Infinite Dimensional Representations of Real Reductive Groups, Spring 2016

1. Locally Convex topological Vector Spaces

Definition (1)**.** *A vector space V is a locally convex topological vector space if it has a topology generated by translations of balanced absorbent convex sets.*

- C **convex**, $\forall x, y \in C, 0 \le t \le 1, (1-t)x + ty \in C$.
- C **balanced**, $\forall x \in C, |\lambda| \le 1, \lambda x \in C$.
- C **absorbent** $\cup_{t>0} tC = V$.

Definition (2)**.** *A seminorm on V is a map $p : V \longrightarrow \mathbb{R}$, such that*

(1) $p(v) \ge 0, \forall v \in V$,
(2) $p(x+y) \le p(x) + p(y)$,
(3) $p(\lambda x) = |\lambda| p(v)$.

A vector space V is called locally convex if its topolgy is defined by a family of seminorms $\{p_\alpha\}$.

For such a space, Hausdorff $\iff$ $p_\alpha(v) = 0 \quad \forall \alpha,$ implies $v = 0$.

Theorem. *The two definitions are equivalent.*

See https://en.wikipedia.org/wiki/Locally_convex_topological_vector_space for a more ample explanation and references.

We will mostly deal with Banach and Hilbert spaces.

2. Differentiable and Analytic maps

Let $f : U \subset \mathbb{R}^n \longrightarrow V$ where V is a Banach space. Define

$$\frac{\partial}{\partial t_i} f(x) = \lim_{t \to 0} \frac{f(x + t_i) - f(x)}{t_i}$$

Definition. *f is called C^k with $k \ge 0$ in the usual way, if all partials up to order k exist and are continuous; similarly for C^∞.*

f is called analytic, if there are $v_m \in V$ with $m = (m_1, \dots, m_n)$ a multi-index such that for any $x^0 \in U$ there is a small ball such that

$$f(x) = \sum v_m (x - x^0)^m \quad \textit{such that} \quad \sum_m |v_m| \cdot |x - x^0|^m < \infty.$$

The notation is $(x - x^0)^m := (x_1 - x_1^0)^{m_1} \dots (x_n - x_n^0) m_n$.

3. Representations

3.1. The groups G are at least Haudorff topological groups, locally compact, with countable base and countable at ∞. They will be real Lie groups, so in fact analytic manifolds. Connected groups admit Haar measures, i.e. left invariant Borel measures $d_l g$. A Haar measure is unique up to a constant factor. Similarly for right invariant Haar measures $d_r g$. The relation between them is given by the modulus function $\delta(g)$. For a connected Lie group, the Haar measure is given by tbe left or right invariant form.

REFERENCE: F. Warner, Lie groups.

3.2. Finite Dimensional Representations. Let $\Phi : G \longrightarrow H$ be a continuous group homomorphism between two real Lie groups. In particular, a continuous representation (π, V) of a (connected for simplicity) real Lie group G where $\dim V < \infty$ is of this form, $\pi : G \longrightarrow GL(V)$. In the general Lie group homomorphism case, Φ is real analytic (by the Baker-Campbell-Hausdorff theorem). There is a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\Phi} & \mathfrak{h} \\ \Big\downarrow{\scriptstyle \exp_G} & & \Big\downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{\Phi} & H \end{array}$$

So we can study finite dimensional representations Φ by studying the representations of the Lie algebras $\mathfrak{g}$, or the universal enveloping algebra $U(\mathfrak{g})$..

Theorem. *Suppose $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism and G is simply connected. Then there is a Lie homomorphism $\Phi : G \longrightarrow H$ such that $d\Phi = \phi$.*

This does not exactly work for infinite dimensional representations. For example the left and right regular representations of a group G on $L^2(G)$ or some of the L^p-spaces will not differentiate:

$$L_g f(x) := f(g^{-1}x), \qquad R_g f(x) := f(xg).$$

The model on how to study such representations are Fourier Analysis on $L^2(S^1)$ and $L^2(\mathbb{R})$.

3.3. Infinite Dimensional Representations.

Definition. *A group homomorphism $\pi : G \longrightarrow V$ into a complete locally convex topological space will be called*

(1) ***continuous*** $\Pi : G \times V \longrightarrow V$ *is continuous*
(2) ***strongly continuoous*** $\forall v \in V, g \to \pi(g)v$ *is continuous*
(3) ***weakly continuous*** $\forall v \in V, \ell \in V^*, g \to \ell(\pi(g)v)$ *is continuous.*
(4) ***norm continuous*** *for* V *a Banach space,* $\pi : G \longrightarrow Aut(V)$ *where* $Aut(V)$ *has the norm topology given by* $||T|| := \sup_{|v|\le 1} |Tv|$.

V^* is the continuous dual of V.
It is straightforward that

$$(4) \Longrightarrow (1) \Longrightarrow (2) \Longrightarrow (3).$$

The fact that

$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3)$$

is also true, but somewhat harder.
We will be concentrating on the case of V a Banach space and π norm continuous.
REFERENCE G. Warner vol. I.

3.4.

Definition. *For* G *a Lie group, we say* $v \in V^\infty$ *if the map* $g \to \pi(g)v$ *is* C^∞*. Similarly* $v \in V^\omega$ *means* $g \mapsto \pi(g)v$ *is analytic.*

If $X \in \mathfrak{g} := Lie(G)$, then we define

$$\pi(X)v := \frac{d}{dt}\bigg|_{t=0} \pi(\exp tX)v.$$

Identify $\mathfrak{g}$ with the space of left-invariant vector spaces. Then the universal enveloping algebra $U(\mathfrak{g})$ is the space of left-invariant differential operators; V^∞ is a representation of $U(\mathfrak{g})$. V^∞ is also a representation of G:

$$\pi(X)\pi(g) = \pi(g)\pi(\mathrm{Ad}g^{-1}X)v.$$

Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$. In this discussion we have used the fact that the maps

$$X \longrightarrow g\exp(X),$$
$$(t_1, \dots t_n) \longrightarrow g\exp(t_1X_1)\cdot\dots\cdot exp(t_nX_n)$$

give charts for the Lie group. Similarly for multiplying with g on the right.

