Two Examples of $R^i_{\mathfrak{q}} W$:

① $\mathfrak{q}$ is the complexification of a $\mathfrak{p}_0 = \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ real

$\mathfrak{q} = \mathfrak{l} + \mathfrak{u} = (\underline{m} + \underline{a}) + \underline{n}$

- W the K-finite vectors of a H-C module for MA, V_γ

(a) $R^i W = 0$ for $i \neq 0$

(b) $R^0 W = \mathrm{Ind}_P^G (Y \otimes D \otimes \mathbb{C}_{-\rho})$ where $D = \Lambda^{\dim \underline{n}} \underline{n}$

(c) $\mathrm{Hom}_{(\mathfrak{g},K)} [Y, X(\gamma)] = \mathrm{Hom}_{\underline{m}+\underline{a}, M\cap K} [H^{\dim \underline{n}}(\underline{n}, Y), W \otimes D^* \otimes \mathbb{C}_\rho]$

$\simeq \mathrm{Hom}_{\underline{m}+\underline{a}, M\cap K} [Y/\underline{n}Y, W \otimes \mathbb{C}_\rho]$

Spectral Sequence:

$$\mathrm{Ext}^p_{\underline{m}+\underline{a}, M\cap K} [H^{\dim \underline{n} - q}(\underline{n}, Y), W \otimes D^* \otimes \mathbb{C}_\rho] \Longrightarrow \mathrm{Ext}^{p+q}_{\mathfrak{g},K} [Y, \mathrm{Ind}_P^G V_\gamma]$$

FOR A FIRST QUADRANT SPECTRAL SEQUENCE

$E_r^{0,0}$ survives: $p = q = 0$ gives

$$\mathrm{Ext}^0_{\underline{m}+\underline{a}, M\cap K} [H^{\dim \underline{n}}(\underline{n}, Y), W \otimes D^* \otimes \mathbb{C}_\rho]$$

$$\mathrm{Ext}^0_{(\mathfrak{g},K)} [Y, \mathrm{Ind}_P^G V_\gamma] \simeq \mathrm{Hom}_{(\mathfrak{g},K)} [Y, \mathrm{Ind}_P^G V_\gamma]$$

$\dim \underline{n} = m$

$H^{\dim \underline{n}}(\underline{n}, Y)$:

$$\mathrm{Hom}[\Lambda^{m-1}\underline{n}, Y] \xrightarrow[d_{m-1}]{} \mathrm{Hom}[\Lambda^m \underline{n}, Y] \longrightarrow 0$$

need the image of d_{m-1}:

$$df(x_0 \cdots x_{m-1}) = \sum (-1)^i x_i f(\ldots \hat{x}_i \ldots) + \sum (-1)^{i+j} f([x_i, x_j] \wedge \cdots \hat{x}_i \cdots \hat{x}_j \cdots)$$

Second term is 0; $\underline{n}$ is nilpotent, ad is strictly upper triangular

First term is $\Lambda^m \underline{n}^* \otimes Y / \underline{n} Y$ $\boxtimes$

Show Vanishing: $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$ (not true if $\theta\mathfrak{q} = \mathfrak{q}$) $\theta\mathfrak{q} = \bar{\mathfrak{q}}$

Restrict $R^i_{\mathfrak{q}}$ to $\mathfrak{k}$.

FORMULA: $\mathrm{Res}^{(\mathfrak{k}, K)}_{(\mathfrak{g}, K)} \circ \Gamma^i = (\Gamma^{\mathfrak{k}, K}_{\mathfrak{k}, M\cap K})^i \circ \mathrm{Res}^{(\mathfrak{q}, M\cap K)}_{(\mathfrak{g}, M\cap K)}$

Module is $Z = \mathrm{pro}^{\mathfrak{g}, B}_{\mathfrak{q}, B}(W \otimes D) = \mathrm{Hom}_{\mathfrak{q}}[U(\mathfrak{g}), W \otimes D]$ B-finite

Restriction $\mathrm{Hom}_{\mathfrak{m}\cap\mathfrak{k}}[U(\mathfrak{k}), \mathrm{res}(W \otimes D)]$ B-finite

$\mathrm{pro}^{\mathfrak{k}, B}_{\mathfrak{m}\cap\mathfrak{k}, B}(\mathrm{res}(W \otimes D))$

This is already injective, so Γ^i gives 0 for $i > 0$

$$m[\delta, R^0 W] = \dim \mathrm{Hom}_{k, M\cap K}\left[\delta, \mathrm{pro}(W \otimes D)\right]$$

$$= \dim \mathrm{Hom}_{\underline{m}\cap k, B}\left[\mathrm{res}\,\delta, W \otimes D\right] \quad \boxtimes$$

BLATTNER FORMULA:

see other notes.

