4-28-2016

Note Title 4/28/2016

RECALL: $\mu \mapsto \overset{G}{\lambda}(\mu) = \mu + 2\rho_c - \rho + \frac{1}{2}\nu$

$\mapsto \mathfrak{q} = \mathfrak{l} + \mathfrak{u}$

$(\pi, \mathcal{X})$ admissible $(\mathfrak{g}, K)$-module

$\pi \in \hat{K}$ lambda lowest for $\mathcal{X}$ if

$\|\mu\|_{lambda} = \langle \lambda(\mu), \lambda(\mu) \rangle$ is minimal

THM (a) μ lambda lowest $\Longrightarrow$

μ is strongly $\mathfrak{u}$-minimal $\square$

(b) $\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}) \mapsto \lambda^L(\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}))$

$\mapsto \mathfrak{q}^L = \mathfrak{l}$..

a K-type $\pi \ni \lambda(\pi) \mapsto \mathfrak{g}$ is called small

They only occur for $\mathfrak{g}$ quasisplit

DEF $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$

max'l abelian

NOTATION $\mathfrak{a}_0 \mapsto \underline{\mathfrak{m}}_0 = C_{\mathfrak{k}}(\mathfrak{a}_0)$

$\mathfrak{t}_0 \subseteq \underline{\mathfrak{m}}_0$ CSA $\mathfrak{h}_0^s = \mathfrak{t}_0^s + \mathfrak{a}_0^s$

max'l split CSA

$\mathfrak{h}_0 \le \mathfrak{k}_0$ CSA, $\mathfrak{g}_0^c = \mathfrak{h}_0^c + \mathfrak{a}_0^c$ A-28-2

fundamental CSA ("compact")

DEF $\mathfrak{g}_0$ is q-split if $\mathfrak{m}_0 = \mathfrak{h}_0^s$.

α called real if $\alpha|_{\mathfrak{h}_0^s} = 0$

Otherwise called complex

RECALL $\Delta = \Delta(\mathfrak{g}/\underline{\mathfrak{m}} + \underline{\mathfrak{a}}^s, \mathfrak{a}^s)$

$\overline{\Delta} = \{\alpha \in \Delta : \frac{1}{2}\alpha \notin \Delta\}$ reduced roots

If α is real, can build a

$$\varphi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}_0 \ni$$

$$\varphi_\alpha(-X^t) = \theta\varphi_\alpha(X)$$

$\varphi_\alpha\begin{pmatrix}1&0\\0&-1\end{pmatrix} \in \mathfrak{a}_0$, $\varphi_\alpha\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\varphi_\alpha\begin{pmatrix}0&0\\1&0\end{pmatrix}$ are root vectors.

$\Phi_\alpha :=$ "exp" of φ_α. It exists because $SL(2,\mathbb{C})$ is s.c.

$Z_\alpha := \varphi_\alpha\begin{pmatrix}0&1\\-1&0\end{pmatrix} \in \mathfrak{k}_0^\alpha$, $\sigma_\alpha = \Phi_\alpha\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

$\sigma_\alpha^2 =: m_\alpha$ and $m_\alpha^2 = 1$. $\sigma_\alpha \in M'$

and $\sigma_\alpha \longrightarrow s_\alpha \in M'/M = W$

These are well defined up to $\mathrm{Ad}\begin{pmatrix}-1&0\\0&1\end{pmatrix}$.

$G^d = [G:G]$, $\mathfrak{g}(G^d) = [\mathfrak{g}_0, \mathfrak{g}_0]$ s.s.

G^d is connected. 4-28-3

Sketch of Proof: $G^d = \bigcap \chi$
$\chi \in \hat{G}$ character

Show that G/G_0 is abelian

$G/G_0 = T^s/(T^s \cap G_0)$ □

DEF • $\delta \in \hat{M}$ is called fine if

$$\delta|_{(G^d \cap M)_0} = \text{triv}$$

• $\pi \in \hat{K} \rightsquigarrow$ h.wt. μ is called fine

if (a) $\mu(iZ_\alpha) = \{0, \pm 1\}$ $\forall \alpha$ real

(b) $\mu|_{\mathfrak{k}_0^\alpha \cap \mathfrak{g}_0^d} = 0$ $\forall \alpha$ cx

THM: (1) μ small $\Longleftrightarrow$ μ fine

(2) $A(\delta) := \{\mu \text{ fine} : [\mu|_M : \delta] \neq 0\}$

for δ fine

(a) $A(\delta) \neq 0$

(b) $\mu \in A(\delta) \Rightarrow \mu = \bigoplus_{w \in W} w \cdot \delta$ □

DEF: $\delta \in \hat{M}$ fine.

