4-19-2016

Note Title 4/19/2016

FILTRATION:

$$\mathrm{Hom}[\Lambda^n \mathfrak{u}, X] = \bigoplus_{I+J=n} \mathrm{Hom}[\Lambda^I \mathfrak{u} \cap \mathfrak{k}, X] \otimes \Lambda^J(\mathfrak{u}\cap\mathfrak{p})^*$$

- $\Lambda^* \mathfrak{u}\cap\mathfrak{p} = \bigoplus_{a=0}^{N} V_a$
- $V_0 = \Lambda^0 \mathfrak{u}\cap\mathfrak{p}$, $V_N = \Lambda^R \mathfrak{u}\cap\mathfrak{p}$
- $V_a \subseteq \Lambda^{r(a)}(\mathfrak{u}\cap\mathfrak{p})$, $a \le a' \Rightarrow r(a) \le r(a')$
- $\mathrm{ad}(\mathfrak{u}\cap\mathfrak{k}) \cdot V_a \subseteq \mathrm{span}\{V_{a'};\ a' \le a\}$

$$A^n = \mathrm{Hom}[\Lambda^n \mathfrak{u}, X]$$

$$A^n_a = \{ f \in A^n ;\ f = 0 \text{ on } \Lambda^I(\mathfrak{u}\cap\mathfrak{k}) \otimes V_{a'} \}_{a' < a}$$

$$A^n_0 = A^n$$

$df(Q \wedge P)$ $\qquad P = \wedge p_j$, $Q = \wedge q_i$

$= d_{\mathfrak{u}\cap\mathfrak{k}}(Q) \wedge P$

$+ \sum (-1)^{I+j} p_j f(Q \wedge \partial(p_j) P)$ (i)

$$+ \sum_{(ii)} (-1)^{i+j+I} f([q_i, p_j]\, z(q_i) Q \wedge z(p_j) P)$$

$$+ \sum_{(iii)} (-1)^{2I+k+l} f([p_k, p_l] \wedge Q \wedge z(p_k, p_l) P)$$

$$f \in A^n_a \Rightarrow df \in A^n_a$$

(i) $z(p_j) P \in V_{a'}$ with $a' < a$

(ii) $[q_i, p_j] \wedge z(p_j) P \in V_{a'}$ $a' < a$

(iii) same as (i)

SPECIAL CASE: $P \in \Lambda^R(\mathfrak{u} \cap \mathfrak{p})$.

Then (i), (ii), (iii) are 0

SO WE GET A MAP

$$H^I(\mathfrak{u} \cap \mathfrak{k}, X) \otimes \Lambda^R(\mathfrak{u} \cap \mathfrak{p})^* \longrightarrow H^{I+R}(\mathfrak{u}, X)$$

Choose $\mu \in \hat{K}$ highest weight.

Want

$$H^0(\mathfrak{u} \cap \mathfrak{k}, X(\mu)) \otimes \Lambda^R(\mathfrak{u} \cap \mathfrak{p})^* \to H^R(\mathfrak{u}, X)$$

to be $\neq 0$.

NECESSARY CONDITION FROM SPECTRAL SEQUENCE OF A FILTRATION

$$\left[H^I(\mathfrak{u}\cap\mathfrak{k}, X(\delta))\otimes\Lambda^J(\mathfrak{u}\cap\mathfrak{p})^*; \mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})\right]$$

$\neq 0$ for som $(I,J)\neq(0,R)$

$\Longrightarrow X(\delta)=0$

SEE NOTES FROM 4-17

<u>Example</u>: $\mathfrak{g}_0 = \mathfrak{su}(1,1) \supset \mathfrak{k}_0 = \mathfrak{s}[\mathfrak{u}(1)\times\mathfrak{u}(1)]$

$$\begin{pmatrix} ix & y \\ \bar{y} & -ix \end{pmatrix} \qquad \rho_c = 0 \qquad \mu = (0,0).$$

$\mu + 2\rho_c = (0,0)$ use $\rho = \left(\frac{1}{2}\ -\frac{1}{2}\right)$

$$\mu + 2\rho_c - \rho = \left(-\frac{1}{2}\ \frac{1}{2}\right) + \left(\frac{1}{2}\ -\frac{1}{2}\right) = (0\,0) = \lambda^G$$

$Z_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so $\mathfrak{h}_0^\beta = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$ $y\in\mathbb{R}$

MORE SPLIT.

$$\mu = \left(\frac{1}{2}\ -\frac{1}{2}\right) \longmapsto \mu + 2\rho_c - \rho = (0\,0) = \lambda^G$$

$$\mu = \left(-\frac{1}{2}\ \frac{1}{2}\right) \longmapsto \mu + 2\rho_c - \rho = (0\,0) = \lambda^G$$

<u>NOTE</u> ρ is different in the last two cases. $\pm\frac{1}{2}\beta$

<u>COHOMOLOGY</u>

$\mu \in \hat{T}_0$ (connected) μ highest weight

μ dominant for $\Delta^+(\underline{k}, \underline{t})$

$\mu \longmapsto \lambda = \mu + 2\rho_c - \rho + \cancel{X} + \frac{1}{2}v$

DEFINE $\|\mu\|_{\text{lambda}} := \langle \lambda, \lambda \rangle$

We say μ is lambda-lowest for X, if $\|\mu\|_{\text{lambda}}$ is minimal for $X(\mu) \neq 0$.

(1) $\lambda - \mu = 2\rho_c - \rho + \frac{1}{2}v$ so

$\langle \lambda - \mu, \lambda - \mu \rangle \leq C_G$

So X_{lambda} is finite $\neq \phi$

(because also λ, μ belong to lattice; $c_i = -2\langle \beta_i, \mu + 2\rho_c - \rho \rangle / \langle \beta_i, \beta_i \rangle$)

PROPOSITION: Let $\mu \in \hat{K}$ be lambda lowest for X; $\mu \mapsto \lambda$ & $\underline{q} = \underline{l} + \underline{u} \longleftrightarrow \lambda$. Then

μ is strongly $\underline{u}$-minimal

$(\Longrightarrow [\mu - 2\rho(u \cap p) : H^R(\underline{u}, X)] \neq 0$