4_26_2016

Note Title 4/26/2016

RECALL:

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \quad \mathfrak{t}_0 \subseteq \mathfrak{k}_0$ CSA

$\mathfrak{h}_0 = C_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{t}_0 + \mathfrak{a}_0$ CSA

$\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ n.c. root

$Z_\beta = X_\beta + \overline{X}_\beta \quad \mathfrak{g}_0^\beta := C_{\mathfrak{g}_0}(Z_\beta)$

$\mathfrak{h}^\perp := \{x \in \mathfrak{h} \mid \beta(x) = 0\} \subseteq \mathfrak{g}^\beta$

$\mathfrak{h}^\beta := \mathfrak{h}^\perp + \mathbb{C} Z_\beta$ a new CSA

THM: $\mu \in \hat{T}$ a highest weight dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$.

$\Delta^+ \ni \mu + 2\rho_c$ is dominant

$\exists!\, \lambda \in \mathfrak{t}^* \ni \lambda$ dominant for Δ^+ &

$\exists\, \beta_1, \dots, \beta_r \in \Delta^+$ imaginary $\ni$

(a) $0 \leq c_i = -\langle \check{\beta}_i, \mu + 2\rho_c - \rho \rangle \leq 1$

$$\lambda = \mu + 2\rho_c - \rho + \frac{1}{2} \sum c_i \beta_i$$

(b) $\alpha \in \Delta_{im}, \langle \alpha, \lambda \rangle = 0 \Rightarrow \exists\, \beta_i \ni \langle \alpha, \beta_i \rangle \neq 0$

(c) β_1 nc simple or $\beta_1 = \alpha + \theta\alpha$, α Cx simple

(d) $\mathfrak{g}^1 = \mathfrak{g}^{\beta_1}$, $\mathfrak{h}^1 = \mathfrak{h}^{\beta_1}$. Then

$\Delta^+ \cap \beta_1^\perp$ & $\{\beta_2, \dots, \beta_r\}$ satisfy (a)-(c)

for $\mu / \mathfrak{g}^1 \cap \mathfrak{h}$

(e) $c_i \neq 0$ & $c_j = 0 \Longrightarrow i < j$

λ is unique but $\beta_1, \dots, \beta_r$ are <u>not</u>.

<u>Example</u> $\mathfrak{g} = \mathfrak{u}(2,1)$ $\mu = (00|0)$

$2\rho_c = (1-1|0)$, $\mu + 2\rho_c = (1-1|0)$

$\mu + 2\rho_c - \rho = (00|0) = \lambda$.

$\beta_1 = (10|-1)$ or $\beta_1 = (0-1|1)$;

<u>COROLLARY</u>: $\mu \longmapsto \lambda^G(\mu) \mapsto \mathfrak{q} = \mathfrak{l} + \mathfrak{u}$

Then $\mathfrak{l}_0$ is quasisplit

- $\lambda^L(\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})) = \mathfrak{l}$

We will need to study $\mu \ni \lambda^G(\mu) = 0$ separately.

$\mu \ni \lambda^G(\mu) = 0$ called SMALL

<u>Example</u> $\mathfrak{g} = \mathfrak{su}(1,1)$

$\mathfrak{h} = \mathrm{diag}(\theta, -\theta)$ $\Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm 2\varepsilon\}$

$2\rho_c = 0$ $\rho = \pm\varepsilon$ $\hat{K} \leftrightarrow \{n\}_{2n \in \mathbb{Z}}$

$$\mu_n(r(\theta)) = e^{2in\theta} \quad \left(r(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right)$$

$(\mathfrak{g}, K)$-modules:

$$V_{+,\nu} = \{\mu_n\}_{n \in \mathbb{Z}}$$

$$V_{-,\nu} = \{\mu_n\}_{n \notin \mathbb{Z},\, 2n \in \mathbb{Z}}$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad X_{-n} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$X_+ \cdot \mu_n = c^+_{n,\nu} \mu_{n+1}$$

$$X_- \cdot \mu_n = c^-_{n,\nu} \mu_{n-1}$$

$$c^+_{n,\nu} = \nu + \tfrac{1}{2} + n, \quad c^-_{n,\nu} = \nu + \tfrac{1}{2} - n$$

$V_{+,\nu}$ has LKT μ_0

$V_{-,\nu}$ has LKT's $\mu_{\pm\frac{1}{2}}$.