Theorem. *$V^\omega \subset V^\infty \subset V$ are dense $G-$invariant subspaces.*

3.5. $V^\infty \subset V$ **is dense.** Let $f \in C_c(G)$. Then we can define

$$\pi(f)v := \int_G f(g)\pi(g)v \; d_l g.$$

This is well defined and bounded:

$$\left| \int_G f(g)\pi(g)v \; d_l g \right| \le sup_{g\in suppf} |\pi(g)v| \int_G |f(g)| \; d_l g.$$

Lemma. *If $f \in C_c^\infty(G)$, then $\pi(f)v \in V^\infty$.*

Proof. This follows from differentiating inside the integral and the fact that

$$\pi(g)\pi(f) = \pi(L_g f),$$

where by definition $L_g f(x) := f(g^{-1}x)$ (and $R_g f(x) := f(xg)$). □

Corollary. *V^∞ is $G-$stable.*

Definition. *The Garding space V^g is the span of all the $\pi(f)v$ with $f \in C_c^\infty(G)$ and $v \in V$.*

It is clear that $V^g \subset V^\infty$ and V^g is $G-$stable.

Proposition. *G has an* ***approximate identity*** *i.e. a sequence of functions $f_n \in C_c^\infty(G)$ such that*

1. *$suppf_n \subset U_n$ where U_n are a sequence of locally compact neighborhoods of the identity satisfying $\cap U_n = \{e\}$,*
2. *$f_n \ge 0$ and $\int_G f_n(g)dg = 1$.*

We can then check that $\pi(f_n)v \to v$:

$$|\pi(f_n)v - v| = \left| \int_G f_n(x)(\pi(x)v - v) \; dx \right| \le \int_G |\pi(g)v - v| f_n(g) \; dg \le$$
$$\le \sup_{g\in U_n} |\pi(g)v - v| \int_G f_n(g) \; dg,$$

and the continuity of π completes the proof.

Since $\pi(f_n)v \in V^g \subset V^\infty$, the density statement follows.

3.6. Convolutions. If $f_1, f_2 \in C_c(G)$ the convolution $f_1 * f_2$ i defined as

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) \; dx = \int_G f_1(gy^{-1}) f_2(y) \; dy.$$

Note that $L : G \longrightarrow End[C_c(G)]$ given by $L_g(f)(x) := f(g^{-1}x)$ is a representation, and this formula is simply $L(f_1)f_2$. Also dx and dy are left-invariant measures only.

Proposition.

(1) $\pi(f_1) \circ \pi(f_2) = \pi(f_1 * f_2)$.
(2) $\left|\int_G f_1 * f_2(g)\, dg\right| \leq \int_G |f_1 * f_2(g)|\, dg \leq \int_G |f_1(x)|\, dx \cdot \int_G |f_2(y)|\, dy$.

So there are extensions to L^1.

3.7. Note on integration. Let G be a Hausdorff, locally compact topological group, countable at ∞ countable base etc, and V a locally convex complete Hausdorff topological space. Let $C_c(G, V)$ be contiuous compact supported functions from G to V.

Theorem. *There exists a continuous map called* ***integration****,*

$$\int : C_c(G, V) \longrightarrow V, \qquad f \mapsto \int_G f(x)\, dx$$

satisfying

(1) $p(\left(\int_G f(x)\, dx\right) \leq \int_G p(f(x))\, dx$ *for any seminorm* p*,*
(2) *if* $L : V_1 \longrightarrow L_2$ *is continuous linear, then*

$$L\left(\int_G f(x)\, d_1x\right) = \int_G L(f(x)\, d_2x.$$

When G *is unimodular,* $\displaystyle\int_G f(x)\, dx = \int_G f(x^{-1})\, dx$*, and when it is compact, we can normalize so that* $\int_G v\, dx = v,\ \forall v \in V$.

REFERENCE Lang ...

3.8. $V^\omega \subset V$ is dense. This was first introduced and proved by Harish-Chandra in TAMS I and II for reductive groups only (see later sections for the notions of reductive, semisimple groups etc). It was proved for arbitrary Lie groups by Nelson, Annals of Math 1959. The proof is similar in spirit to the previous section; use an approximation of the identity with analytic functions. This requires the heat kernel.

For $G = \mathbb{R}^n$, the heat kernel is the function $K(t, x, y) := \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)}$. It satisfies

(1) $\frac{\partial K}{\partial t}(t, x, y) = \Delta_x K(t, x, y),\ \Delta_x = \sum \partial_x^2$.

(2) $\lim_{t \to 0^+} \int_{\mathbb{R}^n} K(t, x, y) f(y)\, dy = f(x)$ for any function $f \in C_c^\infty(\mathbb{R}^n)$. In other words it is a fundamental solution.

The paper of Nelson considers the heat kernel for an arbitrary (connected) Lie group.

4. Reductive Groups

Let G be an arbitrary (connected) real Lie group with Lie algebra $\mathfrak{g}$.

