4-7-2016

Note Title 4/7/2016

Computing $H^*(\underline{u}, X)$.

LEMMA 1: (a) $\alpha \in \Delta(l) \Rightarrow \langle \alpha, \rho(\underline{u}) \rangle = 0$

(b) $\alpha \in \Delta(u) \Longrightarrow \langle \alpha, \rho(u) \rangle > 0$

LEMMA 2 $\mu \in F(\Lambda) \Rightarrow \|\mu + \rho\|^2 \leq \|\Lambda + \rho\|^2$

with "=" $\Longleftrightarrow$ $\mu = w\Lambda$ for some $w \in W$

PROOF: $\mu = \Lambda - \sum_{\beta \in \Delta^+} m_\beta \beta$, $m_\beta \geq 0$

$\langle \mu+\rho, \mu+\rho \rangle = \langle \mu, \mu \rangle + 2\langle \mu, \rho \rangle$

$+ \langle \rho, \rho \rangle \leq \langle \Lambda, \Lambda \rangle + 2\langle \Lambda - \sum m_\beta \beta, \rho \rangle$

$\left.\begin{matrix}\langle \mu, \mu \rangle \\ \leq \langle \Lambda, \Lambda \rangle\end{matrix}\right\}$ $+ \langle \rho, \rho \rangle$

$\leq \langle \Lambda, \Lambda \rangle + 2\langle \Lambda, \rho \rangle + \langle \rho, \rho \rangle = \langle \Lambda + \rho, \Lambda + \rho \rangle$

$\langle \rho, \beta \rangle > 0 \quad \forall \beta \in \Delta^+$.

LEMMA 3: $\{\alpha_i\} \subseteq \Delta(\underline{u})$. Then

$\| w(\Lambda+\rho) - \rho + \sum \alpha_i \|^2 \geq \|\Lambda\|^2$ $\boxed{?}$

COROLLARY:

$[F_{\underline{l}}(w(\Lambda+\rho)-\rho) : Hom(\Lambda^m \underline{u}, F)] = 1$

<u>Proof:</u> A weight of $\mathrm{Hom}(\Lambda^m \underline{u}, F)$ is of the form $\mu = \gamma - \sum \alpha_i$ with $\{\alpha_i\} \subseteq \Delta(\underline{u})$ of size m.

Use $\mathrm{Hom}(A, B) \simeq A^* \otimes B$

$\ell \otimes b \longmapsto \varphi_{\ell, b}(a) = \ell(a)\, b$ $\boxtimes$

In turn $\gamma = \Lambda - \sum_{\beta \in \Delta^+} m_\beta \beta$, $m_\beta \geq 0$

<u>MULTIPLICITY OF μ:</u>

<u># of expressions $\gamma - \sum \alpha_i$ counted with multiplicity of γ</u>

<u>$\mu = w(\Lambda + \rho) - \rho$: Then</u>

$\gamma = w(\Lambda + \rho) - \rho + \sum \alpha_i$ satisfies

(a) $\|\gamma\|^2 \geq \|\Lambda\|^2$ (Lemma 3)

(b) $\|\gamma\|^2 \leq \|\Lambda\|^2$ (F finite dim'l)

So $\|\gamma\|^2 = \|\Lambda\|^2$, therefore

$\{\alpha_i\} = \Delta^+(w)$ $\boxtimes$

$\gamma = w\Lambda + (w\rho - \rho) + \sum \alpha_i = w\Lambda$

- Multiplicity of $w\Lambda$ is 1
- Only one choice of $\{\alpha_i\}$

Need to show $w(\Lambda+\rho)-\rho$ is a highest weight for $\underline{l}$.

