FROBENIUS RECIPROCITY: $(\pi, \mathcal{X})$ $(\mathfrak{g}, K)$-module

$$I(P,\delta,\nu) = \mathrm{Ind}_P^G [\delta \otimes \mathbb{C}_\nu]$$

$$= \{ f: G \to V_\delta \mid f(gman) = \delta(m^{-1}) a^{-\nu-\rho} f(g) \}$$

"Take K-finite Vectors"

$$\mathrm{Hom}_{\mathfrak{g}}[\mathcal{X}, I(P,\delta,\nu)] \simeq \mathrm{Hom}_{(\mathfrak{m}+\mathfrak{a}, M)} [\mathcal{X}/\underline{n}\mathcal{X}, \delta \otimes e^{\nu+\rho}]$$

"Pf": $\varphi \longmapsto F_\varphi(x) := \varphi(x)(\mathbb{1})$

function $G \to V_\delta$

$$\varphi(\pi(g)x) = L_g \varphi(x)$$

$$L_g \varphi(x)(h) = \varphi(x)(g^{-1}h).$$

$$X \in \underline{n}, \quad F_\varphi(\pi(X)x) = \left.\frac{d}{dt}\right|_{t=0} \varphi(x)(\exp -tX) = 0.$$

$$m \in M \quad F_\varphi(\pi(m)x) = L_m \varphi(x)(1) = \varphi(x)(m^{-1}) =$$

$$= \delta(m)\varphi(x)(e)$$

$H \in \mathfrak{a}$ ... $(\nu+\rho)(H)\varphi(x)(e)$

<u>Exercise:</u> Show this is an $\simeq$.

<u>Infinitesimal Character of $I(P,\delta,\nu)$</u>

$\underline{t}_0 \subseteq \underline{m}_0$ CSA for $\underline{m}_0$, $t_0 + \alpha_0 = \mathfrak{h}_0$ CSA for $\mathfrak{g}_0$.

$\Delta^+(\underline{m}, t)$ choice of positive system.

$\Delta^+(\mathfrak{g}, \mathfrak{h}) := \{\alpha \mid \alpha \in \Delta(\underline{m}, t) \text{ or } \alpha|_{\alpha_0} \in \Delta^+\}$

λ_0 highest weight of δ

PROPOSITION: $\chi_{I(P,\delta,\nu)} = (\lambda + \rho(\Delta^+(\underline{m}, t), \nu)$.

<u>Proof:</u> $\gamma : Z(\mathfrak{g}) \to Z(\underline{m} + \underline{\alpha} = \mathfrak{h})$ is the restriction of z to $\mathcal{U}(\mathfrak{h}) \perp$ to $\mathcal{U}(\mathfrak{g}) \cdot \underline{n}$.

Then $z \cdot f = \gamma(z) \cdot f = \xi^{m+\alpha}(\gamma(z))(\lambda, \nu + \rho) f$

where ρ is $\rho(\Delta^+)$ for the "restricted roots".

$\therefore \quad z \cdot f = \xi^{\mathfrak{g}}(z)(\lambda, \nu) f \quad \boxtimes$

$\underline{u}$ nilradical of $\underline{q} = \underline{l} + \underline{u} \supseteq \underline{h} + \underline{n}$

$\mathrm{ad}: u \longrightarrow \mathrm{End}(\underline{u})$, $\mathrm{ad}\,x$ is nilpotent.

$\exists$ a basis $x_0, x_1, \ldots, x_{n-1}$ $\quad n = \dim \underline{u}$

$\ni \quad [x_i, x_j] = \sum_{k > \max(i,j)} c_{ij}^k x_k$

$f: \Lambda^{n-1}\underline{u} \longrightarrow X$ of the form $f_{i,v}(x_0 \wedge \cdots \hat{x}_j \cdots x_{n-1})$

$= \delta_{i,j} v$, then $f([x_i, x_j] \wedge x_0 \cdots \wedge \hat{x}_i \cdots \wedge \hat{x}_j \wedge \cdots x_{n-1}) = 0$

So $df_{i,v}(x_0 \wedge \ldots \wedge x_{n-1}) = \sum (-1)^k f_{i,v}(x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{n-1})$

$= (-1)^i v$

So $H^n(\underline{n}, \mathcal{X}) \simeq \mathcal{X}/\underline{n}\mathcal{X}$ as vector spaces.

<u>Question</u>: How does $\underline{l}$ act?

<u>ANSWER</u>: On $\mathrm{Hom}_{\mathbb{C}}[\Lambda^i \underline{u}, \mathcal{X}]$

$(H \cdot \psi)(\varpi) = \psi(-H \cdot \varpi) + \pi(H) \cdot \psi(\varpi)$

Note that $x_0, \ldots, x_{n-1}$ are root vectors. So

on $\Lambda^n \underline{u}$, get $-2\rho(H) = -\sum_{\alpha \in \Delta(\underline{u})} \alpha(H)$

FROM 4-5-2016 (not covered at that time)

$\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supseteq \mathfrak{g} + \underline{\mathfrak{n}}$ $\mathfrak{g} = \mathfrak{h} + \mathfrak{a}$ but this time

$\mathfrak{q}$ is θ-stable & $\mathfrak{g}$ is fundamental.