Theorem (Ado)**.** *Every finite dimensional Lie algebra $\mathfrak{g}$ has a faithful finite dimensional representation $\rho : \mathfrak{g} \longrightarrow gl(V)$.*

Theorem (Levi)**.** *Any finite dimensional Lie algebra $\mathfrak{g}$ is the semidirect product of a semisimple Lie algebra and a solvable ideal (called the solvable radical).*

Definition. *A real Lie group is called **reductive** if*

(1) *$\mathfrak{g} := Lie(G)$ is reductive. This means $\mathfrak{g} = \mathfrak{c} + [\mathfrak{g}, \mathfrak{g}]$ where the second term is semisimple. Semisimple means a direct sum of simple ideals, and simple means dimension > 1 and no nontrivial ideals.*

(2) *G has a maximal compact subgroup $K \subset G$ given by the fixed points of an involution θ. Furthermore*

  a: $\mathrm{Ad}g : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is inner,*

  b: *G has a faithful finite dimensional representation (not always assumed)*

Properties include (some of them are consequences, others are assumptions)

(1) $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$, the ± 1 eigenspaces of θ. Write θ for the involution on the group as well as the algebra. Then $\exp : \mathfrak{s} \longrightarrow G$ is an isomorphism on a submanifold S, such that $K \times \mathfrak{s} \longrightarrow G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism.

(2) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra (a maximal abelian subalgebra formed of ad$-$semisimple elements. Then $H := Cent_G(\mathfrak{h})$ is abelian. Every such subalgebra/subgroup is conjugate under K to one which is $\theta-$stable.

Main Example: $G = GL(n, \mathbb{R}) \supset K = O(n)$, $\theta(g) := {}^T g^{-1}$.

A large class is obtained as follows. Let $G_c \subset GL(n, \mathbb{C})$ be an affine algebraic group defined over $\mathbb{R}$, ie. a group given by the zeroes of polynomial equations on $gl(n, \mathbb{C})$. We say G_c is **symmetric**, if it is stabilized by $g \mapsto {}^T g$.

A **real reductive group** G is a finite cover of an open subgroup of the real points of such a G_c.

The class of real points of a reductive connected linear algebraic group form a subclass with special properties.

Examples: $G = GL(n, \mathbb{R}), SL(n, \mathbb{R}), U(p, q), O(p, q), So(p, q)$, $Sp(2n, \mathbb{R}), Sp(2p, 2q)$ and also $U^*(2n)$ and $O^*(2n)$. $So(p, q)_e$ the identity component of $So(p, q)$ is in this class, but not the real points of a reductive ... group. Its representation theory is *somewhat less reasonable*. See Helgason for a list of all simple Lie algebras and their real forms.

REFERENCE Helgason 1962 or later edition.

4.1. $SL(2,\mathbb{R})$**.** This is a typical case of the real points of a semisimple linear algebraic group. The irreducible finite dimensional representations are parametrized by $n \in \mathbb{N}$. If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det g = ad - bc = 1$,

$$V = span\{x^k y^{n-2k}\}, \pi_n(g)x^k y^{n-2k} = (ax+cy)^k (bx+dy)^{n-2k}.$$

The maximal compact subgroup is $\left\{ r(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}$, and it has eigenvalues $e^{i(n-2k)\theta}$.

Since $\pi_1(SL(2,\mathbb{R})) = \pi_1(SO(2)) = \mathbb{Z}$, there is a nontrivial cover for each $n \in \mathbb{N}$. The universal cover is realized as

$$\widetilde{SL(2,\mathbb{R})} = \{(g,\phi) \; : \; e^{\phi(z)} = cz + d\}$$

where $\phi : \mathcal{P}^+ \longrightarrow \mathbb{C}$ is a holomorphic function from the upper half-plane $\mathcal{P}^+$.

The $n-$fold cover is

$$\widetilde{SL(2,\mathbb{R})}_n = \{(g,\phi) \; : \; \phi(z)^n = cz + d\}.$$

Note that $SL(2,\mathbb{R})$ acts on $\mathcal{P}^+$ by conformal transformations:

$$g \cdot z : \frac{az+b}{cz+d}.$$

The stabilizer of i is K, and the stabilizer of ∞ is

$$B = AN = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \right\} \cdot \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\},$$

the action is transitive, and you can use it to prove that $G = B \cdot K$.

Let $\phi_n : \widetilde{Sl(2,\mathbb{R})}_n \longrightarrow SL(2,\mathbb{R})$ be the cover map. The $\widetilde{Sl(2,\mathbb{R})}$ are **not** linear groups for $n > 1$. Any finite dimensional representation $\widetilde{\pi}$ differentiates to a finite dimensional representation $d\widetilde{\pi}$ of the Lie algebra $sl(2,\mathbb{R})$. It *complexifies* to a representation of $sl(2,\mathbb{C})$. Since the Lie group of $sl(2,\mathbb{C})$ is $SL(2,\mathbb{C})$ which is connected simply connected, it exponentiates to a representation π of $SL(2,\mathbb{C})$ which we can restrict to $SL(2,\mathbb{R})$; it follows that $d\widetilde{\pi} = d\pi$. Via the exponential map there are open sets (isomorphic under ϕ_n) $\widetilde{e} \in \widetilde{U} \subset \widetilde{Sl(2,\mathbb{R})}_n$ and $e \in U \subset Sl(2,\mathbb{R})$ so that $\widetilde{\pi} = \pi \circ \phi_n$. But then this holds for the entire groups because they are generated by open sets containing the identities. So $\widetilde{\pi}$ cannot be genuine.

