group

$\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ $\quad\Delta^+ \subset \Delta$ a positive system $\leftrightarrow \underline{n}$

$\hat{A}_0 \longleftrightarrow \alpha_0^*$ $\quad \nu \in \alpha_0^* \longleftrightarrow \nu(a) = e^{\nu(\log a)}$ $(= a^\nu)$

W acts on $\hat{M}_0$, $\hat{A}_0$ the obvious way:

$w.\delta(m) = \delta(x^{-1} m x)$

DEF: $(\delta, V_\delta) \in \hat{M}$, $\nu \in \hat{A}$

$$I(P,\delta,\nu) = \left\{ f : G \to V_\delta \;\middle|\; \begin{array}{l} f|_K \in L^2(K) \\ f(gman) = a^{-(\nu+\rho)} \delta(m^{-1}) f(g) \end{array} \right\}$$

$$\|f\|^2 := \int_K \|f(k)\|^2 \, dk$$

Action of G: $\pi_{P,\delta,\nu}(g) \cdot f(x) = f(g^{-1}x)$

$I(P,\delta,\nu)^\infty$, $I(P,\delta,\nu)_K$ obvious subspaces

PROPERTIES ($P = \bar{P}_0$ and $\dim \delta < \infty$)

1. $I(P_0,\delta,\nu)$ is admissible
   Finite Composition Series

2. $I(P_0,\delta,\nu)$ & $I(P_0,\delta',\nu')$ have same composition factors $\iff \exists w \in W \ni \left\{ \begin{array}{l} w\delta = \delta' \\ w\nu = \nu' \end{array} \right\}$

3. $I(\delta) = \{ f|_K : f \in I(P,\delta,\nu) \}$

   $[\mu : I(P,\delta,\nu)] = [\mu : I(\delta)] \le \dim \mu$

Proof of 3: $L^2(K) = \sum V_\mu \otimes V_\mu^*$

$I(\delta) \simeq \sum V_\mu \otimes [\delta : V_\mu^*/M]$

So $I(P,\delta,\nu)$ has a distibution character ?

This accounts for 2.

Infinitesimal Character

$\mathfrak{t}_0 \subseteq \mathfrak{m}_0$ CSA; then $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ is a CSA of $\mathfrak{g}$

$\Delta^+(\mathfrak{m}_0, \mathfrak{t}_0)$ positive system. Choose

$\Delta(\mathfrak{g}, \mathfrak{h}_0) = \{ \alpha : \alpha \in \Delta^+(\mathfrak{m}_0, \mathfrak{t}_0) \text{ or } \alpha|_{\mathfrak{a}_0} \in \Delta^+ \}$

λ_0 highest weight of δ. Then

$$\chi(I(P,\delta,\nu)) = (\lambda_0 + \rho(\Delta^+(\underline{m},\underline{t}_0), \nu).$$

Pf: $\gamma : Z(\underline{g}) \longrightarrow Z(\underline{m}_0 + \underline{a}_0)$

restriction $\perp$ to $\quad U(\underline{g})\underline{m}_0$.

Then $z \cdot f = \gamma(z) \cdot f = \xi_{\underline{g}}^{\underline{m}_0 + \underline{a}_0}(\gamma(z))(\lambda, \nu+\rho) \cdot f$

so $\quad z \cdot f = \xi(z)(\lambda, \nu)$

SUBQUOTIENT THEOREM (H-C)

Every irreducible admissible rep'n is a subquotient of an $I(P,\delta,\nu)$.

($P = P_0$ the minimal parabolic subgp)

The proof is somewhat involved. We are trying to prove this as a consequence of a more general result $\square$

FROBENIUS RECIPROCITY

$(\pi, \mathcal{H})$ a $(\underline{g}, K)$-module

$$\mathrm{Hom}_{\underline{g}}[\mathcal{H}, I(P,\delta,\nu)] \simeq \mathrm{Hom}_{\underline{m}_0 + \underline{a}_0}[\mathcal{H}/\underline{n}\mathcal{H}, \delta \otimes e^{(\nu+\rho)}]$$

"Pf": $\Phi : \mathcal{H} \longrightarrow I(P,\delta,\nu)$

gives $\Phi(x) = f_x : G \longrightarrow V_\delta$

$$\Phi(\pi(g)x) = f_{\pi(g)x} = (L_g f)$$

$$\Phi \longmapsto \varphi_\Phi : \mathcal{H} \longrightarrow V_\delta \otimes e^{(\nu+\rho)}$$

$\varphi_\Phi(x) := f_x(1)$.

$$\varphi_\Phi(\pi(X)x) = \frac{d}{dt}\Big|_{t=0} f\big(\pi(\exp(-tX))\big) = 0$$

So

1. $\varphi_\Phi : \mathcal{X}/\underline{n}\mathcal{X} \longrightarrow V_\delta \otimes e^{\cdot(\nu+\rho)\cdot}$

2. $\varphi_\Phi\big(\pi(ma)x\big) = f_x(a^{-1}m^{-1}) = \delta(m)e^{(\nu+\rho)}(a)$

In Particular, Recall

$H^0(\underline{n}, \mathcal{X}) \simeq \mathcal{X}/\underline{n}\mathcal{X}$

So if we can find a $\delta \otimes e^{(\nu+\rho)}$ in $H^0(\underline{n}, \mathcal{X})$, we will have proved the stronger SUBREPRESENTATION THM

<u>REMARK</u>: $H^n(\underline{n}, \mathcal{X})$ with $n = \dim \underline{n}$ is related to $H^0(\underline{n}, \mathcal{X})$, and this might be more suited for generalizations

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$$\mathrm{Hom}[\Lambda^{n-1}\underline{n}, \mathcal{X}] \xrightarrow{d} \mathrm{Hom}[\Lambda^n \underline{n}, \mathcal{X}]$$

Let $x_0, x_1, \dots, x_{n-1}$ be a basis for $\underline{n}$

$\ni$ ad $\underline{n}$ acts strictly upper triangular

$\omega_i = x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{n-1}$ is a basis for $\Lambda^{n-1}\underline{n}$. $f_{i,v}(\omega_j) := \delta_{ij} v$ generate $\mathrm{Hom}[\Lambda^{n-1}\underline{n}, \mathcal{X}]$

$$df_{i,x}(x_0 \wedge \cdots \wedge x_{n-1}) = (-1)^i x_i \cdot v$$

So $H^n(\underline{n}, \mathcal{X}) \simeq \{ f_{\bar v}(x_0 \wedge \cdots \wedge x_{n-1}) = \bar v \in \mathcal{X}/\underline{n}\mathcal{X} \}$

The map $f_{\bar{v}} \longmapsto \bar{v} \otimes (x_0 \wedge \cdots \wedge x_{n-1})$ is an isomorphism

$$H^n(\underline{n}, \mathcal{H}) \simeq H^0(\underline{n}, \mathcal{H}) \otimes \bigwedge^n(\underline{n}).$$

Only need to worry about action of $\underline{\dot{m}} + \underline{a}$