Integral Formula $G = KMAN$ $g = \mathcal{K}(g)a(g)n(g)$

$f \in C(K/K\cap M)$ $$\int_K f(k)\,dk = \int_{\bar N} f(\mathcal{K}(\bar n))\,a(\bar n)^{-2\rho}\,d\bar n$$

Pf: Extend f to $F \in I(\mathrm{triv} \otimes -\rho)$

$$F(gman) = a^{-2\rho} F(g)$$

$\varphi > 0$, $\varphi(kman) = \varphi(an)$ & right invariant under $M\cap K$ $\quad \int_{MAN} \varphi(gman)\,dm\,da\,dn = 1 \quad \forall g$

$$\int_K f(k)\,dk = \int_K f(k)\int_{MAN} \varphi(kman)\,dm\,da\,dn\,dk =$$

$$= \int_G F(x)\,\varphi(x)\,dx$$

$$\int_{\bar N} F(\mathcal{K}(\bar n))\,a(\bar n)^{-2\rho}\,d\bar n =$$

$$\int_{\bar N} F(\mathcal{K}(\bar n)a(\bar n)n(\bar n))\,d\bar n = \int_{\bar N} F(\bar n)\int_{MAN} \varphi(\bar n man)\,dm\,da\,dn\,d\bar n = \int_G F(x)\,\varphi(x)\,dx$$

$$\int_G f(x)\,dx = \int_{\bar N MAN} f(\bar n man)\,a^{2\rho}\,dm\,da\,dn\,d\bar n$$

$$= \int_{\bar{N}MAN} F(\mathcal{K}(\bar{n}))\, a(\bar{n})^{-2\rho}\, \varphi(\mathcal{K}(\bar{n})\, man)\, dm\, da\, dn\, d\bar{n}$$

Intertwining Operators

$P = MAN \qquad P' = MAN' \qquad P'' = MAN''$

$\mathfrak{n}'' \cap \bar{\mathfrak{n}} \subseteq \mathfrak{n}' \cap \bar{\mathfrak{n}} \qquad \langle \mathrm{Re}\ \nu, \beta \rangle > 0\ \forall \beta \in \Delta(\mathfrak{n}), -\Delta(\mathfrak{n}'')$

$$A(P'', P, \delta \otimes \nu) = A(P'', P', \delta \otimes \nu) \circ A(P', P, \delta \otimes \nu)$$

Proposition 7.9 Kn

PROOF OF "UNIQUE QUOTIENT" ~~5-5-2~~

FOR G split, $I_P(\delta\otimes\nu)$ with $\mathrm{Re}\,\nu>0$

Want to show: If $\mathcal{Y}\subseteq I_P(\delta\otimes\nu)$ is a proper subspace, $A\mathcal{Y}=0$

Since $\mathcal{Y}$ is proper, $\exists\, g\not\equiv 0$ in $I_P(\delta\otimes\nu)$ $\ni$ $g\perp\overline{\mathcal{Y}}$. From the limit formula

$$\langle \pi(x)f, g\rangle = 0 \quad \forall x = k_1 m a k_2$$

$$\langle \pi(m a k_2)f, \pi(k_1^{-1})g\rangle = 0 \quad \forall k_1, m, a, k_2$$

$$\langle \delta(m) A(\bar{P},P)\pi(k_2)f(1), \pi(k_1^{-1})g(1)\rangle_\delta = 0$$

δ irreducible $\Longrightarrow$

$$0 = A(\bar{P},P)\pi(k_2)f(1) = \pi(k_2)A(\bar{P},P)f(1) = A(\bar{P},P)f(k_2) \quad \forall k_2$$

or

$$0 = \pi(k_1^{-1})g(1) = g(k_1^{-1}) \quad \forall k_1$$

SO $A(\bar{P},P)f \equiv 0$ QED $\square$

5-5-2016

5-5-4

G split $P = MAN$, $\underset{P}{I}(\delta \otimes \nu)$. ~~5-5-1~~

$\mathrm{Re}\,\nu \rightsquigarrow P^1 \supset M^1 A^1 N^1$: $\Delta(M^1) \leftrightarrow \langle \mathrm{Re}\,\nu, \alpha \rangle = 0$