If not, $w(\Lambda+\rho)-\rho+\alpha_0$ is a weight in $\mathrm{Hom}(\Lambda^m \underline{u}, F)$ where $\alpha_0 \in \Delta^+(\underline{l})$. Then

$$w(\Lambda+\rho)-\rho = \gamma - \sum_{i=0}^{m} \alpha_i$$

As before, $\{\alpha_i\} = \Delta^+(w)$

But $\Delta^+(w)$ cannot contain an $\alpha_0 \in \Delta(\underline{l})$; by assumption $w \in W^1$

$\boxtimes$

Proof of Theorem

- $\mathrm{Hom}\left[F_{\underline{l}}(\mu) \,;\, H^m(\underline{u}, F)\right] \neq 0$

$$\Rightarrow \mu + \rho(\underline{l}) = w(\Lambda+\rho) - \rho(\underline{u})$$

inf'l Character

So $\mu = w(\Lambda+\rho) - \rho$

$$\Leftrightarrow \mu + \rho = w(\Lambda+\rho)$$

- μ dominant for $\Delta^+(\mathfrak{l}) \Rightarrow$
$\mu+\rho$ dominant for $\Delta^+(\mathfrak{l}) \Rightarrow$
$w(\Lambda+\rho)$ dominant for $\Delta^+(\mathfrak{l}) \Rightarrow w \in W^1$
- By the Corollary, $F_{\mathfrak{l}}(w(\Lambda+\rho)-\rho)$ occurs exactly once in

$$\mathrm{Hom}\left[\Lambda^{\ell(w)}\underline{u}, F\right] \quad \text{QED}$$

SEE NOTES FROM 4/5
FOR NEXT MATERIAL

SUPPLEMENTS

Cartan Subalgebras

$\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ real CSA's

Fundamental CSA

assume $K \subseteq G$ connected for simplicity

Let $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ be a CSA of $\mathfrak{k}_0$
$T = C_K(\mathfrak{t}_0)$ is the CSG

PROPOSITION: $\mathfrak{h}_0 := C_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is a CSA of $\mathfrak{g}_0$ ($\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{t})$ complexification)

Pf: $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{p}_0^0$, $\mathfrak{g}$ has a real form $\mathfrak{u}_0 = \mathfrak{k}_0 + i\mathfrak{p}_0$. Then $\mathfrak{h}$ is the complexification of $\mathfrak{t}_0 + i\mathfrak{p}^0 \subseteq \mathfrak{u}_0$. So $\mathfrak{h}$ is reductive. Then $[\mathfrak{h},\mathfrak{h}]$ is semisimple or 0. If we show

$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{t}$,

then $[\mathfrak{h},\mathfrak{h}] = 0$.

$$[\mathfrak{h},\mathfrak{h}] = \underbrace{[\mathfrak{t},\mathfrak{t}]}_{=0} + \underbrace{[\mathfrak{t},\mathfrak{p}^0]}_{=0} + [\mathfrak{p}^0,\mathfrak{p}^0]$$

But $[\mathfrak{p}^0,\mathfrak{p}^0] \subseteq C_{\mathfrak{k}}(\mathfrak{t}) = \mathfrak{t}$. QED

- Note that $\theta\mathfrak{h} = \mathfrak{h}$. $\mathfrak{h}_0 = \underline{\mathfrak{t}}_0 + \underline{\mathfrak{a}}_0$

Let $\Delta \subseteq \mathfrak{h}^*$ be the roots,

$\theta\Delta = \Delta$ & $\Delta \subseteq i\mathfrak{t}_0 + \mathfrak{a}_0$.

Let B be the K-invariant form <0 on $\mathfrak{k}_0$, >0 on $\mathfrak{p}_0$.

- Any $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{t}} \neq 0$.

Otherwise $X_{\pm\alpha} \in C_{\mathfrak{g}}(\mathfrak{t})$.

$\Delta = \Delta_{im} \cup \Delta_{cx}$

$\Delta_{im} = \{\alpha : \alpha|_{\underline{\mathfrak{a}}} = 0\}$

$\Delta_{cx} = \{\alpha ; \alpha|_{\underline{\mathfrak{a}}} \neq 0\}$

- X a root vector $\Longrightarrow \theta X$ a root vector.

- Restricted Roots.

$(\mathfrak{t} \neq \mathfrak{h})$.