More explicitly one can to examine the action of $\widetilde{K}$ in more detail using the realization of the cover. Let $\widetilde{K}, \widetilde{B}$ be the inverse images. We can replace $\widetilde{B}$ by a copy isomorphic to B by choosing $\phi_b(z) := e^{-t/n}$. Then $\widetilde{G} = B \cdot \widetilde{K}$, and $\ker \phi_n \subset \widetilde{K}$. Both $\widetilde{K}$ and K are isomorphic to S^1, but ϕ_n is raising to power n. The inverse image of $r(\theta)$ is $(r(\theta), \phi_\theta)$ where ϕ_θ is determined by its value at i, satisfying $\phi_\theta(i)^n = i\sin\theta + \cos\theta = e^{i\theta}$. To see this, let $\phi_0(i) := 1$, and deform continuously in θ. The exponential map is $\exp_{\widetilde{G}} : \theta \mapsto (r(n\theta), \phi_\theta)$. The points $(r(2i\pi/n), \phi_{2i\pi/n})$ are the kernel of ϕ_n. The values on any irreducible representation can be computed from the Lie algebra via the exponential map; they are the identity, so $\widetilde{\pi}$ is trivial on the kernel and factors to $SL(2,\mathbb{R})$. it follows that this holds for any finite dimensional representation,

4.2. The Metaplectic Group. Let $Mp(2,\mathbb{R}) := \widetilde{SL(2,\mathbb{R})}_2$. The Metaplectic/Oscillator/Segal-Shale-Weil representation of the Lie algebra is as follows. Let

$$\mathcal{H} := span\{p_n(x)e^{-x^2/2}\}$$

be the vector space spanned by the Hermite functions. Then

$$\begin{aligned} E &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longleftrightarrow \frac{i}{2} m_{x^2}, \\ H &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longleftrightarrow x\partial_x + \frac{1}{2}, \\ F &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longleftrightarrow \frac{i}{2}\partial_x^2, \end{aligned}$$

defines a representation of the Lie algebra. It **does not** exponentiate to $Sl(2,\mathbb{R})$. However there is an inner product $\langle f_1, f_2 \rangle := \int_{\mathbb{R}} f_1(x)\overline{f_2(x)}\,dx$ for which $sl(2,\mathbb{R})$ acts by skew symmetric matrices, ie. they satisfy

$$\langle X \cdot f_1, f_2 \rangle + \langle f_1, X \cdot f_2 \rangle = 0.$$

It follows that this representation exponentiates to a representation of a group, **not** $SL(2,\mathbb{R})$, but $Mp(2,\mathbb{R})$ acting on the completion of $\mathcal{H}$, which is $L^2(\mathbb{R})$. The group action can be written down for the group B of upper triangular matrices, but is rather complicated otherwise.

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot f(x) := e^{t/2} f(e^t x) \qquad \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot f(x) := e^{it^2/2} f(x).$$

The action of the maximal compact group is given by *integral operators* with some constants that are a bit difficult to deal with. One of them is the usual Fourier transform:

$$e^{\pi/2(E-F)} \longleftrightarrow FT$$

This representation appears in Mathematical Physics, and in Number theory via the Theta-function. It is easier to study the representation on the Lie algebra.

5. Representations of Compact Lie Groups

5.1. Compact Lie Groups. The Lie algebra decomposes $\mathfrak{k} = \mathfrak{c} + \mathfrak{k}_{ss}$ the central and semisimple part. The Lie algebra admits an invariant negative definite form $B(X, Y)$ which coincides with the Cartan-Killing form on $\mathfrak{k}_{ss}$. If K is a connected semisimple simply connected Lie group, then the universal cover is also compact. In particular the simply connected cover of a compact connected Lie group fits in a sequence

$$1 \longrightarrow \kappa \longrightarrow (\mathbb{R}^+)^n \times \widetilde{K} \longrightarrow K \longrightarrow 1$$

with κ a discrete subgroup.

We will restrict attention to connected semisimple groups at some point; the modifications for a general group follow from the structure above in a straightforward manner.

5.2. Structure of Compact Lie Groups. Assume the group K is connected semisimple; the general case is not much harder. Let $\mathfrak{k}$ be the Lie algebra. The Cartan-Killing form $B(X, Y) := \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$ is negative definite. Let $\mathfrak{t} \subset \mathfrak{k}$ be aCartan subalgebra . This means $\mathfrak{t}$ is a maximal abelian subalgebra; it is automatic that $\mathrm{ad}X$ is semisimple for any $X \in \mathfrak{t}$ (in fact for any $X \in \mathfrak{k}$). Let $T \subset K$ be the Cartan subgroup corresponding to $\mathfrak{t}$. In this case $T = C_K(\mathfrak{t})$ as well. Let $\mathfrak{g} := \mathfrak{k}_{\mathbb{C}}$. This is a semisimple Lie algebra, and $\mathfrak{h} := (\mathfrak{t})_{\mathbb{C}}$ is a Cartan subalgebra (a maximal abelian subalgebra formed of ad−semisimple elements). The Cartan subgroup is denoted H. We write $\mathfrak{h} = \mathfrak{t} + \mathfrak{a} := \mathfrak{t} + i\mathfrak{t}$.

Proposition. *There exists a connected group G with Lie algebra $\mathfrak{g}$ such that $K \subset G$.*

This follows from the fact that K has a faithful finite dimensional representation; a consequence of the Peter-Weyl theorem in the next section.

REFERENCE Helgason, Bröcker-tomDieck.

Let $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \ : \ [h, X] = \alpha(h)X \ \forall h \in \mathfrak{h}\}$ where $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$ is a linear functional. When $\mathfrak{g}_\alpha \neq 0$, α is called a root; the set of all roots is denoted $\Delta(\mathfrak{g}, \mathfrak{h})$.