$\Delta(N^1) \leftrightarrow \langle \mathrm{Re}\,\nu, \alpha \rangle > 0$

$P_1 = M^1 \cap P$ is a minimal parabolic

$$\underset{P_1}{I}(\delta \otimes \mathrm{Im}\,\nu) = \bigoplus \delta^i \otimes \nu^i$$

$$I(\delta \otimes \nu) = \bigoplus I(\delta^i \otimes \nu^i)$$

THM: Each $I(\delta^i \otimes \nu^i)$ has a
(L-M) unique $J(\delta^i \otimes \nu^i)$ irreducible quotient

$\simeq$ Image of an $A(\overline{P^1}, P^1)$.

This can be generalized to give a classification of the irreducible $(\mathfrak{g}, K)$-modules called Langlands Classification

---

The δ^i are replaced by DISCRETE SERIES / TEMPERED REP'S

We aim to do this via LKT's and $J(\delta^i \otimes \nu^i)$ for δ^i small

G split $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, $\mathfrak{k}_0 \supseteq \mathfrak{t}_0$ $\mathfrak{p}_0 \supseteq \mathfrak{a}_0$

$P = MAN$ $N \leftrightarrow \Delta^+(\mathfrak{g}_0/\mathfrak{a}_0, \mathfrak{a}_0)$

DEF: $\pi \in \hat{K} \rightsquigarrow \mu$. μ fine means $\mu(iZ_\alpha) \in \{0, \pm 1\}$

all α are real

$\delta \in \hat{M}$ is fine/small means $\delta|_{(G^d \cap M)_0} = \text{triv}$

$A(\delta) := \{\mu \text{ fine} \ni [\mu|_M : \delta] = 0\}$

THM (4.3.16) (a) $A(\delta) \neq 0$

(b) $\mu|_M = \bigoplus_{w \in W/W_\delta} w \cdot \delta$

(c) $\hat{R}_\delta$ acts "naturally" on $A(\delta)$.

action given by an automorphism of G commuting with θ and acting trivially on MA.

PROP: π small $\Longleftrightarrow$ π fine

(all δ are small for G split)

THM (L-M) identifies Langlands (sub)quotients $J(\delta^i \otimes \nu^i)$.

THM: Every $J(\delta^i \otimes \nu^i)$ contains a fine/small K-type; necessarily a LKT, with $m = 1$.

The way the LKT's are distributed depends on ν (and δ)

Parametrized by an $R_{\delta,\nu}$.

Example $G = Sp(2n,\mathbb{R})$ $\quad \mathfrak{g}_0 = \left(\begin{array}{c|c} a & b \\ \hline c & -a^t \end{array}\right)$ $\quad b^t = b$, $c^t = c$

$\mathfrak{a}_0 = \mathrm{diag}$ $\quad M = \mathrm{diag}(\pm 1 \cdots \pm 1; \pm 1 \cdots \pm 1) = \prod \mathbb{Z}_2$

$W = M'/M$ (acts by) permutations and sign changes.

$\delta \leftrightarrow (\varepsilon_1 \cdots \varepsilon_n)$ $\quad \varepsilon_i \pm 1$ $\quad \delta_k = (\underbrace{1 \cdots 1}_{k}\ -1 \cdots -1)$

$w.\delta = (\varepsilon_{\sigma(1)}, \cdots, \varepsilon_{\sigma(n)})$ $\quad \sigma \in S_n$

FINE K-TYPES: $\mu_k = (\underbrace{1 \cdots 1}_{k}\, 0 \cdots 0)$, $\mu_{-k} = (0 \cdots 0\, \underbrace{-1 \cdots -1}_{k})$

$$\mu_{\pm k}\big|_M = \bigoplus_{w \in W/W_{\delta_k}} w.\delta$$

μ_k and μ_{-k} related by $Ad\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \notin Sp(2n,\mathbb{R})$

REMARK: An alternate is to define

$$A(\delta) := \{\pi \text{ small} \mid [\pi : \delta] \neq 0\}$$

and prove everything using the subquotient theorem;

THM (H-C) Every admissible (π, V) is a subquotient of an $I_P(\delta \otimes \nu)$. G arbitrary.

V-book does without it.

