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Note Title 4/5/2016

Given an admissible $(\mathfrak{g}, K)$-module (π, X) want to compute a cohomology $H^*(\underline{\mathfrak{u}}, X)$. From this we construct a "standard module" $\mathcal{X}$ such that when (π, X) is irreducible, it is a subquotient of $\mathcal{X}$. This leads to a "classification" of irreducible admissible modules.

REFERENCES: Vogan, Algebraic Structure of Representations Annals of Math

Vogan "Green Book"

Knapp-Vogan

- Type of parabolic subalgebra

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ Cartan decomposition

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ complexification

$\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ a CSA of $\mathfrak{k}_0$

$\mathfrak{h}_0 := C_{\mathfrak{g}_0}(\mathfrak{t}_0)$ a CSA of $\mathfrak{g}_0$

- $\mathfrak{t}, \mathfrak{h}$ complexifications

$\lambda \in i\mathfrak{t}_0$. $\mathfrak{q}_\lambda = \mathfrak{q} = \underline{l} + \underline{u}$ parabolic subalgebra

$\underline{l}$ = the centralizer of λ, $\underline{l} = C_{\mathfrak{g}}(\lambda)$

Choose a $\Delta^+(k, t)$.

ad λ has real eigenvalues

$$u = \bigoplus_{\xi(\lambda) > 0} \mathfrak{g}_\xi$$

$\mathfrak{q}$ is θ stable, $\theta\lambda = \lambda \Longrightarrow \theta\underline{l} = \underline{l}$

$\theta u = u$.

There is also $\bar{u} = \bigoplus_{\xi(\lambda) < 0} \mathfrak{g}_\xi$

$$\mathfrak{g} = \underline{u} + \underline{l} + \underline{u}^-$$

Since λ is imaginary, $\bar{\underline{l}} = \underline{l}$ and $\bar{u} = u^-$

$$l = (l \cap k) + (l \cap p), \quad u = (u \cap k) + (u \cap p)$$

FROM KOSTANT'S THEOREM we know

$$H^*(u \cap k, X) = \sum_{\gamma \in \hat{K}} m_\gamma H^*(u \cap k, F(\gamma))$$

Want to deduce information about X from some $m_\mu \neq 0$

Find an $H^*(\underline{u}, X) \neq 0$ for a special $\mathfrak{q} = l + u$

① Find necessary condition for $H^*(\underline{u}, X) \neq 0$

② Construct the $\underline{u}$ from μ.

② is the more technical one.

$S = \dim(\underline{u} \cap \underline{k})$, $R = \dim(\underline{u} \cap \underline{p})$

$$\Lambda^n(u) = \bigoplus_{I+J=n} \Lambda^I(u \cap k) \otimes \Lambda^J(u \cap p)$$

$$\mathrm{Hom}[\Lambda^n \underline{u}, X] = \sum \mathrm{Hom}[\Lambda^I(u \cap k), X] \otimes \Lambda^J(u \cap p)^*$$

- Need to keep track of $\underline{l} \cap \underline{k}$ and $\underline{k}$
- Need to use d in these coordinates

- $$\Lambda^*(u \cap p) = \bigoplus_{0 \leq a \leq N} V_a \quad (\underline{l} \cap \underline{k})\text{-modules}$$

(a) $V_0 = \Lambda^0(u \cap p)$, $V_N = \Lambda^R(u \cap p)$

(b) $V_a \subseteq \Lambda^{r(a)}(u \cap p)$, $a \leq a' \Rightarrow r(a) \leq r(a')$

(c) $\mathrm{ad}(\underline{u} \cap \underline{k}) V_a \subseteq \mathrm{span}\{V_0, \dots, V_{a'}\}_{a' < a}$

We can use λ to achieve this $\boxtimes$

$(\underline{u} \cap \underline{k})$ acts nilpotently on $(u \cap p)$.
Lie-Engel's theorem implies action is upper triangular; use appropriate labels

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$$\mathrm{Hom}[\Lambda^* u, X] = \sum \mathrm{Hom}[\Lambda^* u \cap k, X] \otimes V_a^*$$

$$Q \in \Lambda^{m-r(a)}(u \cap k) \quad P \in V_a \subseteq \Lambda^{r(a)} u \cap p.$$

$df(Q \wedge P) =$ sum of

(i) $\sum (-1)^i q_i f(\imath(q_i) Q \wedge P)$

(ii) $\sum\limits_{i<j} (-1)^{i+j} f([q_i, q_j] \wedge \imath(q_i, q_j) Q \wedge P)$

(iii) $\sum (-1)^i p_i f(Q \wedge \imath(p_i) P)$

(iv) $\sum (-1)^{i+j} f([p_i, p_j] \wedge Q \wedge \imath(p_i, p_j) P)$

(v) $\sum (-1)^{i+j} f([q_i, p_j] \wedge \imath(q_i) Q \wedge \imath(p_j) P)$

Use the appropriate ranges for i and j in each sum

NOTE: (i) + (ii) is $d_{\underline{u} \cap \underline{k}} \otimes 1$

CONSIDER:

$f \in \mathrm{Hom}[\Lambda^{I+R}\,\underline{u} : X]$ which is 0 except on $\Lambda^I(u \cap k) \otimes \Lambda^R(u \cap p)$.

apply d:

(i) + (ii) give $d_{\underline{u} \cap k} \otimes 1$

(iii), (iv), (v) are zero.

We get a map

$$\pi_I : H^I(u \cap k, X) \otimes \Lambda^R(u \cap p)^* \to H^{I+R}(u, X)$$

A SPECIAL CASE is $I = 0$.

$\mu \in \hat{K}$ is a highest

LHS has weight $\mu - 2\rho(u \cap p)$.

If $\pi_I \neq 0$, then $H^{I+R}(u, X)^{\mu - 2\rho(u \cap p)} \neq 0$

NOTE: $\Lambda^R(u \cap p)$ is a 1-dimensional $(l \cap k)$-module; $2\rho(u \cap p) = \sum_{\alpha \in \Delta(u \cap p)} \alpha$

FILTRATION

$$A^n = \mathrm{Hom}\left[\Lambda^n \underline{u}, X\right], \quad A = \bigoplus A^n$$

$$A^n_a = \left\{ f(Q \wedge P) = 0 \ \forall Q \in \Lambda^{\alpha}(u \cap k) \right.$$
$$\left. P \in V_{a'} \text{ for } a' < a, \ \deg Q + \deg P = n \right\}$$

MAIN FACT:

d preserves the filtration

DEF: $\mu \in \hat{K}$ is called strongly $\underline{u}$-minimal in X

if $\left[H^{j}(u \cap k, \gamma) \otimes \Lambda^{j'}(u \cap p)^{*}\right]^{\mu - 2\rho(u \cap p)} \neq 0$

$\Rightarrow X(\gamma) = 0$

THM: If $\mu \in \hat{T}$ is strongly $\underline{u}$-minimal, then

$$\pi_j : H^0(\underline{u} \cap k, X)^{\mu} \otimes \Lambda^R(u \cap p)^{*} \xrightarrow{\sim} H^R(\underline{u}, X)^{\mu - 2\rho(u \cap p)}$$