5_10_2016

Note Title 5/10/2016

G real reductive, $\mathcal{L}(G) = \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$

$K \subseteq G$ max'l cpct, $B \subseteq K$ closed

$\mathcal{C}(\mathfrak{g}, K)$, $\mathcal{C}(\mathfrak{g}, B)$ compatible modules (π, V)

$\ni \forall v \in V$, $\dim\{\pi(K)v\} < \infty$ (same for B)

& K (B) acts continuously on $\{\pi(K)v\}$

$V = \bigoplus V_\gamma$ where $V_\gamma :=$ sum of irred's $\simeq V(\gamma)$

(Def 6.2.3 in [V], $\mathcal{C}$ is $\mathcal{M}$)

$\mathcal{F} = \mathcal{F}^{\mathfrak{g},K}_{\mathfrak{g},B}$ "forgetful" functor

ASSUME K connected.

<u>DEF</u>: $V \in \mathcal{C}(\mathfrak{g}, B)$.

ΓV: $\left\{ v \in V \,/\, \dim\{\pi(K)v\} < \infty, \text{ \& continuous action} \right\}$

called K-finite vectors $\square$

<u>MAIN PROPERTY</u>

$$\operatorname{Hom}_{(\mathfrak{g},K)}[X, \Gamma Y] \simeq \operatorname{Hom}_{(\mathfrak{g},B)}[\mathcal{F}X, Y]$$

$\Gamma V = 0$ for most V

Consider <u>derived functors</u> $\Gamma^i V$

Need enough injectives/projectives in $\mathcal{C}$

$(0 \to V \to I^*) \longmapsto \Gamma^i V = H^i(I^*)$

$\Gamma^0 V = \Gamma V$

5-10-2

$\mathrm{ind}\, V := \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{k}} V$, $\mathrm{pro}\, V = \mathrm{Hom}_{\mathbb{C}}[\mathcal{U}(\mathfrak{g}), V]$ K-finite

projective injective

FOR $\mathrm{pro}(V)$, action of K is via Ad

action of $\mathfrak{g}$ is on the right on $\mathcal{U}(\mathfrak{g})$)

LEMMA (6.2.12) Γ^i are functors.

(6.2.13) $z \in Z(\mathfrak{g}) \in \mathrm{Hom}_{(\mathfrak{g},B)}(V,V)$ $v \mapsto z.v$

z acts on $\Gamma^i V$ via $\Gamma^i z$ $\boxed{\times}$

FROBENIUS RECIPROCITY

$X \in \mathcal{C}(\mathfrak{g},K)$, $V \in \mathcal{C}(\mathfrak{g},B)$ $\square$ There is a 1st quadrant spectral sequence

$$E_n^{p,q} \Longrightarrow \mathrm{Ext}^{p+q}_{(\mathfrak{g},B)}(\mathcal{F}X, V)$$

with

$$E_2^{p,q} = \mathrm{Ext}^p_{(\mathfrak{g},K)}[X, \Gamma^q V]$$

d_n has degree $(n, 1-n)$ $\boxed{\times}$

Proof: $0 \to V \to I^*$ injective resolution

$P_* \to X \to 0$ projective resolution

$$C^{p,q} := \mathrm{Hom}_{(\mathfrak{g},B)}[\mathcal{F}P_p, I^q] \simeq \mathrm{Hom}_{(\mathfrak{g},K)}[P_p, \Gamma I^q]$$

$d_1 :=$ differential for P_*

$d_2 :=$ differential for I^*

$\partial = d_1 + (-1)^{p+q} d_2$ 5-10-3

$$\begin{array}{ccc} C^{p+1,q} & \xrightarrow{d_2} & C^{p+1,q+1} \\ \uparrow d_1 & & \uparrow d_1 \\ C^{p,q} & \xrightarrow{d_2} & C^{p,q+1} \end{array}$$

commutes

$(-1)^{p+q}$ makes $\overline{\partial}^2 = 0$.

<u>Two Filtrations:</u>

$\bigoplus_{h \geq q} C^{p,h}$ $\qquad$ $\bigoplus_{k \geq p} C^{k,q}$

Compute

$$H^*(d_2, H^*(d_1, C^*)), \; H^*(d_1, H^*(d_2, C^*))$$

For d_2 everything collapses

$\mathrm{Ext}^q_{(\mathfrak{g},B)}[\mathcal{F}X, V]$ $\quad p=0$

$0 \quad p \neq 0$

$$H^q(\partial, C^*) = \mathrm{Ext}^q_{(\mathfrak{g},B)}[\mathcal{F}X, V]$$

For d_1 get $\mathrm{Hom}_{(\mathfrak{g},K)}[P_p, \Gamma^q V]$.

with differential induced by d_2

<u>COR</u> (6.2.15) $\quad (\delta, Z) \in \hat{K}$, $\mathcal{F}Z = \mathrm{res}\, Z$ a $(\underline{k}, B)$ module

$$\dim \mathrm{Ext}^q_{(\underline{k},B)}[\mathcal{F}Z, \mathrm{res}\, V] = [\delta : \Gamma^q V] \quad \boxtimes$$

Pf: Let $X := \operatorname{ind}_{(\mathfrak{k},K)}^{(\mathfrak{g},k)} Z = \mathcal{U}(\mathfrak{g}) \otimes Z$. 5-10-4

$$\operatorname{Ext}^{p}_{(\mathfrak{g},K)}[X, \Gamma^{\mathfrak{g}} V] = 0 \text{ for } p > 0$$

$$= \operatorname{Hom}_{(\mathfrak{g},K)}[X, \Gamma^{\mathfrak{g}} V] = [\delta : \Gamma^{\mathfrak{g}} V] \text{ for } p = 0$$

PARABOLIC INDUCTION

Remark: Original Ideas due to Zuckerman References are [Vogan] and [Knapp-Vogan]

$\mathfrak{q} \subseteq \mathfrak{g}$ such that $\mathfrak{q} = \underline{\mathfrak{l}} + \underline{\mathfrak{u}}$ with $\underline{\mathfrak{l}}$ θ and σ stable. $L \ni \mathcal{L}(L) = \underline{\mathfrak{l}}$, $L_0 = G \cap L$ and $B = L \cap K$.