Implies W admissible $\Longrightarrow R^i W$ admissible

& growth of multiplicities.

$\Delta^+(\underline{k}, \mathfrak{t}^c)$ rank K = rank G $\mathfrak{t}^c = \mathfrak{h}^c$ 5-11-4

Other Case: μ such that $\mu + 2\rho_c'$ dominant for

$\mathfrak{b} \ni \Delta(\mathfrak{b}, \mathfrak{t}^c) \supseteq \Delta^+(\underline{k}, \mathfrak{t}_c)$. Assume that

$\lambda = \mu + 2\rho_c - \rho$ is still dominant for $\mathfrak{b}$.

Take $\mathcal{R}^i_{\mathfrak{b}}(\mathbb{C}_{\lambda-\rho}) = \begin{cases} \text{irreducible if } i = S = \dim(\underline{u} \cap \underline{k}) \\ 0 \quad \text{otherwise} \end{cases}$

NOTE: ① If χ is inf'l character of W, then inf'l character of $\mathcal{R}^i_{\mathfrak{q}}(W)$ is $\lambda + \rho(u)$.

② $\mathcal{R}^S_{\mathfrak{b}}(\mathbb{C}_{\lambda-\rho})$ is a (limit of) discrete series

Its H-C parameter is $\lambda = \mu + \rho_c - \rho_n = \mu + 2\rho_c - \rho$.

DISCRETE SERIES $\langle \pi(g)v, w \rangle \in L^2(G)$

LIMIT OF DISCRETE SERIES $\in L^{2+\varepsilon}(G)\ \forall \varepsilon > 0$

Example: $G = SL(2,\mathbb{R})$ $k = \mathfrak{t}$ & so the $\mathcal{R}^i$ degenerate.

Hermitian Symmetric $\mathfrak{g} = \underline{k} + \mathfrak{p} = k + \mathfrak{p}^+ + \mathfrak{p}^-$

$\mathfrak{q}^{\pm} = k + \mathfrak{p}^{\pm}$ are parabolic subalgebras; $\underline{k}$ has a center

e.g. $G = Sp(2n,\mathbb{R})$, $SO(n,2)$, $SO^*(2n)$ etc

Vanishing Follows From a Duality between

Γ^i and Π_j : ($\mathfrak{h} = \mathfrak{g}$ in thm 3.5)

$(\mathfrak{g}, B) \longrightarrow (\mathfrak{g}, K)$

$\Pi_j\left(V \otimes \bigwedge^m (\mathfrak{k}/\mathfrak{b})^*\right) \simeq \Gamma^{m-j}(V)$

$\left[\Pi_j\left(V \underset{\mathbb{C}}{\otimes} \bigwedge^m (\mathfrak{k}/\mathfrak{b})\right]^c \simeq \Pi_{m-j}(V^c)$ (m is twice S from before)

$\left[\Gamma^i\left(V \underset{\mathbb{C}}{\otimes} \bigwedge^m (\mathfrak{k}/\mathfrak{b})\right)\right]^c \simeq \Gamma^{m-i}(V^c)$

Irreducibility is another matter.

Theorem (5.99) Z admissible $(\mathfrak{l}, L\cap K)$-module

$\chi_Z = \lambda \in \mathfrak{h}^* \ni (\mathrm{Re}\,\lambda + \rho(\mathfrak{u}), \alpha) \geq 0 \quad \forall \alpha \in \Delta(\mathfrak{u})$

Then (b) $\mathcal{L}_j(Z), \mathcal{R}^j(Z) = 0$ for $j < S$

(c) $\mathcal{L}_S(Z) = \mathcal{R}^S(Z)$

NOTE: There is an interplay between Z and $\bar{Z}$

<u>Theorem</u>: $\langle \mathrm{Re}\,\lambda + \rho(u), \alpha \rangle \geq 0 \ \forall \alpha \in \Delta(u)$

Then $\mathcal{L}_S(Z)$, $\mathcal{R}^S(Z)$ are irreducible or 0.

> 0 nonzero.