Structure of Semisimple Lie Algebras

(1) $\dim \mathfrak{g}_\alpha \leq 1$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
(2) $B \mid_{\mathfrak{a}}$ abbreviated $(\ , \ )$, is positive definite. Let h_α be such that $B(h, h_\alpha) = \alpha(h)$. Denote by $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. The reflection associated to α is $s_\alpha(h) := h - \alpha(h)h_{\alpha^\vee}$. Let $W := N_G(\mathfrak{h})/C_G(\mathfrak{h}) = N_K(\mathfrak{a})/C_K(\mathfrak{a})$. This is a finite group, generated by the s_α.
(3) The set $\mathfrak{h}' := \{h \in \mathfrak{a} \ : \ \alpha(h) \neq 0\}$ is a finite union of open sets called chambers. They are in $1-1$ correspondence with

W. A choice C of chambers gives rise to a positive root system P. Then $\Delta = P \cup -P$. Each psotive root system contains a set of simple roots $\Pi \subset P$, the roots in P which cannot be written as integer sums of roots in P.

The weights are the span of $\lambda \in \mathfrak{h}^*$ which take integer values on the $h_{\alpha^\vee}$. The fundamental weights ϖ_α are the dual functionals to the $h_{\alpha^\vee}$ with $\alpha \in \Pi$.

(4) Choose $0 \neq X_\alpha \in \mathfrak{g}_\alpha$. There is a choice such that each $\{X_\alpha, h_{\alpha^\vee}, X_{-\alpha}\}$ is isomorphic to an $sl(2,\mathbb{C})$, denoted $sl(2,\mathbb{C})_\alpha$. By a theorem of Weyl, there is a choice of X_α such that $(X_\alpha - X_{-\alpha}), i(X_\alpha + X_{-\alpha}) \in \mathfrak{k}$. This amounts to saying that if $N_{\alpha,\beta}$ are defined by $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$, then there is a choice such that $N_{\alpha,\beta}$ are integers satisfying $N_{\alpha,\beta} = -N_{-\beta,-\alpha}$.

REFERENCE Helgason

Theorem. *There is a $1-1$ correspondence between irreducible **finite dimensional** representations of connected simply connected K and*

$$\{\Lambda \in \mathfrak{h}^* \ : \ \Lambda(h_{\alpha^\vee}) \in \mathbb{N}\}$$

Any such Λ is a linear combination of ϖ_α with nonnegative integer coefficients.

Let V_Λ be the finite dimensional irreducible representation associated to Λ. Let $\rho := \frac{1}{2}\sum_{\alpha \in P} \alpha$. Then

$$\dim V_\Lambda = \frac{\prod_{\alpha \in P} (\Lambda + \rho, \alpha^\vee)}{\prod_{\alpha \in P} (\rho, \alpha^\vee)}.$$

The operator $\Omega \in U(\mathfrak{k})$ essentially acts by the scalar $||\Lambda + \rho||^2 - ||\rho||^2$.

REFERENCES Dixmier, Jacobson
EXAMPLE: $SU(2) \subset Sl(2,\mathbb{C})$, $SU(m) \subset Sl(m,\mathbb{C})$.

5.3. Peter-Weyl Theorem. Let (ρ, V) be a finite dimensional representation. Then (ρ^*, V^*) is also a representation, where V^* is the linear dual space, and $\rho^*(g)(\ell)(v) := \ell(\rho(g^{-1})v)$. The hermitian dual (ρ^h, V^h) is the representation with the same formula, but V^h is the complex space of conjugate linear maps. A nondegenerate invariant hermitian product can be thought of as a choice of $K-$equivariant automorphism

$$\iota : V \longrightarrow V^h.$$

Every linear map $f : V \longrightarrow W$ induces a map $f^h : W^h \longrightarrow V^h$ via $f^h(\lambda)(v) := \lambda(f(v))$. In particular, $\iota^h = \iota$ is the skew-hermitian aspect of the hermitian form. Assume $(V, \langle\ ,\ \rangle)$ and $(W, \langle\ ,\ \rangle)$ are *hermitian* spaces. Then $\mathrm{Hom}_{\mathbb{C}}[V, W]^h \cong \mathrm{Hom}[W^h, V^h] \cong \mathrm{Hom}[W, V]$ inherits an action of $K \times K$, and an invariant hermitian form $\langle A, B\rangle := \mathrm{tr}(A \circ B^h)$.

If the hermitian products are positive definite, so is the ensuing one on $\mathrm{Hom}_{\mathbb{C}}[V,V]$. In particular, if (π,V) is unitary, (π^*,V^*) does too because,

$$V^* = \mathrm{Hom}[V,\mathbb{C}]$$

We obtain a $K\times K-$equivariant unitary map

$$\mathrm{Hom}[V,V] \longrightarrow C(K), \qquad A \mapsto f_A(g) := \mathrm{tr}(A\circ\pi(g)).$$

A matrix entry is a function of the form $f_A(g) := \mathrm{tr}(A\circ\pi(g))$. The more standard way is to fix an invariant inner product on (π,V), and to write $f_{x_1,y_1}(g) := \langle\pi(g)x_1,y_1\rangle$. This identification has to do with the following fact. Let V^{conj} be the vector space V with action of $\mathbb{C}$ given by multiplication by the complex conjugate,

$$\lambda\cdot_{conj} v := \overline{\lambda}\cdot v.$$

V^{conj} identifies with V^* with the usual complex multiplication via the inner product. Complex multiplication in V^h and V^* is the usual multiplication coming from V.