Define functors $\mathcal{R}^i$ and $\mathcal{L}_j$

$$\mathcal{R}^i(*) = \left(\Gamma_{(\mathfrak{g},B)}^{(\mathfrak{g},K)}\right)^i \left[\operatorname{pro}_{(\mathfrak{q},B)}^{(\mathfrak{g},B)}\left[* \otimes \Lambda^{\dim \bar{\mathfrak{u}}} \underline{\mathfrak{u}}\right]\right]$$

$$\mathcal{L}_j(*) = \left(\Pi_{(\mathfrak{g},B)}^{(\mathfrak{g},K)}\right)_j \left[\operatorname{ind}_{(\mathfrak{q},B)}^{(\mathfrak{g},B)}\left[* \otimes \Lambda^{\dim \bar{\mathfrak{u}}} \underline{\mathfrak{u}}\right]\right]$$

- Π is like Γ, only use $P_* \to V \to 0$
- $\underline{\mathfrak{u}}$ is assumed to act trivially on $*$

FROBENIUS RECIPROCITY

There are two spectral sequences with common limit

5-10-5

$$\mathrm{Ext}^{p+q}_{(\mathfrak{g},B)}\left[\mathcal{F}X, \mathrm{pro}(W\otimes\Lambda^{top}\underline{u})\right]$$

$$I_2^{p,q} = \mathrm{Ext}^{p}_{(\mathfrak{g},K)}\left[X, \mathcal{R}^q W\right]$$

$$II_2^{p,q} = \mathrm{Ext}^{p}_{(\mathfrak{l},B)}\left[H^{\dim\underline{u}-q}(\underline{u}, \mathrm{res}X), W\right]$$

This is used to relate I_2 with II_2

SHAPIRO'S LEMMA (6.1.27)

$$\mathrm{Ext}^{i}_{(\mathfrak{g},B)}\left[\mathcal{F}X, \mathrm{pro}(W\otimes\Lambda^{top}\underline{u})\right] \simeq$$

$$\simeq \mathrm{Ext}^{i}_{(\mathfrak{h},B)}\left[\mathrm{res}X, W\otimes\Lambda^{top}\underline{u}\right]$$

COROLLARY If $R^i W = 0$ except for $i = q_0$

$$\mathrm{Ext}^{p}_{(\underline{\mathfrak{l}},B)}\left[H^{\dim\underline{u}-q_0-q}(\underline{u}, \mathrm{res}X), W\right]$$

$$\Longrightarrow \mathrm{Ext}^{p+q}_{(\mathfrak{g},K)}\left[X, R^{q_0}W\right]$$

(May not be first quadrant)

Background and Notation

$\mathfrak{b} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ $\quad \mathfrak{b} \longleftrightarrow B$ acting by Ad

5-10-6

- $\mathrm{res}^{(\mathfrak{g},B)}_{(\mathfrak{l},B)} Z$ obvious def'n

- $\mathrm{ind}^{(\mathfrak{g},B)}_{(\mathfrak{l},B)}(X) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{l}} X$; $\mathrm{pro}(X) = \mathrm{Hom}_{\mathfrak{l}}[\mathcal{U}(\mathfrak{g}) X]_{B\text{-finite}}$

$0 \to X \to I^* \ (\text{injective } (\mathfrak{l}, B))$

$\mathrm{pro}(I^i)$ is injective $(\mathfrak{g}, B)$

$0 \to \mathrm{pro}(X) \to \mathrm{pro}(I^i)$ injective $(\mathfrak{g}, B)$

Computes $\mathrm{Ext}^i_{(\mathfrak{g},B)}[Z, \mathrm{pro}\, X]$ via

$\to \mathrm{Hom}_{(\mathfrak{g},B)}[Z, \mathrm{pro}(I^*)] \longrightarrow$

$\mathrm{S}\wr$

$\to \mathrm{Hom}_{(\mathfrak{l},B)}[\mathrm{res}\, Z, I^*] \longrightarrow$

$\mathrm{Ext}^i_{(\mathfrak{g},B)}[\mathcal{F}X, \mathrm{pro}(W \otimes \Lambda^{top}\mathfrak{u})] \simeq$

$\mathrm{Ext}^i_{(\mathfrak{l},B)}[\mathrm{res}\, X, W \otimes \Lambda^{top}\mathfrak{u}]$.

MISSING INGREDIENT (u acts by 0)

$\exists\ E^{p,q}_{\mathfrak{u}} \Longrightarrow \mathrm{Ext}^{p+q}_{(\mathfrak{l},B)}(X, Y)$

$E^{p,q}_2 = \mathrm{Ext}^p_{(\mathfrak{l},B)}[\mathrm{res}\, X, H^q(\mathfrak{u}, Y)]$

called Hochschild-Serre

$$\mathrm{Ext}^{p}_{(\mathfrak{l},B)}\left(H^{\dim \underline{\mathfrak{u}}-q}(\mathfrak{u}, X \otimes \Lambda^{top}\underline{\mathfrak{u}}), \mathrm{res}(Y)\right)$$

$$\Longrightarrow \mathrm{Ext}^{p+q}_{\mathfrak{g},B}(X, Y)$$

<u>NOTE</u>: $H^{m-q}(\mathfrak{u}, X \otimes D) \simeq H_q(\mathfrak{u}, X)$