Examples:

$\underline{\mathfrak{g} = \mathfrak{u}(p,q)}$: $\underline{\mathfrak{t}}$ is the diagonal.

$\mathfrak{h} = \mathfrak{t}$. These are called "equal rank groups"

$\underline{\mathfrak{g} = \mathfrak{gl}(2n, \mathbb{R})}$:

$\mathfrak{h} = \mathrm{diag}\left\{\begin{pmatrix} t_i & \theta_i \\ -\theta_i & t_i \end{pmatrix} t_i, \theta_i \in \mathbb{R}\right\}$

$\mathfrak{t} = \mathrm{diag}\left\{\begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix}\right\}$

$\underline{\mathfrak{a}} = \mathrm{diag}\left\{\begin{pmatrix} t_i & 0 \\ 0 & t_i \end{pmatrix}\right\}$

Other Classical Equal Rank Gps

$sp(2n,\mathbb{R})$, $so(2p,2q)$, $so(2p+1,2q)$

$sp(2p,2q)$, $so^*(2n)$

Unequal Rank: $gl(n,\mathbb{R})$, $sl(n,\mathbb{R})$

$u^*(2n)$

$\alpha \in \Delta \longmapsto (a,b)$

$\theta\alpha \in \Delta \longmapsto (a,-b)$

Restricted Roots are the $(a,0)$.

$b=0 \longleftrightarrow$ imaginary roots

$\theta X_\alpha = c X_\alpha \qquad c=\pm 1$

$c=1$ compact root $\to c$

$c=-1$ noncompact root $\to nc$

$b \neq 0$ $X+\theta X \in \mathfrak{k}$, $X-\theta X \in \mathfrak{p}$

cpct ncpct $\mathfrak{t}$-roots

! same restriction to $\mathfrak{t}$!

CLAIM: $\dim \mathfrak{g}_a = 2$ $(b \neq 0)$

PROOF: Need (a, b') a $\mathfrak{t}$-root then $b' = \pm b$.

$\langle (a, b), (a, b') \rangle = (a, a) + (b, b')$

Changing (a, b) to $(a, -b)$ if necessary,

$\langle (a, b), (a, b') \rangle \geq (a, a) > 0$

$\Longrightarrow$

$(a, b) - (a, b') = (0, b - b') \in \Delta$.

But $\mathfrak{h}$ is fundamental, any root has $\neq 0$ restriction to $\mathfrak{t}$.

• NOTE: $\dim \mathfrak{g}_\alpha = 1$ if $\alpha \in \Delta_{im}$.

θ-invariant Positive System

Fix $\gamma \in i\mathfrak{t}_0 \ni \langle \gamma, a \rangle \neq 0$ for any $\mathfrak{t}$-root a.

$\Delta^+ = \{ \alpha \in \Delta ; \langle \alpha|_{\mathfrak{t}}, \gamma \rangle > 0 \}$

Δ^+ is a positive system $\ni \theta \Delta^+ = \Delta^+$

Compact Weyl Group

$W_K := W(\underline{k}, \mathfrak{t})$. $\quad a = \alpha \in \Delta_{im}$

a a cpct $\mathfrak{t}$ root $\leftrightarrow$ $\alpha \leftrightarrow X_\alpha + \theta X_\alpha$

$\dot{a} = \alpha \in \Delta_{im} \longmapsto s_\alpha$

$\alpha = (a, b), \quad (a, b) - \theta(a, b) \notin \Delta$

(a) $(2a, 0 = (a, b) + (a, -b) \in \Delta$

(b) $(2a, 0)$ NOT A ROOT. Then

$\langle \alpha, \theta\alpha \rangle = 0 \Longleftrightarrow \langle a, a \rangle_{\mathfrak{h}} = (b, b)|_{\mathfrak{h}}$

(a) $s_\alpha \longmapsto s_{(2a, 0)}$

(b) $s_\alpha \mapsto s_{(a,b)} \circ s_{(a,-b)} = s_{(a,-b)} \circ s_{(a,b)}$ $\boxtimes$