Good Roots: $\Delta_\delta = \{\alpha \text{ real}, \delta(m_\alpha) = 1\}$

4-28-4

<u>PROPERTY</u> $\bar{\Delta}_\delta$ is a rootsystem & $W(\bar{\Delta}_\delta)$ stabilizes δ

- $W^0_\delta = W(\bar{\Delta}_\delta)$
- $M'_\delta = \mathrm{Stab}_{M'}(\delta)$
- $W_\delta = \mathrm{Stab}_W(\delta)$
- $R_\delta = W_\delta / W^0_\delta$

<u>LEMMA</u>: $W^0_\delta \subseteq W_\delta$ is normal & R_δ is a product of $\mathbb{Z}_2$'s

(c) $\hat{R}_\delta$ acts transitively & naturally on $A(\delta)$

<u>Example</u>: $G = Sp(2n, \mathbb{R})$, $\mathfrak{g}_0 = sp(2n, \mathbb{R})$.

$= \{ X \in gl(2n, \mathbb{R}) : XJ + JX^t = 0 \}$

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. $\left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} : \begin{matrix} b = b^t \\ c = c^t \end{matrix} \right\}$

$\mathfrak{a}_0^s$ = diagonal. Roots $\{ \pm \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \}$

$\mathfrak{k}_0 \simeq \mathfrak{u}(n) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : \begin{matrix} x + x^t = 0 \\ y - y^t = 0 \end{matrix} \right\}$

all roots are real; <u>split</u> group

$M = \mathrm{diag}(\pm 1, \cdots, \pm 1 | \pm 1 \cdots \pm 1)$ 4-28-5

with the required symmetry

$G^d = G$ & $(G^d \cap M)_0 = \{I\}$ so <u>all</u> $\delta \in \hat{M}$ are fine.

$M \simeq \mathbb{Z}_2^n$, $\delta \leftrightarrow (\varepsilon_1, \dots, \varepsilon_n)$ $\quad \varepsilon_i = \pm 1$

<u>Orbits under W:</u> $(\underbrace{1 \cdots 1}_{k} \underbrace{-1 \cdots -1}_{n-k}) = \delta_k$

$$\bar{\Delta}_\delta = \left\{ \begin{array}{ll} \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i & i,j \leq k \\ \pm\varepsilon_l \pm \varepsilon_m & k < l,m \end{array} \right\}$$

$$W_\delta^0 = W(C_k) \times W(D_k)$$

$$W_\delta = W(C_k) \times W(C_k)$$

$R_\delta \simeq \mathbb{Z}_2$ unless $k = n$ when $R_\delta = 1$

Size of $W \cdot \delta_k$ is $\binom{n}{k}$. The rep'n $\Lambda^k \mathbb{C}^n$ of $U(n)$ decomposes as $\oplus W \cdot \delta_k$

Same for $\Lambda^{n-k} (\mathbb{C}^*)^n$

$(\underbrace{1 \cdots 1}_{k} 0 \cdots 0)$ & $(0 \cdots 0 \underbrace{-1 \cdots -1}_{k})$

The two are related by $\mathrm{Ad}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ which is <u>not</u> in $Sp(2n, \mathbb{R})$, but normalizes it.

Fine K-types. Enough to look at $Sp(4,\mathbb{R})$: 4-28-6

$\alpha = 2\varepsilon_1$: $Z_\alpha = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\alpha = 2\varepsilon_2$: $Z_\alpha = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

$\alpha = \varepsilon_1+\varepsilon_2$: $Z_\alpha = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$ $\alpha = \varepsilon_1-\varepsilon_2$: $Z_\alpha = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$

FACT: $Z_{\varepsilon_1-\varepsilon_2}$ & $Z_{\varepsilon_1+\varepsilon_2}$ are conjugate under K $\square$

$\pi \in \hat{K} \longleftrightarrow \mu = (a_1 \overset{\geq}{;} a_2)$ $a_1, a_2 \in \mathbb{Z}$

$a_1 - a_2 = \{0, \pm 1\}$ $a_1, a_2 = \{0, \pm 1\}$

ONLY CHOICES: $(0\ 0)$

$(1\ 0), (0\ {-1})$

$(1\ 1), ({-1}\ {-1})$

$G = SU(n,n)$: $\mathfrak{h}^s = \mathfrak{t}^s + \mathfrak{a}^s = \left[\begin{array}{c|c} 0 & t \\ \hline t & 0 \end{array}\right]$

$\pi \in \hat{K} \leftrightarrow \mu \in \mathfrak{t}^{c*}$

The condition for fine K-type requires $\mu|_{\mathfrak{t}^s} = 0$ (from cx roots)

Together with the condition on real roots, get $\pi(\mathrm{diag}(g_1, g_2)) = \begin{cases} 1 \\ \det g_1 \\ \det g_2 \end{cases}$

$$\left\{\begin{matrix} W\text{-orbit of} \\ \delta\in\hat{M}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\pi\in\hat{K} \text{ fine} \\ [\pi:\delta]\neq 0\end{matrix}\right\}.$$

Given a $\delta\in\hat{M}$ fine, want to construct a rep'n with LKT $\pi\in A(\delta)$.

