5-3-2016

Note Title 5/2/2016

G split: $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ max'l abelian 1

$\mathfrak{a}_0$ a CSA for $\mathfrak{g}_0$.

$P = MAN$ a minimal parabolic, M finite

$\delta \in \hat{M}$, $\nu \in \hat{A}$ $\qquad \rho = \frac{1}{2} \sum_{\alpha \in \Delta(\underline{n})} \alpha$

$$I(\delta \otimes \nu) = \{ f : G \to V_\delta \mid f(gman) = \delta(m)^{-1} a^{-(\nu+\rho)} f(g) \}$$

$\pi(g) f(x) := f(g^{-1}x)$.

f measurable, $f|K \in L^2(K, V_\delta)$.

K-finite functions.

$A(\delta) := \{ \pi \in \hat{K} \rightsquigarrow \text{hwt } \mu, \text{ LKT of } I(\delta \otimes \nu) \}$

LKT $\|\mu\|_{\text{lambda}}$ minimal $\Leftrightarrow$ $\|\mu + 2\rho_c\|^2$ minimal

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THM: $\mu \in A(\delta) \Leftrightarrow \mu$ is small, $[\mu : \delta] \neq 0$

Intertwining Operators

$G = SL(2, \mathbb{R})$ $\qquad \delta = \text{trivial}$

$P = MAN$ $\quad$ & $\quad \bar{P} = MA\bar{N}$

$I(P, \delta, \nu)$ $\qquad I(\bar{P}, \delta, \nu)$

An intertwining operator is a map

$A : (\pi, V) \longrightarrow (\pi', V')$

$A \circ \pi(g) = \pi'(g) \circ A$

If A has nontrivial kernel, (π, V) is reducible

If A has proper image, (π', V') reducible

$$A(\bar{P}, P, \delta, \nu) f(x) := \int_{\bar{N}} f(x\bar{n})\, d\bar{n}$$ 2

$$(Af)(xa) = \int_{\bar{N}} f(xa\bar{n})\, d\bar{n} =$$

$$= \det\left(\mathrm{Ad}\, a\big|_{\bar{\mathfrak{n}}}\right)^{-1} \int_{\bar{N}} f(x\bar{n}a)\, d\bar{n} =$$

$$= a^{2\rho} \cdot a^{-\rho-\nu} Af(x).$$

Similar for $\bar{N}$.

So $A : I(P, \delta, \nu) \longrightarrow I(\bar{P}, \delta, \nu)$

PROBLEM WITH CONVERGENCE

$$f(r(\theta)) = e^{in\theta} \quad (n \text{ even}).$$

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \cdot \infty = \frac{1}{x} = \cot\theta \Rightarrow \tan\theta = x$$

$$\cos\theta = (1+x^2)^{-1/2}, \quad \sin\theta = x(1+x^2)^{1/2}$$

$$e^{in\theta} = \left(\frac{1+ix}{(1+ix)^{1/2}(1-ix)^{1/2}} \right)^n = \left(\frac{1+ix}{1-ix} \right)^{\frac{n}{2}}$$

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \cdot i = \frac{i}{-xi+1} = e^{-2t}(i+z)$$

$$\frac{i(1+ix)}{1+x^2} \Rightarrow e^{-t} = (1+x^2)^{-1/2}$$

$$Af_n(1) = \int_{-\infty}^{\infty} \frac{(1+ix)^{\frac{n-\nu-1}{2}}}{(1-ix)^{\frac{n+\nu+1}{2}}}\, dx$$

Function in ν $(\neq 0)$ $\cdot \dfrac{1}{\Gamma\left(\frac{\nu+1+n}{2}\right)\Gamma\left(\frac{\nu+1-n}{2}\right)}$.

$$= \int_{-\infty}^{\infty} \frac{(1+xi)^{\frac{n+1-\nu}{2}}}{(1-xi)^{\frac{n+1+\nu}{2}}}\,dx$$

(3)

This integral converges for $\mathrm{Re}\,\nu > 0$

Has kernel $\neq 0$ at $\nu = k+1$

In general, $P = MAN$, $P' = MAN'$

$$A(P', P, \delta, \nu) f(x) = \int_{\bar{N} \cap N'} f(x\bar{n})\, d\bar{n}$$

$N' = wNw^{-1}$

$\bar{N} \cap wNw^{-1} \longleftrightarrow \{\alpha \in \Delta^- : w^{-1}\alpha \in \Delta^+\}$

Can factor into simpler operators

$\alpha \in \Delta^+$ simple $\ni w^{-1}\alpha \in \Delta^-$

$$N \qquad s_\alpha N s_\alpha^{-1} \qquad wNw^{-1}$$

$\bar{N} \cap s_\alpha N s_\alpha^{-1} = N_\alpha \ni$ span $N_\alpha, N_{-\alpha}$ is an $SL(2)$.

$s_\alpha N s_\alpha^{-1} \longrightarrow wNw^{-1}$ is shorter and factors further into $SL(2)$-intertwiners.

COROLLARY: If $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ $\forall \alpha \in \Delta^+ \ni w^{-1}\alpha \in \Delta^-$ then the operator converges.

THM If $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ $\forall \alpha \in \Delta(\mathfrak{n})$ then $I(\delta, \nu)$ has a unique irreducible quotient.

If $\langle \mathrm{Re}\,\nu, \alpha\rangle < 0$ unique irreducible submodule

Image of $A(\bar{P}, P, \delta, \nu)$

Integral Formulas

THM: If $\delta\otimes\nu$ is unitary, so is $I(\delta\otimes\nu)$.