$\underline{\mathfrak{u}} = (\mathfrak{u} \cap \mathfrak{k}) + (\mathfrak{u} \cap \mathfrak{p})$

$S = \dim(\mathfrak{u} \cap \mathfrak{k})$ $R = \dim(\mathfrak{u} \cap \mathfrak{p})$.

n arbitrary (<u>not</u> dimension of $\underline{\mathfrak{u}}$)

$$\Lambda^n(\underline{\mathfrak{u}}) = \bigoplus_{I+J=n} \Lambda^I(\mathfrak{u} \cap \mathfrak{k}) \otimes \Lambda^J(\mathfrak{u} \cap \mathfrak{p}).$$

$$\operatorname{Hom}_{\mathbb{C}}\left[\Lambda^n \underline{\mathfrak{u}}, \mathcal{X}\right] = \sum_{I+J=n} \operatorname{Hom}\left[\Lambda^I(\mathfrak{u} \cap \mathfrak{k}), X\right] \otimes \Lambda^J(\mathfrak{u} \cap \mathfrak{p})^*$$

The differential d <u>does not</u> respect this decomposition

<u>FILTER $\Lambda^*(\mathfrak{u} \cap \mathfrak{p})$ with $(\mathfrak{l} \cap \mathfrak{k})$-modules</u>

$$\Lambda^*(\underline{\mathfrak{u}} \cap \mathfrak{p}) = \bigoplus_{0 \le a \le N} V_a \longleftarrow (\mathfrak{l} \cap \underline{\mathfrak{k}})\text{-modules}$$

(a) $V_0 = \Lambda^0(\mathfrak{u} \cap \mathfrak{p})$ $V_N = \Lambda^R(\mathfrak{u} \cap \mathfrak{p})$

(b) $V_a \subset \Lambda^{r(a)}(\mathfrak{u} \cap \mathfrak{p})$, $a \le a' \Longrightarrow r(a) \le r(a')$

(c) $\operatorname{ad}(\mathfrak{u} \cap \mathfrak{k}) V_a \subseteq \operatorname{span}\{V_0, \dots, V_{a'}\}$ $a' < a$.

(The reverse ordering from before)

$\mathfrak{q} = \mathfrak{l} + \underline{\mathfrak{u}} \longleftrightarrow \lambda \in \mathfrak{t}_0^*$ can be used to achieve this.

Let $Q \in \Lambda^{n-r(a)}(\underline{\mathfrak{u}} \cap \underline{k})$, $P \in V_a \subseteq \Lambda^{r(a)} \mathfrak{u} \cap \mathfrak{p}$

COMPUTE $df(Q \wedge P)$: $\quad Q = q_1 \wedge \cdots \wedge q_{n-r(a)}$, $P = P_1 \wedge \cdots \wedge P_{r(a)}$

$$\left.\begin{array}{l} \text{(i) } \sum (-1)^i q_i f(\imath(q_i) Q \wedge P) + \\ \text{(ii) } \sum (-1)^{i+j} f([q_i, q_j] \wedge \imath(q_i, q_j) Q \wedge P) \end{array}\right\} d_{\underline{\mathfrak{u}} \cap k}$$

(iii) $\sum (-1)^i P_i \cdot f(Q \wedge \imath(P_i) P) +$

(iv) $\sum (-1)^{i+j} f([P_i, P_j] \wedge Q \wedge \imath(P_i, P_j) P)$

(v) $\sum (-1)^{i+j} f([q_i, P_j] \wedge \imath(q_i) Q \wedge \imath(P_j) P)$.

Let $f \in \mathrm{Hom}[\Lambda^{I+R} \underline{\mathfrak{u}}, X]$ 0 on all terms except $\Lambda^I(\mathfrak{u} \cap k) \otimes \Lambda^R(\mathfrak{u} \cap \mathfrak{p})$

FOR df:

(i) & (ii) are $d_{\underline{\mathfrak{u}} \cap k} f$.

(iii), (iv), (v) are all 0.

(iii) $Q \wedge \imath(P_i) P \in \Lambda^{I+?}(\underline{\mathfrak{u}} \cap \underline{k}) \otimes \Lambda^{R-I-?}(\underline{\mathfrak{u}} \cap \mathfrak{p})$ with $? \geq 1$

(iv) same argument

(v) $[q_i, P_j] \wedge \imath(P_j) P$ is zero when in degree R.

So $df=0 \iff d_{\mathfrak{u}\cap\mathfrak{k}} f = 0$. Get a map

$$\pi_I : H^I(\underline{u}\cap\underline{k}, X) \otimes \Lambda^R(\mathfrak{u}\cap\mathfrak{p})^* \longrightarrow H^{I+R}(\underline{u}, X)$$

SPECIAL CASE: $I=0$.

LHS has highest weight $\mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})$, $\mu \in \hat{K}$ from Kostant's Theorem

$$\pi_I \neq 0 \implies H^{I+R}(\underline{u}, X)^{\mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})} \neq 0.$$

We can push this further; make a filtration

$$A^n := \mathrm{Hom}[\Lambda^n \underline{u}, X], \quad A = \bigoplus A^n$$

$$A_a^n = \{ f(Q \wedge P) = 0 \ \forall Q \in \Lambda^*(\mathfrak{u}\cap\mathfrak{k}),\ P \in V_{a'}\ a' < a \}$$

FACT: d preserves the filtration

DEFINITION: $\mu \in \hat{K}$ is called strongly $\underline{u}$-minimal IF

$$[H^J(\underline{u}\cap\underline{k}, \gamma) \otimes H^{J'}(\mathfrak{u}\cap\mathfrak{p})^*]^{\mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})} \neq 0 \implies X(\gamma) = 0$$

THM: If $\mu \in \hat{K}$ is strongly $\underline{u}$-minimal,

$$\pi_I : H^0(\underline{u}\cap\underline{k}, X)^{\mu} \otimes \Lambda^R(\mathfrak{u}\cap\mathfrak{p})^* \xrightarrow{\sim} H^R(\underline{u}, X)^{\mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})}$$