Example

$$\begin{array}{l} (\overbrace{1 \cdots 1}^{k} \quad 0 \cdots 0) \\ + (n-1 \quad n-2k+1 \; n-2k-1 \cdots -n+1) \\ \hline (n, \dots, n-2k+2, n-2k-1, \dots, -n+1) \end{array}$$

$$\begin{array}{r|r} (1110) & (1000) \\ +(31{-1}{-3}) & (31{-1}{-3}) \\ \hline (420{-3}) & (41{-1}{-3}) \\ (42{-1}{-3}) & -(42{-1}{-3}) \\ \hline (00{-1}0) & (0{-1}00) \\ +\frac{1}{2}(0020) & +(0\ \tfrac{1}{2}\ \tfrac{1}{2}\ 0) \\ \hline (0000 & +(0\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\ 0) \\ & \overline{(0000).} \end{array}$$

MODIFICATION FOR G $\underline{q}$-split

$\Delta^+(\mathfrak{g}_0/\underline{m}_0 + \mathfrak{a}_0, \mathfrak{a}_0)$ has real and complex roots

(Roots of $(\mathfrak{g}, m+\mathfrak{a})$ which are 0 on $\underline{m}$ (real)
$\neq 0$ on $\underline{m}$ (cx)

FOR REAL ROOTS, same condition.

For a cx root $\alpha \in \Delta^+$ look at $\pm\alpha$ and $\pm\theta\alpha$; the $\pm\alpha$ root vectors in $\mathfrak{g}_0$ generate a real algebra $\underline{m}_0^\alpha$ equal to $su(2,1)$ or $sl(2,\mathbb{C})$ as a real algebra

It is θ-stable, and we require same for α real,

μ fine <u>means</u> $\mu|_{k_0^\alpha \cap \mathfrak{g}_0^d} = \text{triv}$

α cx

δ fine <u>means</u> $\delta|_{\underline{m}_0^\alpha \cap \mathfrak{g}_0^d} = \text{triv}$

Example $\mathfrak{g}_0 = \mathfrak{su}(2,1)$

$$\left[\begin{array}{c|c} \alpha & \beta \\ \hline \beta^* & \delta \end{array}\right] \qquad \alpha + \alpha^* = 0 \qquad \delta + \delta^* = 0$$

$$\mathfrak{a}_0 = \mathbb{R}\left[\begin{array}{cc|c} & & \\ & 0 & 1 \\ \hline & 1 & 0 \end{array}\right] \qquad \mathfrak{m}_0 = \begin{bmatrix} i\alpha & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & +i\theta \end{bmatrix} \qquad \alpha + 2\theta = 0$$

Roots $\pm 2\lambda$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & i & +i \\ 0 & +i & -i \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & -i & -i \end{bmatrix}$$

$\pm\lambda$

$$J = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \qquad JX + X^* J = 0$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} + \begin{bmatrix} a_{31}^* & a_{21}^* & a_{11}^* \\ a_{32}^* & a_{22}^* & a_{12}^* \\ a_{33}^* & a_{23}^* & a_{13}^* \end{bmatrix} = 0$$

$$\begin{matrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{matrix} \qquad \begin{bmatrix} a_{11} & a_{12} & ia_{13} \\ a_{21} & ia_{22} & -a_{12}^* \\ ia_{31} & -a_{21}^* & -a_{11}^* \end{bmatrix} \qquad \theta(X) = -X^*$$

CSA $\mathfrak{m}_0 + \mathfrak{a}_0$ is the diagonal; $\mathfrak{a}_0 = \mathbb{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

ROOT VECTORS: $\pm 2\lambda$ $\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$

$\pm\lambda$: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix}$ & lower triangular ones $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{bmatrix}$

The reduced roots are $\pm\lambda$

k_0^α is spanned by $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{bmatrix}$

and the brackets. Get $su(2) \times su(1) \subseteq s[u(2) \times u(1)]$

Only fine K-types are $\mathrm{diag}(a,b) \mapsto \det a = (\det b)^{-1}$

! G - split ! 5-5-10

DEF: $\delta \in \hat{M}$. $\Delta_\delta = \{ \alpha \in \Delta \ni \delta(m_\alpha) = 1 \}$

LEMMA (4.3.12): Δ_δ is a root system

DEF: $\overset{\circ}{W}_\delta = W(\Delta_\delta)$

SEE EXAMPLE OF $sp(2n, \mathbb{R})$ & $\delta = \delta_k$

$M'_\delta = \mathrm{Stab}_{M'}\, \delta$

$W_\delta = \mathrm{Stab}_W\, \delta$

$R_\delta = W_\delta / \overset{\circ}{W}_\delta$

Lemma (4.3.14) $\overset{\circ}{W}_\delta$ is normal in W_δ & every element of R_δ has order 2.