Pf: $F\in I(\delta\otimes\nu)$, $F(gman)=\delta(m)^{-1}a^{-\nu-\rho}F(g)$

$|F(g)|^2$ satisfies $\mathcal{F}(gman)=a^{-2\rho}\mathcal{F}(g)$

because M is cpct so δ unitary

$\nu=\mathrm{Re}\,\nu+i\,\mathrm{Im}\,\nu$ unitary $\iff \mathrm{Re}\,\nu=0$

& so $\bar{\nu}=-\nu$.

Let $\psi\geq 0$, $\psi(kan)=\psi(k)$ be $\ni$

$$\int\psi(gan)\,da\,dn=1 \quad \forall g\in G$$

$$\int_K\mathcal{F}(g^{-1}k)\,dk=\int_K\mathcal{F}(g^{-1}k)\int_{AN}\psi(kan)\,dk\,da\,dn$$

$$=\int_{KAN}\mathcal{F}(g^{-1}kan)\,a^{2\rho}\psi(kan)\,dk\,da\,dn=$$

$$=\int_G\mathcal{F}(g^{-1}x)\psi(x)\,dx=\int_G\mathcal{F}(x)\psi(gx)\,dx$$

$$=\int_{KAN}\mathcal{F}(kan)\,a^{2\rho}\psi(gkan)\,dk\,da\,dn=$$

$$\int_K\mathcal{F}(k)\,dk$$

$$\int_G F(x)\,dx=\int_{KAN}F(kan)\,a^{2\rho}\,dk\,da\,dn=$$

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$$= \int_{KNA} F(kna)\, dk\, dn\, da$$

$g = \mathcal{K}(g)\, a(g)\, n(g)$ Iwasawa Decomposition

$G = KAN$, $\bar{N} = \Theta N$ opposite

$$\int_{M\bar{N}} f(m\mathcal{K}(\bar{n}))\, a(\bar{n})^{-2\rho}\, d\bar{n} = \int_K f(k)\, dk$$

normalized so $\int_{\bar{N}} a(\bar{n})^{-2\rho}\, d\bar{n} = 1 = \int_K dk$

A BASIC FORMULA (Langlands–Miličić)

$$\lim_{a\to\infty} e^{-(\nu-\rho)(\log a)} \langle \pi(ma)f, g\rangle$$

$$= \langle \delta(m)\, A(\bar{P}, P, \delta, \nu) f(1), g(1)\rangle_\delta$$

$a \to \infty$ so that $a^\alpha \to \infty$ $\forall \alpha \in \Delta(\mathfrak{n})$

COROLLARY: If $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ $\forall \alpha \in \Delta(\mathfrak{n})$ then $I(\delta\otimes\nu)$ has a unique irreducible quotient; image of A.

see Knapp, Lemma 7.23 for the formula.

We need to show that for any proper $Y \subsetneq I(\delta\otimes\nu)$, $A(\bar{P}, P)Y = 0$

Say $f \in I(\delta \otimes \nu) \ni \text{span}\, \overline{\pi(x) f} \neq I(\delta\otimes\nu)$

$\exists\, g \in I(\delta \otimes \nu)$ $g \not\equiv 0 \ni \langle \pi(x) f, g \rangle_\delta = 0$

$\forall x \in G$. $x = k_1 m a k_2$.

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$\langle \pi(ma)\pi(k_2) f, \pi(k_1^{-1}) g \rangle_\delta = 0$ $\forall$ ...

Take the limit:

$\langle \delta(m) A \circ \pi(k_2) f(1), \pi(k_1^{-1}) g(1) \rangle_\delta = 0$

$\forall k_1, k_2, m$. Because δ is irred,

$A \circ \pi(k_2) f(1) = A f(k_2^{-1}) = 0$ $\forall k_2$ or

$\pi(k_1^{-1}) g(1) = g(k_1^{-1}) = 0$ $\forall k_1$

$g \not\equiv 0 \implies Af \equiv 0$. QED.

Image of A called Langlands quotient $\overline{I(\delta \otimes \nu)}$.

More general, P^1 given by $\operatorname{Re} \nu$:

$P^1 = M^1 A^1 N^1$, $M^1 \leftrightarrow \{\alpha \in \Delta : \langle \operatorname{Re}\nu, \alpha \rangle = 0\}$

$\operatorname{Ind}_{MA}^{M^1 A^1} (\delta \otimes \nu) = \bigoplus \delta^i \otimes \nu^i$

Each $\operatorname{Ind}_{P^1}^{G} (\delta^i \otimes \nu^i)$ has a unique irreducible quotient.

GOAL: Match $\overline{I(\delta^i \otimes \nu^i)}$ with lowest K-types

VARIANT OF INDUCTION

$P' = wPw^{-1}$. $\quad P' = MA\underbrace{wNw^{-1}}_{N'}$

$$T_w : I_P(\delta \otimes \nu) \xrightarrow{\sim} I_{P'}(w\delta \otimes w\nu)$$

$$(T_w f)(g) := f(gw)$$

$$T_w f(gwnw^{-1}) = f(gwnw^{-1}w) = f(gw) = T_w f$$

$$T_w f(gma) = f(gmaw) = \delta(w^{-1}mw)^{-1} a^{-w\nu - w\rho} f(g)$$

Can use $T_w \circ A : I(\delta \otimes \nu) \to I(w\delta \otimes w\nu)$