Theorem (Schur orthogonality relations)**.** *Let (ρ_1,V_1) and (ρ_2,V_2) be irreducible finite dimensional modules with invariant inner products $\langle\ ,\ \rangle_{1,2}$, and let $x_1,y_1\in V_1$ and $x_2,y_2\in V_2$. Then*

$$\int_K \langle\pi_1(k)x_1,y_1\rangle\cdot\overline{\langle\pi_1(k)x_2,y_2\rangle}\,dk = \begin{cases} 0 & \text{if } V_1\not\simeq V_2,\\ \frac{1}{\dim V},\langle x_1,x_2\rangle\cdot\overline{\langle y_1,y_2\rangle} & \text{if } V_1\simeq V_2\end{cases}.$$

In particular, if V_1,V_2 are inequivalent, $\int \mathrm{tr}(\pi_i(k))\cdot\overline{\mathrm{tr}(\pi_j(k))}\,dk = \delta_{i,j}$.

Furthermore, let $\phi_i(k) := \frac{1}{\dim V_i}\mathrm{tr}(\pi_i(k))$. Then $\phi_i*\phi_j = \delta_{i,j}\phi_i$. More general, if $f_{v,w}$ is a matrix entry, from V_j, then $\phi_i * f_{v,w} = \delta_{i,j}f_{v,w}$.

Theorem (Peter-Weyl)**.** *The image of $\oplus_{V\in\widehat{K}}\mathrm{Hom}_{\mathbb{C}}[V,V]$ is dense in $C(K)$ in the L^∞ and L^2-topology. The individual spaces are orthogonal. The image coincides with the space of $K-$finite vectors.*

Definition. *Let (π,V) be a continuous representation on a locally convex topological space. The space of $K-$finite vectors is*

$$V^K := \{v\in V \mid \dim\pi(K)v<\infty\}.$$

The Lie algebra $\mathfrak{k}$ acts on V^K.

Corollary. *Let (π,V) be a representation on a complete locally convex space. Then $V^K\subset V$ is dense.*

If V is irreducible then $\dim V<\infty$.

Proof. Recall $\{f_n\}$ the approximate identity. Then $\pi(f_n)v\to v$. Every f_n is approximated by $K-$finite functions. ∎

Theorem. *Assume K is a connected compact Lie group and (π, V) is a Banach space representation. Then the Fourier series of every v converges absolutely to v.*

Example: $G = S^1$. If $f \in C^\infty(S^1)$, then the Fourier coefficient satisfies

$$a_n(f) = \int_0^{2\pi} e^{-inx} f(x)\, dx = \frac{1}{n} \int_0^{2\pi} e^{-inx} f'(x)\, dx = \dots$$

So there is a constant $C_k > 0$ depending on $f^k(x)$ such that

$$|a_n| \le \frac{C_k}{n^k}.$$

For a general group, the Fourier series of $v \in V$ is defined as follows. Let (ξ, E) be a finite dimensional irreducible representation of K with character χ_E. The Schur orthogonality relations show that

$$\phi_E * \phi_E = \phi_E.$$

So $\pi(\phi_E)$ is a projector onto the space

$$span\{T(f)\}_{T \in \mathrm{Hom}_K[E,V], f \in E}.$$

The Fourier series of v is $\sum \pi(\phi_E)v$.

The following facts are used in the proof:

(1) The irreducible representations of K are parametrized by a lattice in a vector space of dimension $\dim K$. Actually it is a cone.
(2) Let $X_1, \dots, X_n$ be an orthonormal basis of the Lie algebra. The operator $\Omega = 1 - \sum X_i^2 \in U(\mathfrak{k})$ is central and acts by a scalar $c(\xi) > 0$ on E which grows like a polynomial. For any integer $s >> 0$ the series
$$\sum_{\xi \in \widehat{K}} c(\xi)^{-s} \dim(E_\xi)^2 < \infty.$$

This implies that the Fourier series of v converges to a $v^0 \in V$, and that all the Fourier components of $v - v^0$ are zero. The key inequality is

$$|\pi(\phi_E)v| = \left| \int_K \phi_E(k)\pi(k)\, dk \right| \le c(\xi)^{-s} \int_K |\phi_E(k)| \cdot |\pi(\Omega^s k)v|\, dk.$$

Then $|\phi_E(k)| \le (\dim E)^2$, and the rest of the integral is bounded by a constant that only depends on v. (You may need another ϕ_E using $\phi_E * \phi_E = \phi_E$ in the formula to replace v by the differentiable $\pi(\phi_E)v$ on the right side of the formula).

Remains to show that if $v \in V$ satisfies $\pi(\phi_E)v = 0$ for all $E \in \widehat{K}$, then $v = 0$. This follows from the fact that $\pi(f)v = 0$ for any $K-$finite function, and then by the Peter-Weyl theorem, $\pi(f)v = 0$ for any

$f \in C(K)$. The existence of the approximate identity completes the proof.

REFERENCES Bröcker-tomDieck

6. Representations on Banach and Hilbert spaces

6.1. Let (π, V) be a continuous (in the operator norm mostly) representation of a reductive group G. We define V^K to be the space of analytic $K-$finit vectors. V^K defined earlier consists of analytic vectors for K, but it is not clear they are analytic for V. The previous sections show that $\mathfrak{g}$ acts on V^K, and every vector $v \in V$ has a Fourier series which converges absolutely to v. Furthermore using the projections $\pi(\phi_E)$,

$$V^K = \oplus V_\xi;$$

the V_ξ are called the $\xi-$isotypic components of V. Every vector $v \in V_\xi$ is contained in a finite dimensional representation of K which in turn is a direct sum of irreducible representations isomorphic to V_ξ. In the case of a Hilbert space, the V_ξ are orthogonal.