When $\nu = k - \frac{1}{2}$ $\quad k \in \mathbb{N}_+$

$$X_- \cdot \mu_k = (k - \tfrac{1}{2}) + \tfrac{1}{2} - k = 0$$

$$X_+ \cdot \mu_{-k} = (k - \tfrac{1}{2}) + \tfrac{1}{2} + (-k) = 0$$

There are two submodules

$\{\mu_{k+i}\}$ $\quad \{\mu_{-k-i}\}$ LKT's $\pm k$

For $V_{-,\nu}$ look at $\nu = k \in \mathbb{N}$

Two submodules $\{\mu_{k+i}\}, \{\mu_{-k-i}\}$.

At $\nu = 0$ 2 factors

$\nu \neq 0$ 3 factors

For $V_{+,\nu}$ 3 factors

Example: $\mathfrak{g} = \mathfrak{u}(5,3)$

$\mu = (2\,2\,0\,0\,0 \,|\, 2\,0\,0)$ $2\rho_c = (4\,2\,0\,{-2}\,{-4} \,|\, 2\,0\,{-2})$

$$\mu + 2\rho_c = (6\,4\,0\,{-2}\,{-4} \,|\, 4\,0\,{-2})$$
$$-\left(\tfrac{7}{2}\,\tfrac{5}{2}\,\tfrac{1}{2}\,{-\tfrac{3}{2}}\,{-\tfrac{7}{2}} \,\middle|\, \tfrac{3}{2}\,{-\tfrac{1}{2}}\,{-\tfrac{5}{2}}\right)$$
$$\left(\tfrac{5}{2}\,\tfrac{3}{2}\,{-\tfrac{1}{2}}\,{-\tfrac{1}{2}}\,{-\tfrac{1}{2}} \,\middle|\, \tfrac{5}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\right)$$
$$+\left(\quad \tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2} \qquad {-\tfrac{1}{2}}\,{-\tfrac{1}{2}}\,{-\tfrac{1}{2}}\right)$$
$$\left(\tfrac{5}{2}\,2\,0\,0\,{-\tfrac{1}{2}} \,\middle|\, 2\,0\,0\right)$$

$$\mathfrak{l}_0 = \mathfrak{u}(1,0) \times \mathfrak{u}(1,1) \times \mathfrak{u}(2,2) \times \mathfrak{u}(1,0)$$

$$2\rho(\mathfrak{u} \cap \mathfrak{p}) = (3\,2\,{-1}\,{-1}\,{-3} \,|\, 2\,{-1}\,{-1})$$

$$\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}) = (-1; 0; 1\,1; 3 \,|\, 0; 1\,1)$$

FINE K-TYPES

Quasisplit groups $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$

$\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ max'l abelian.

$\underline{m}_0 = C_{k_0}(\mathfrak{a}_0) \supseteq \mathfrak{t}_0$ a CSA 4-26-3

$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ a CSA of $\mathfrak{g}_0$ (most split)

$\overline{\Delta} := \{\alpha \in \Delta(\mathfrak{g}/\underline{m}+\underline{\mathfrak{a}}, \mathfrak{a}) : \tfrac{1}{2}\alpha \notin \Delta\}$

Choose a positive system $\Leftrightarrow \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \underline{n}_0$

DEF: $\mathfrak{g}$ is quasisplit if $\underline{m}_0$ is abelian

$\mathfrak{h}_0 = \underline{m}_0 + \mathfrak{a}_0$ is a CSA

- α real $:= \alpha|\underline{m}_0 = 0$; cx otherwise

- $G^d := [G:G]$ $\mathcal{L}(G^d) = [\mathfrak{g}_0, \mathfrak{g}_0]$

is semisimple. (always connected)

- G is called split if $\underline{m}_0 = 0$;

same as $\mathfrak{a}_0$ is a CSA

Examples:

- $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}), \mathfrak{sp}(2n, \mathbb{R}), \mathfrak{so}(n,n)$,

  $\mathfrak{so}(n+1, n)$ are split.