$GL(2n, \mathbb{R})$: $\quad \delta \longleftrightarrow (\varepsilon_1, \dots, \varepsilon_{2n}) \qquad \varepsilon_i = \pm 1$

M the diagonal

Only use 2n for convenience

$K = O(2n)$: $\quad \pi \longleftrightarrow (a_1, \dots, a_n) \qquad a_1 \geq \dots \geq a_n \geq 0$
$a_i \in \mathbb{N}$

If $a_n \neq 0$ restriction to $SO(2n)$ is a sum

$$(a_1, \dots, a_n) + (a_1, \dots, a_{n-1}, -a_n).$$

If $a_n = 0$ two rep's $(a_1, \dots, 0; +)$ & $(a_1, \dots, 0; -)$

- Need to normalize what is + and what is $-$

Fine K-types: $(1 \dots 1\, 0 \dots 0; \pm)$ & $(1 \dots 1)$.

Can realize these as $\bigwedge^k \mathbb{C}^{2n}$:

$k < n \quad (\underbrace{1 \dots 1}_{k}\, 0 \dots 0; +) \quad k = n \quad (1 \dots 1) \quad k > n \quad (\underbrace{1 \dots 1}_{n-k}\, 0 \dots 0; -)$

Each decomposes as $\sum_{w \in W / W_{\delta_k}} w \cdot \delta_k$.

Lowest K-types are unique

Can derive $SL(2n, \mathbb{R})$ from this with some work

$(0; +)$ trivial $\quad (0; -)$ determinant

$G = Sp(2n, \mathbb{R}) \supseteq U(n) = K$. M is the diagonal

$\pi \in \hat{K} \longleftrightarrow (a_1 \cdots a_n) \quad a_i \geq a_{i+1} \quad a_i \in \mathbb{Z}$

Fine K-types: $(\underbrace{1 \cdots 1}_{k} 0 \cdots 0) \quad (0 \cdots 0 \underbrace{-1 \cdots -1}_{k})$.

Parameter $\underbrace{(\nu_1 \cdots \nu_n)}_{\nu} \quad \underbrace{(\varepsilon_1, \cdots, \varepsilon_n)}_{\delta} \quad \varepsilon_i = \pm 1$

Assume ν real; $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0$

$(\underbrace{1 \cdots 1}_{k} 0 \cdots 0) = \Lambda^k \mathbb{C}^n \quad (0 \cdots 0 -1 \cdots -1) = \Lambda^k (\mathbb{C}^n)^*$

$\mathbb{C}^n \longleftrightarrow$ highest weight $(1 0 \cdots 0) \quad g \cdot v$

$(\mathbb{C}^n)^* \longleftrightarrow$ — " — $(0 \cdots 0 -1) \quad {}^t g^{-1} \cdot v$

There is an element that stabilizes (G, K) which take μ_k^+ to μ_k^- $\quad \underline{Ad\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}$

If $\nu_n \neq 0$ the μ_k & μ_{-k} in the same irreducible

If $\nu_n = 0 \quad \varepsilon_n = 1 \quad \mu_k$ and μ_{-k} together

$\varepsilon_n = -1 \quad \mu_k$ and μ_{-k} separate.

Second one comes from $SL(2, \mathbb{R}) \simeq Sp(2, \mathbb{R})$.

$\nu \longleftrightarrow P^1 = M^1 A^1 N^1$ with $M^1 \simeq GL(k_1) \times \cdots \times GL(k_r) \times Sp(2m, \mathbb{R})$

! CORRECTED FROM LECTURE!

G q-split, $\mathfrak{g}_0 = \underline{\mathfrak{k}_0} + \underline{\mathfrak{p}_0}$ $\quad \mathfrak{a}_0^s \subseteq \mathfrak{p}_0$

$\mathfrak{h}_0^s = \mathfrak{m}_0^s + \mathfrak{a}_0^s \longleftrightarrow MA$ & choose $P_0 = MAN \leftrightarrow \Delta^+(\mathfrak{g}/\mathfrak{m}+\mathfrak{a}, \mathfrak{a})$

δ a fine M-type $\quad \delta|_{(G^d \cap M)} = triv$.

μ a fine/small K-type $[\mu : \delta] \neq 0$. $A(\delta)$

$I(\delta \otimes \nu) = \operatorname{Ind}_P^G [\delta \otimes \nu]$.