Let $\Delta_\delta^+ \subset \Delta_\delta$ be a positive system, $\longmapsto \rho_\delta$

$\Delta_S = \{ \alpha \in \Delta : \langle \alpha, \rho_\delta \rangle = 0 \}$

Then Δ_S is formed of strongly orthogonal roots with $\delta(m_\alpha) = -1$.

$R_\delta^c := \{ w \in W_\delta : w(\Delta_\delta^+) = \Delta_\delta^+ \}$

LEMMA (4.3.29) $W_\delta = R_\delta^c \cdot \overset{\circ}{W}_\delta$ semidirect product

$R_\delta^c \subseteq W(\Delta_S)$.

(SHORT) SUMMARY OF PROOF OF (C).

(1) $W = M'/M$. Let $M'_\delta := \mathrm{Stab}_{M'}(\delta)$.

Then if μ is in $A(\delta)$, $\mu|_{M'}$ is irreducible; in fact

$\mu|_{M'} = \mathrm{Ind}_M^{M'}(\delta)$.

$M'_\delta := \mathrm{Stab}_{M'}\delta$ $\qquad \mu_\delta :=$ δ-isotypic component of μ

μ_δ is irreducible under M'_δ.

$\mu, \mu' \in A(\delta) \longmapsto r(\mu,\mu') := (\mu_\delta) \otimes (\mu'_\delta)^*$

a rep'n of M'_δ. This is trivial on M'_δ.

So it is a rep'n of R_δ:

$M_\theta := \{ x \in A^d_{\mathbb{C}} : x^2 = 1 \}$ (G^d is the adjoint group)

<u>Lemma</u>: $x \in M_\theta$. Then $\mathrm{Ad}x \circ \theta = \theta \circ \mathrm{Ad}x$, exponentiates
$\mathrm{Ad}x \cdot \mathfrak{g}_0 = \mathfrak{g}_0$
to $\mathrm{Aut}(G_0)$. There is a choice that centralizes M;
this extends uniquely to an automorphism of G
which preserves K and centralizes MA.

5-5-12

<u>Example:</u> $G = Sp(2n, \mathbb{R})$ $\quad \delta = \delta_k$

$\Delta_\delta = \{ \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \;\; i,j \leq k, \;\; \pm\varepsilon_l \pm \varepsilon_m \;\; l,m > k \}$

system of type $C_k \times D_{n-k}$ $\quad W_\delta^0 = W(C_k) \times W(D_{n-k})$

M_δ' is $W(C_k) \times W(C_{n-k})$

you can see $M_\delta' / M_\delta \simeq \mathbb{Z}_2$

$\Delta_\delta^+ = \{ \varepsilon_i \pm \varepsilon_j, 2\varepsilon_i \;\; i < j \leq k, \;\; \varepsilon_l \pm \varepsilon_m \;\; k < l < m \}$

$\rho_\delta = (k, \dots, 1, n-k-1, \dots, 0)$

$\Delta_S = \{ \varepsilon_i - \varepsilon_{n-i}, 2\varepsilon_n \}_{1 \leq i \leq k \text{ or } n-k}$ depending which is smaller

$R_\delta^c := \{ w$ needs to be $s_{2\varepsilon_n} \}$

5-5-13

M_θ is given by $\{\operatorname{diag}(e^{\pi k_1 i} \cdots e^{\pi k_n i}, e^{-\pi k_1 i} \cdots e^{-\pi k_n i})\}$

with $k_i \in \mathbb{Z}$. Need $x^2 \in$ Center, so this allows

$k_1 = \cdots = k_n = 1/2$ $\quad \operatorname{diag}(i \cdots i, -i \cdots -i)$.