- $\mathfrak{g} = \mathfrak{u}(p,q)$ $|p-q| \le 1$ quasisplit

- $\mathfrak{g} = \mathfrak{u}(p,q)$ $|p-q| > 1$ neither

α real ($\alpha|\mathfrak{a}_0 = 0$ not an option for $\mathbb{Q}$-split groups)

$\phi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}$ which commutes

$x \mapsto -x^t$ with Θ

$\phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}_0$, $Z_\alpha := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}_0$

$\Phi_\alpha :=$ "exp" of ϕ_α

$\sigma_\alpha = \Phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $m_\alpha = \sigma_\alpha^2$, $m_\alpha^2 = 1$

$\sigma_\alpha \in M' \rightsquigarrow s_\alpha \in M'/M = W$.

These are well defined up to conjugation by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in $SL(2,\mathbb{R})$

<u>DEF</u>: $\mu \in \hat{K}$ is called fine if

(a) $\mu(iZ_\alpha) \in \{0, \pm 1\}$ $\forall \alpha$ real

(b) $\mu|_{\mathfrak{k}_0^\alpha \cap \mathfrak{g}_0^d} = 0$ $\forall \alpha$ complex

(Roots of $\Delta(\mathfrak{g}, \mathfrak{h} = \underline{\mathfrak{m}} + \underline{\mathfrak{a}})$)

<u>THM</u>: Fine K-tyes "=" Small K-types

<u>Examples</u>:

$\mathfrak{g} = sp(2n, \mathbb{R})$ $\mu = (a_1 \geq \cdots \geq a_n)$
$a_i \in \mathbb{Z}$

Fine K-types

<u>$n=1$</u> is $sl(2,\mathbb{R})$ are $\mu \pm \frac{1}{2}, 0$

$\mathfrak{a}_0 = \mathrm{diag}(t_1 \dots t_n, -t_n, \dots, -t_1)$
is a CSA (split group)

α is always real.

α short Z_α is $\varepsilon_i \pm \varepsilon_j$

α long Z_α is ε_i

So μ fine $\longleftrightarrow$ $(\underbrace{1 \dots 1}_{k}, 0 \dots 0)$

$(0 \dots 0 \underbrace{-1 \dots -1}_{k})$

$\underline{\mathfrak{g} = \mathfrak{u}(n,n)}$:

$$\left[\begin{array}{c|c} & \begin{matrix} & & t_1 \\ & \cdot^{\cdot^{\cdot}} & \\ t_n & & \end{matrix} \\ \hline \begin{matrix} & & t_n \\ & \cdot^{\cdot^{\cdot}} & \\ t_1 & & \end{matrix} & \end{array}\right] = \mathfrak{a}_0$$

$$\mathfrak{t}_0 = \left[\begin{array}{c|c} \begin{matrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{matrix} & \\ \hline & \begin{matrix} \theta_n & & \\ & \ddots & \\ & & \theta_1 \end{matrix} \end{array}\right]$$

So there are real <u>and</u> complex roots.

$\mathfrak{h} = \mathfrak{t} + \mathfrak{a} =$

$\{\theta_1 + t_1 \dots \theta_n + t_n \mid \theta_n - t_n, \dots, \theta_1 - t_1\}$

Real Roots: $\varepsilon_i - \varepsilon_{2n+1-i}$

$\longleftrightarrow 2e_i$

Cx Roots: all the rest; $e_i \pm e_j$

$\mu = (a_1 \cdots a_n \mid b_1 \cdots b_n)$

Real Roots: $|a_i + b_i| \leq 2$

Cx Roots: $a_i = b_i$