Proposition. *Assume G is connected. Let $\mathcal{W} \subset V^K$ be a $U(\mathfrak{g})-$invariant subspace. Then $W := Cl(\mathcal{W})$ is a $G-$invariant subspace of V, and $W^K = \mathcal{W}$.*

Proof. Suppose there is $x_0 \in G$ and $v' \in W$ such that $v := \pi(x_0)v' \notin W$. By the Hahn-Banach theorem there is a continuous functional ℓ such that $\ell(W) = 0$ but $\ell(v) = 1$. Let

$$f(g) := \ell(\pi(g)v').$$

Then f is analytic, and $\pi(u)f(e) = 0$ for all $u \in U(\mathfrak{g})$. This implies that $f \equiv 0$, in contradiction to $f(x_0) = 1$. $\square$

This establishes a 1-1 correspondence/functor between Banach space representations, and representations of $\mathfrak{g}$ with an action of K. Precisely the ensuing modules are called $(\mathfrak{g}, K)-$modules. The terminology is that $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$, the complexification of a real reductive Lie algebra $\mathfrak{g}_0$ of a real redfctive group G with maximal compact subgroup K.

6.2.

Definition. *A representation of $(\mathfrak{g}, K)$ on a Banach/Hilbert space V is called a $(\mathfrak{g}, K)-$module if it has an action π_K of K and an action $\pi_{\mathfrak{g}}$ of $\mathfrak{g}$ which are compatible:*

1. $\pi(k)\pi(X) = \pi(\mathrm{Ad}kX)\pi(k)$,
2. $V = \bigoplus V_\xi$,
3. *the differential of π_K coincides with the action $\pi_{\mathfrak{g}} \mid_{\mathfrak{k}}$ of $\mathfrak{k} \subset \mathfrak{g}$.*

The representation is called ***admissible*** *if* $\dim V_\xi < \infty$ *for all* $\xi \in \widehat{K}$.

Remarks. *Given a* $(\mathfrak{g}, K)-$*module, it is not clear that there is a Banach space representation that it comes from; results of Casselmann, Kashiwara, Schmid, Wallach, independently and together, show that there are several possible completions of* V*. For the case when* V *admits an invariant positive definite inner product, Harish-Chandra shows that there exists a (unique) irreducible unitary representation whose* $K-$*finite vectors is the original module.*

6.3. The Lie algebra $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0 := Lie(G)$.

Definition. *A hermitian form on a representation* (π, V) *of* $\mathfrak{g}$ *is called invariant if it satisfies*

$$\langle \pi(X)v_1, v_2\rangle + \langle v_1, \pi(X)v_2\rangle, \qquad \forall X \in \mathfrak{g}_0.$$

For $U(\mathfrak{g})$*, define an anti-automorphism by* $X^\dagger := -\overline{X}$ *where conjugation is with respect to the real Lie algebra* $\mathfrak{g}_0 \subset \mathfrak{g}$*. Then invariance means*

$$\langle \pi(u)v_1, v_2\rangle = \langle v_1, \pi(u^\dagger)v_2\rangle, \qquad \forall u \in U(\mathfrak{g}).$$

We will study *admissible* modules. To make any headway we will have to study irreducible rerpesentations (for G and for $(\mathfrak{g}, K)$) and involve the action of center of the universal envelopoing algebra
$\mathcal{Z} := U(\mathfrak{g})^G$.

EXAMPLE: $Sl(2, \mathbb{R})$.

6.4. Unitary Irreducible Modules. We need some results about the spectral decomposition of bounded self-adjoint operators on Hilbert spaces (separable).

REFERENCE Simon-Reed

A bounded operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is called self-adjoint if

$$\langle Tv_1, v_2\rangle = \langle v_1, Tv_2\rangle.$$

In general, for a bounded operator one defines T^* by the formula

$$\langle Tv, w\rangle = \langle v, T^* w\rangle.$$

If $f(x)$ is a real valued polynomial function, the definition of $f(T)$ is clear, and it is also bounded self-adjoint. Using Stone-Weierstrass, the definition extends to continuous functions. We get a functional from $C_C(\mathbb{R})$ (or rather continuouos functions on the spectrum of T whcih is a compact set), and the Riesz representation theorem insures that it is given by a Borel measure. Thus the definition of $f(T)$ can be extended to Borel measurable functions. The **spectral projection** of T is the family $\{P_\Omega = \chi_\Omega(T)\}$ where χ_Ω is the indicator function of the Borel measurable set Ω. These operators satisfy the following:

(1) P_Ω is a projection.
(2) $P_\emptyset = 0$, $P_{(-a,a)} = Id$ for some $a > 0$.
(3) If $\Omega = \cup\Omega_n$ disjoint union, then $\lim \sum_1^N P_{\Omega_i} = P_\Omega$.

Such a family is called a projection valued measure. For each $v \in \mathcal{H}$, one can define a measure by the formula $\lambda \to \langle v, P_\lambda v\rangle$. One can attach to such a family a self adjoint operator by the formula

$$\langle v, Tv\rangle := \int_{\mathbb{R}} \lambda \; d(\langle v, P_\lambda v\rangle.$$

In summary,

Theorem. *There is a 1-1 correspondence between projection valued measures and bounded self-adjoint operators. In this correspondence,* $\{P_\Omega\} \longleftrightarrow < v, Tw >= \int_{\mathbb{R}} \lambda \; d(v, P_\lambda w)$.