LKT's of $I(\delta \otimes \nu)$ are the $A(\delta)$ occuring with

$m = 1$

$Re\, \nu \longleftrightarrow P^1 = M^1 A^1 N^1 \supseteq P$

not P. Series

$$\operatorname{Ind}_{M^1 \cap P}^{M^1} (\delta \otimes \nu) = \bigoplus I_j = \bigoplus \delta_j' \otimes \nu_j'$$

$$\operatorname{Ind}_P^G (\delta \otimes \nu) = \bigoplus \operatorname{Ind}_{P^1}^G (\delta_j' \otimes \nu_j')$$

LKT's of $\operatorname{Ind}_P^G(\delta \otimes \nu)$ distribute among the $\operatorname{Ind}_{P^1}^G(\delta_j' \otimes \nu_j')$.

Each $\operatorname{Ind}_{P^1}^G(\delta_j' \otimes \nu_j')$ has a unique irreducible quotient $\mathcal{J}(\delta' \otimes \nu')$ determined by the LKT's; also image of the intertwining operator.

Called Langlands quotients.

<u>Examples:</u> $G = GL(2n, \mathbb{R})$, $Sp(2n, \mathbb{R})$.

<u>Next Step:</u> X irreducible. μ a LKT.

$$\mu \longmapsto \lambda^G(\mu) \longmapsto \mathfrak{q} = \mathfrak{l} + \mathfrak{u}.$$

$H^R(\mathfrak{u}, X)^{\mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})} \neq 0$. a cohomology class

is a $J(\delta^i \otimes \nu^i)$ quotient of an $I(\delta^i \otimes \nu^i)$

μ^L L.K.T. of $I(\delta^i \otimes \nu^i)$ satisfies $\mu^L + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ dominant

$\lambda^L(\mu^L)$ $\qquad$ $\lambda^L(\mu^L) + \rho(\mathfrak{u}) = \lambda^G$ dominant

$$R^i_{\mathfrak{q}}[I(\delta \otimes \nu)] = \begin{cases} 0 & i \neq S = \dim(\mathfrak{u} \cap \mathfrak{k}) \\ X(\mathfrak{q}, \delta, \nu). & i = S \end{cases}$$

$X(\mathfrak{q}, \delta, \nu)$ contains a set of LKT's among which the lowest K-type(s) of X.

ν is "negative", $X \hookrightarrow X(\mathfrak{q}, \delta, \nu)(\mu)$.

In fact, $X(\mathfrak{q}, \delta, \nu) = \bigoplus X(\mathfrak{q}, \delta, \nu)(\mu)$

Example: $\mu = (1\ -1)$ in $U(2) \subseteq Sp(4,\mathbb{R})$.

$\mu + 2\rho_c = (1\ -1) + (1\ -1) = (2\ -2) - (2\ -1) = (0\ -1)$

$\lambda^G = (\frac{1}{2}\ -\frac{1}{2})$. So $\mathfrak{Z}(\lambda^G) = \underline{\mathfrak{l}} + \underline{\mathfrak{u}}$ where $2\rho(\mathfrak{u}\cap\mathfrak{p}) = (2\ -2)$

$\mathfrak{l} = \mathfrak{u}(1,1)$ and $\mathfrak{u} \longleftrightarrow \{\varepsilon_1 - \varepsilon_2, 2\varepsilon_1, -2\varepsilon_2\}$. $\rho(\mathfrak{u}) = (\frac{3}{2}\ -\frac{3}{2})$

$\mathfrak{l} \longleftrightarrow \pm\{\varepsilon_1 + \varepsilon_2\}$

A rep'n with LKT μ must come from one on $\mathfrak{u}(1,1)$. λ^G is on the center. The fine K-types (up to the center of $\mathfrak{u}(1,1)$) are $(0|0)$ $\varepsilon_+ = (1|0)$ and $\varepsilon_- = (0|1)$. In the appropriate coordinates, we take $\delta \otimes \nu$

$(\frac{1}{2}|-\frac{1}{2}) + (\nu|\nu)$ is infinitesimal character.

$\lambda^G - \rho(\mathfrak{u}) = (-1|1)$ $\quad \mu - 2\rho(\mathfrak{u}\cap\mathfrak{p}) = (-1|1)$.

