Ext and $(\mathfrak{g},K)$-cohomology

Note Title 5/10/2016

$\mathfrak{k} \subseteq \mathfrak{g}$ a reductive subalgebra

(Coming from $G \supset K$ max'l cpct, but could be a $B \subseteq K$ closed)

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p}$ is a K-stable complement

$$C^q(\mathfrak{g}, V) = \mathrm{Hom}_{\mathbb{C}}[\Lambda^q \mathfrak{g}, V] \simeq \Lambda^q(\mathfrak{g}^*) \otimes V$$

$$df(x_0 \cdots x_q) = \sum (-1)^i x_i \cdot f(x_0 \cdots \hat{x}_i \cdots x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], \hat{x}_i \cdots \hat{x}_j \cdots)$$

$d^2 = 0$.

$$x \in \mathfrak{g} \mapsto \theta_x f(x_1 \cdots x_q) = \sum f(\cdots \mathrm{ad}\, x(x_i) \cdots)$$

$$i_x f(x_1 \cdots x_{q-1}) = f(x, x_1 \cdots x_{q-1})$$

Then $\theta_x = d \circ i_x + i_x \circ d$

x_i basis of $\mathfrak{g}$, x_i^* basis of $\mathfrak{g}^*$

$$\varepsilon(x^i)\omega = x^i \wedge \omega$$

$$2d_0 = \sum \varepsilon(x^i) \cdot \theta_{x_i}$$

$$d = d_0 \otimes 1 + \sum \varepsilon(x^i) \otimes \pi(x_i)$$

$$C^q(\mathfrak{g}, \mathfrak{k}; V) \subset C^q(\mathfrak{g}; V)$$

elements annihilated by i_x, θ_x

$H^i(\mathfrak{g}, \mathfrak{k}; V) :=$ Cohomology of $C^q(\mathfrak{g}, \mathfrak{k}; V)$

<u>PROP</u>: $H^i = 0$ for $i > \dim(\mathfrak{g}/\mathfrak{k})$

DEFINITION OF EXT

5-10-2

$X, Y \in \mathcal{C}(\mathfrak{g}, B)$ $\quad P_* \to X \to 0$ projective resolution

$$\to \mathrm{Hom}_{(\mathfrak{g},B)}[P_i, Y] \to \mathrm{Hom}_{(\mathfrak{g},B)}[P_{i+1}, Y] \quad (**)$$

is a complex

$$\mathrm{Ext}^i_{(\mathfrak{g},B)}[X,Y] := H^i(**)$$

- Can be computed via $0 \to Y \to I^*$ injective resolution

- Note that $\mathrm{Hom}_{\mathbb{C}}[X,Y]_{B\text{-finite}}$ is a $(\mathfrak{g},B)$-module

$$\mathrm{Hom}_{(\mathfrak{g},B)}[X, \mathrm{Hom}_{\mathbb{C}}[Y,Z]_{B\text{-finite}}] \simeq$$

$$\simeq \mathrm{Hom}_{(\mathfrak{g},B)}[X \otimes_{\mathbb{C}} Y, Z]$$

$$\Rightarrow \mathrm{Ext}^i_{(\mathfrak{g},B)}[X, \mathrm{Hom}_{\mathbb{C}}[Y,Z]] \simeq \mathrm{Ext}^i_{(\mathfrak{g},B)}[X \otimes Y, Z]$$

SPECIAL CASE: $X = \mathbb{C}$

$$\mathrm{Ext}^i_{(\mathfrak{g},B)}[\mathbb{C}, Y] = H^i(\mathfrak{g}, B; Y)$$

B connected, $H^i(\mathfrak{g}, B, *) = H^i(\mathfrak{g}, \mathfrak{b}, *)$

disconnected, change def'n slightly

So $\mathrm{Ext}^i_{(\mathfrak{g},B)}[X,Y] \simeq H^i(\mathfrak{g}, B; \mathrm{Hom}_{\mathbb{C}}[X,Y]_B)$

vanishes for $i > \dim(\mathfrak{g}/\mathfrak{b})$.

REFERENCE: Borel-Wallach