$\delta\in\hat{M}$, $\nu\in\hat{A} \longmapsto \mathrm{Ind}_P^G(\delta\otimes\nu) := X(\delta,\nu)$

The K-types in $A(\delta)$ occur with mult 1 ($\pi = \oplus w\delta$).

<u>WANT</u>: $\pi\in A(\delta)$ are <u>THE</u> lambda lowest for $X(\delta,\nu)$.

TRUE, and this implies that for each $\mu\in A(\delta)$ there is a subquotient $\overline{X}(\delta,\nu,\mu)$.
The example of $SL(2,\mathbb{R})$ shows that it can happen that

$$\overline{X}(\delta,\nu,\mu_1)=\overline{X}(\delta,\nu,\mu_2)$$

This depends on ν.

PRINCIPAL SERIES

QUASI-SPLIT GROUP G

$H^S = T^S A^S$ $\quad$ $T^S = M$, $A^S = A$

$H^S = H$

4-28-8

$\Delta = \Delta(\mathfrak{g}/\underline{m}+\mathfrak{a}, \mathfrak{a})$

$\Delta^+ \subseteq \Delta$ positive system

gives $G = KNA$ $\quad N \longleftrightarrow \Delta^+$

$= KAN$

$\bar{\Delta} = \{\alpha \in \Delta : \frac{1}{2}\alpha \notin \Delta\}$.

$W = W(\bar{\Delta}) = M'/M$

$\delta \in \hat{M}$, $\nu \in \hat{A}$ $\quad \mathfrak{a}^{\nu}$

w comes from $m' \in M'$ $\quad (w\delta)(m) = \delta(m' m m'^{-1})$.

independent of the representative

$$\mathcal{H}^{P}_{\delta\otimes\nu} := \{f : G \to V_\delta \mid f(gmna) = a^{-\nu-\rho}\delta(m^{-1})f(g)\}$$

$\pi(x)f(g) = f(x^{-1}g)$.

Generalize to $P^1 \supseteq P$ where

$P^1 = M^1A^1N^1 \quad N^1 \subseteq N, A^1 \subseteq A, M^1 \supseteq M$

PROPOSITION: If $\delta_1 \otimes \nu_1$ is unitary, then so is $\mathcal{H}^{P^1}_{\delta_1\otimes\nu_1}$

Integral Formulas

(a) $\displaystyle\int_G f(g)\,dg = \int_{KNA} f(kna)\,dk\,dn\,da$

4-28-9

$$= \int_{KAN} f(kan)\, a^{2\rho}\, dk\, da\, dn$$

(b) $$\int_K f(k)\,dk = \int_{\bar{N}\times M} f(m\,k(\bar{n}))\, a(\bar{n})^{-2\rho}\, d\bar{n}\, dm$$

Here, $\int_{\bar{N}} a^{-2\rho}\, d\bar{n} = 1$ (normalization)

$\bar{n} = k(\bar{n})\, a(\bar{n})\, n(\bar{n})$

Iwasawa decomposition □

$$\int_K f_1(gk)\cdot\overline{f_2(gk)}\, dk \overset{?}{=} \int_K f_1(k)\cdot\overline{f_2(k)}\, dk$$

$f := f_1 \cdot \overline{f_2}$ satisfies (ν imaginary & unitary)

$$f(gman) = a^{-2\rho} f(g)$$

$$\int_K f(gk)\, dk = \int_K f(k)\, dk$$

Let $g = n_0 a_0$ and $\psi(kna) = \psi_0(na)$

4-28-10

$$\int_G f(g^{-1}x)\psi(x)\,dx \overset{?}{=} \int f(x)\psi(x)\,dx$$

$\|$

$$\int_G f(x)\psi(gx)\,dx = \int_K f(k)\,dk \int_{NA} a_0^{-2\rho}\psi_0(n_0 a_0 n a)\,dn\,da$$

$$= \int_K f(k)\,dk \int_{NA} \psi(n_0 n\, a_0 a)\,dn\,da = \int_G f(x)\psi(x)\,dx \qquad \square$$

δ can be assumed unitary when using $P_0 = M_0A_0N_0$ minimal parabolic subgroup $\square$

$\nu = \mathrm{Re}\,\nu + i\,\mathrm{Im}\,\nu$. Use induction in stages. $P^1 \longleftrightarrow \mathrm{Re}\,\nu$.

$P^1 = M^1A^1N^1$ and use

$\delta^1 = \mathrm{Ind}_P^{P^1}[\delta \otimes i\mathrm{Im}\,\nu]$ $\quad \nu^1 = \mathrm{Re}\,\nu$

$\delta^1 = \bigoplus \pi_i$ each unitary,

$\mathcal{H}_P(\delta\otimes\nu) = \bigoplus \mathcal{H}_{P^1}(\pi_i \otimes \mathrm{Re}\,\nu)$.