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Note Title 4/17/2016

STRUCTURE AND CLASSIFICATION OF REAL CSA'S

Recall the fundamental CSA:

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \qquad \mathfrak{t}_0 \subseteq \mathfrak{k}_0$ CSA

$\Delta^+(\mathfrak{k}_0, \mathfrak{t}_0)$. $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{t} + \mathfrak{a}$

a CSA of $\mathfrak{g}$ (all complexified)

$\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ a noncompact imaginary root; $X_\beta \in \mathfrak{p}$ and $\beta/_{\mathfrak{a}} = 0$

Then $\overline{X}_\beta$ (conjugation w.r.to $\mathfrak{g}_0$) is the root vector for $-\beta$

$Z_\beta := X_\beta + \overline{X}_\beta \in \mathfrak{p}_0$

$\mathfrak{g}_0^\beta := C_{\mathfrak{g}_0}(Z_\beta)$. $\theta Z_\beta = -Z_\beta$ so

$\mathfrak{g}_0^\beta$ is θ-stable as well, and reductive. $G_0^\beta \longleftrightarrow \mathcal{L}(G_0^\beta) = \mathfrak{g}_0^\beta$ is in the same class as G.

$$\mathfrak{h}^\perp := \{x \in \mathfrak{h} \mid \beta(x) = 0\} \subseteq \mathfrak{g}^\beta$$

is a CSA.

$\mathfrak{h}_0^\beta := \mathfrak{h}^\perp + \mathbb{C} Z_\beta$

LEMMA: • $\mathfrak{g}_0^\beta$ is a fundamental CSA of $\mathfrak{g}_0^\beta$

• $\Delta(\mathfrak{g}_0^\beta, \mathfrak{h}_0^\beta) \simeq \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \alpha, \beta \rangle = 0\}$

cx, im $\longleftrightarrow$ cx, im

$\alpha \perp \beta$ cpct $\longleftrightarrow$ $\tilde{\alpha}$ cpct $\Longleftrightarrow$ $\alpha \pm \beta$ not roots

$\alpha \perp \beta$ nc $\longleftrightarrow$ $\tilde{\alpha}$ nc $\Longleftrightarrow$ $\alpha \pm \beta$ not roots

Pf by example:

$\mathfrak{g}_0 = sp(4, \mathbb{R}) \supseteq \mathfrak{h}_0$ cpct CSA

β short $\Longleftrightarrow$ $\varepsilon_1 - \varepsilon_2$ cpct, $\varepsilon_1 + \varepsilon_2$ nc

β long $\Longleftrightarrow$ nc. $2\varepsilon_1, 2\varepsilon_2$

MAIN THEOREM

$\mu \in \hat{T}$ (cx torus) a highest weight

$\Delta^+(\mathfrak{g}, \mathfrak{h})$ fixed positive system

$\ni \mu + 2\rho_c$ dominant for Δ^+

There exists a unique $\lambda \in \mathfrak{h}^*$ such that

• λ is dominant for Δ^+

• $\exists\ \beta_1, \ldots, \beta_r \in \Delta^+$ ~~nc~~ imaginary such that

(a) $0 \le c_i = -\langle \check{\beta}_i, \mu + 2\rho_c - \rho \rangle \le 1$

& $\lambda = \mu + 2\rho_c - \rho + \frac{1}{2}\sum c_i \beta_i$

(b) $\alpha \in \Delta$ imaginary & $\langle \alpha, \lambda \rangle = 0$, then $\langle \alpha, \beta_i \rangle \neq 0$ for some i

(c) β_1 is n.c. im. simple or $\exists$ cx simple $\alpha \in \Delta^+ \ni \beta_1 = \alpha + \theta\alpha$

(d) $\mathfrak{g}^1 := \mathfrak{g}^{\beta_1}$, $\mathfrak{h}^1 := \mathfrak{h}^{\beta_1}$. Then $\Delta^+(\mathfrak{g}, \mathfrak{h}) \cap \beta_1^\perp$ & $\{\beta_2, \dots, \beta_r\}$ satisfy (a)–(c) for $\mu|_{\mathfrak{g}^1 \cap \mathfrak{k}}$

(e) $c_i \neq 0$ & $c_j = 0 \Rightarrow i < j$

<u>Example</u>: $\mathfrak{g}_0 = sp(4, \mathbb{R}) \supseteq \mathfrak{k}_0 = u(2)$

$\mu = (a_1\ a_2)$, $2\rho_c = (1\ -1)$ $\quad a_1 \ge a_2$

$\mu + 2\rho_c = (a_1 + 1 \underset{>}{,} a_2 - 1)$

$a_1, a_2 \in \mathbb{Z}$

FOUR CHOICES OF Δ^+

(I) $\{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$ $\quad \rho = (2, 1)$

(II) $\{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, -2\varepsilon_2\}$ $\quad \rho = (2, -1)$

(III) $\{\varepsilon_1 - \varepsilon_2, -\varepsilon_1 - \varepsilon_2, 2\varepsilon_1, -2\varepsilon_2\}$ $\quad \rho = (1, -2)$

$(IV)\ \{\varepsilon_1-\varepsilon_2, -\varepsilon_1-\varepsilon_2, -2\varepsilon_1, -2\varepsilon_2\}\ \rho=(-1,-2)$

$a_1+1 > a_2-1 \geq 0$ Choose (I)

$a_1 \geq a_2 \geq 1$

(Ia) $a_1-1 > a_2-2 > 0$

(Ib) $a_1-1 > a_2-2 = 0$

(Ic) $a_1-1 > 0 > a_2-2$

(Id) $0 = a_1-1 > a_2-2$

(Ia) $\lambda=(a_1-1, a_2-2)$

(Ib) $a_2=2 \quad \lambda=(a_1-1, 0)$

(Ic) $a_2=1 \quad \lambda=(a_1-1,-1)+\frac{1}{2}(0,2)$

(Id) $a_1=1 \quad a_2=1 \quad \lambda=(0\ -1)+\frac{1}{2}(0\ 2)$

$a_1+1 \geq 0 > a_2-1$ & $a_1+1 \geq 1-a_2$

Choose (II) $\mu+2\rho_c-\rho=(a_1-1, a_2)$

(IIa) $a_1-1 \geq -a_2 \geq 0 \quad \lambda=(a_1-1, a_2)$

(IIb) $\left.\begin{array}{l} -a_2 > a_1-1 > 0 \\ \mu=(a\ -a) \end{array}\right\}$ $\lambda=(a-1,-a)+\frac{1}{2}(1\,1)=$
$=(a-\frac{1}{2}, -a+\frac{1}{2})$

(IIc) $\left.\begin{array}{l} -a_2 \geq 0 \geq a_1-1 \\ a_1=0,\ a_2=0 \end{array}\right\}$ $\lambda=(-1\ 0)+(\frac{1}{2}\ \frac{1}{2})+(\frac{1}{2}\ -\frac{1}{2})$
$=(0\ 0)$