$SU(1,1)$ is $SL(2,\mathbb{R})$, has 3 fine/small K-types

To get the same λ^G need to look at $Mp(2,\mathbb{R})$

$\mu^+ = (\frac{3}{2}\ -\frac{1}{2}) + (2\rho_c = (1\ -1)) = (\frac{5}{2}\ -\frac{3}{2}) - (\rho = (2\ -1)) = (\frac{1}{2}\ -\frac{1}{2})$

$\mu^- = (\frac{1}{2}\ -\frac{3}{2}) + (2\rho_c) - (\rho = (1\ -2)) = (\frac{1}{2}\ -\frac{1}{2})$

$(\frac{3}{2}\ -\frac{1}{2}) + 2\rho_c = (\frac{5}{2}\ -\frac{3}{2})$, $(\frac{1}{2}\ -\frac{3}{2}) + 2\rho_c = (\frac{3}{2}\ -\frac{5}{2})$

$(1\ -1) + 2\rho_c = (2\ -2)$

do not give the same $\|\mu + 2\rho_c\|^2$; but $(1\ -1)$ and $\{(\frac{3}{2}\ -\frac{1}{2}), (\frac{1}{2}\ -\frac{3}{2})\}$ never show up in the same X.

Parameter $(\lambda^G, \nu) = \{(\frac{1}{2}\ -\frac{1}{2}) + (\nu\ \nu)\}$.

When $\nu = 0$, $\mu^{\pm}$ give different irreducible modules

$\nu \neq 0$ just one.

Can also look at $MAN \longleftrightarrow \mathrm{Re}\,\underline{\nu}$.

If $\underline{\nu} = (\nu\ \nu; -\nu\ -\nu)$ in standard realization, centralizer is a $GL(2,\mathbb{R})$ (a cover thereof for $Mp(2,\mathbb{R})$)

$\underline{\nu} = (0\ 0; 0\ 0)$ gives all of $Sp(4,\mathbb{R})$.

$\mu_0 = (a\ -a) \mapsto \mu + 2\rho_c = (a+1, -a-1)$

Can choose $\{2\varepsilon_1, \varepsilon_1+\varepsilon_2, +\varepsilon_1-\varepsilon_2, -2\varepsilon_2\}$ $(2,-1)$
two systems:
$\{2\varepsilon_1, -\varepsilon_1-\varepsilon_2, \varepsilon_1-\varepsilon_2, -2\varepsilon_2\}$ $(1-2)$.

Get the same λ. Only one LKT. $\lambda^G = (a-\frac{1}{2}\ -a+\frac{1}{2})$

$\mu^+ = (a+\frac{1}{2}\ -a+\frac{1}{2}) \mapsto \mu + 2\rho_c = (a+\frac{3}{2}, -a-\frac{1}{2}) \mapsto \lambda^G = (a-\frac{1}{2}, -a+\frac{1}{2})$

$\mu^- = (a-\frac{1}{2}, -a-\frac{1}{2}) \mapsto \mu + 2\rho_c = (a+\frac{1}{2}, -a-\frac{3}{2}) \mapsto \lambda^G = (a-\frac{1}{2}, -a+\frac{1}{2})$

Only one positive system for each $\mu \mapsto (a-\frac{1}{2}, -a+\frac{1}{2})$.

Two LKT's: $2\rho(\mathfrak{u}\cap\mathfrak{p}) = (2\ -2)$.

$\mu_0 - 2\rho(\mathfrak{u}\cap\mathfrak{p}) = (a-2, -a+2) + (0\ 0)$

$$\left.\begin{array}{l}\mu^+ - 2\rho(\mathfrak{u}\cap\mathfrak{p}) = (a-2, -a+2) + (\frac{1}{2}\ \frac{1}{2}) \\ \mu^- - 2\rho(\mathfrak{u}\cap\mathfrak{p}) = (a-2, -a+2) + (-\frac{1}{2}\ -\frac{1}{2})\end{array}\right\}$$

These are fine$(L\cap K)$-types $\longleftrightarrow$ fine δ's

The example of $\mu_0 = (a, 1)$, $\mu_+ = (a, 2)$ and $\mu_- = (a, 0)$ does not involve covers:

$(a\ 0) + (1\ -1) = (a+1, -1) - (2\ -1) = (a-1, 0)$

$(a\ 1) + (1\ -1) = (a+1, 0) - (2\ -1) = (a-1, 1) + (0\ -1) = (a-1, 0)$

$(a\ 2) + (1\ -1) = (a+1\ 1) - (2\ 1) = (a-1, 0)$

$\mathfrak{q}(\lambda) = \mathfrak{l} + \mathfrak{u}$ with $\mathfrak{l} \simeq \mathfrak{u}(1) \times \mathfrak{sp}(2,\mathbb{R})$

and $\mu_0, \mu_\pm$ correspond to the fine K-types of

$\mathfrak{sp}(2,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R})$.

Parameters are (λ^G, ν) & $\mu_0, \mu_\pm$.

$\mu_\pm$ occur together except for $\nu = 0$.