6.5. Schur's Lemma.

Theorem (Schur's lemma for a Hilbert space representation)**.**
Let $(\pi, \mathcal{H})$ *be an irreducible representation on a Hilbert space. Denote by* $\mathrm{Hom}_G[\mathcal{H}, \mathcal{H}]$ *the space of bounded operators commuting with the action of* $\mathcal{H}$*. Then*

$$\mathrm{Hom}_G[\mathcal{H}, \mathcal{H}] = \mathbb{C}Id.$$

Proof. Any $T = \frac{T+T^*}{2} - i\frac{T-T^*}{2i}$, so a linear combination of self-adjoint bounded operators If T is $G-$equivariant, so is T^*. It is therefore enough to prove the statement for bounded self-adjoint operators. In this case, since $\pi(g)T = T\pi(g)$, the same holds for the projectors P_Ω. Since $\ker P_\Omega$ and $\mathrm{im} P_\Omega$ are closed invariant subspaces, $P_\Omega = Id$ or 0. We know $P_{(-a,a)} = Id$. Divide the interval in two equal parts. One of the $P's = Id$ by (3). We obtain a point λ such that $P_\lambda = Id$. Then $T = \lambda Id$. ☐

This implies that the center of the group acts by scalars on a unitary irreducible representation. This is not a rich enough set of operators to derive any substantial conclusion. The enveloping algebra $U(\mathfrak{g})$ has a larger center $\mathcal{Z} := U(\mathfrak{g})^G \subset U(\mathfrak{g})^{\mathfrak{g}}$. For a connected group, $U(\mathfrak{g})^G = U(\mathfrak{g})^{\mathfrak{g}}$. We will often assume this, the modifications for the more general case are not very hard.

These operators do not act on the whole space. What we can say is that for $z \in \mathcal{Z}$, $z^* \in \mathcal{Z}$ as well. Here z^* is defined algebraically via the anti involution in Section 6.3. They satisfy

$$\langle \pi(z)v_1, v_2\rangle = \langle v_1, \pi(z^*)v_2\rangle \qquad \forall v_1, v_2 \in \mathcal{H}^K.$$

6.6. The next theorem is closely related to the Jacobson density theorem and the analogous result of von Neumann for C^*-algebras.

Theorem (Schur's Lemma, infinitesimal version)**.** *Assume* $(\pi, \mathcal{H})$ *is an irreducible unitary representation. There is an algebra homomorphism* $\chi_{\mathcal{H}} : \mathcal{Z} \longrightarrow \mathbb{C}$, *such that*

$$\pi(z) = \chi_{\mathcal{H}}(z)Id \qquad \forall z \in \mathcal{Z}.$$

We have two operators S, T, satisfying $\langle Tv_1, v_2\rangle = \langle v_1, Sv_2\rangle$ for $v_1, v_2 \in \mathcal{H}'$ a dense subspace ($\mathcal{H}^\infty$ or $\mathcal{H}^\omega$) on which G still acts. We want to show that if $\pi(g)T = T\pi(g)$ $\forall g \in G$, then $T = \lambda Id$.

Lemma. *Assume* $(\pi, \mathcal{H})$ *unitary irreducible Hilbert space representation. Let* $\mathcal{A}$ *be the algebra generated by the* $\pi(g)$. *It is invariant under* $*$. *Let* Z *be a bounded operator on* $\mathcal{H}$, $x, y \in \mathcal{H}$ *linearly independent,* $\delta > 0$. *There is* $U \in \mathcal{A}$ *such that*

$$||Ux - Zx||,\ ||Uy - Zy|| < \delta.$$

We prove the lemma in the next section. The fact that $\mathcal{A}$ is invariant under * holds because of the assumption of unitarity. Suppose there exist $x, Tx \in \mathcal{H}'$, that are linearly independent. Let $x_1 = x, y_1 = Tx$ and $x_2 = y_2 = x$. The lemma implies that there is a sequence $U_j \in \mathcal{A}$ such that

$$U_jx \to x, \qquad U_jTx \to x.$$

Then

$$\langle x, w\rangle = \lim\langle U_jTx, w\rangle = \lim\langle TU_jx, w\rangle = \lim\langle U_jx, Sw\rangle = \langle Tx, w\rangle.$$

Since $\mathcal{H}'$ is dense, we conclude $Tx = x$, a contradiction. So for any $x \in \mathcal{H}'$, there is λ_x such that $Tx = \lambda_x x$. The linearity of T implies $\lambda_{x+y} = \lambda_x = \lambda_y$ $\forall x, y \in \mathcal{H}$.

6.7. Proof of the Lemma. Set $V = \mathcal{H} \oplus \mathcal{H}$. Let

$$B = \{(U, U)\ ;\ U \in \mathcal{A}\}.$$

This is a subalgebra of all the bounded operators on V; **it contains the identity operator, and if $u \in B$ then $u^* \in B$ as well**. Define

$$B' := \{T : V \longrightarrow V\ \ :\ Tb = bT \quad \forall b \in B\}.$$

We can compute what this is:

$$B' = \{T : V \longrightarrow V\ \ :\ T(x, y) = (Zx, Zy)\}$$

for some $Z : \mathcal{H} \longrightarrow \mathcal{H}$. This follows from Schur's lemma for a Hilbert space representation. The proof of the lemma is completed by an observation of von Neumann:

If $Id \in B$ and $T \in B \Longrightarrow T^* \in B$, then

$$(B')'v \subset Cl[Bv] \quad \forall v \in V.$$

Indeed, let $\Omega := Cl[Bv]$. Then $V = \Omega + \Omega^\perp$, and both Ω and $\Omega^\perp$ are $B-$invariant. The orthogonal projection P_Ω satisfies $P_\Omega \in B'$. If $T \in (B')'$, then $TP_\Omega = P_\Omega T$, so T preserves Ω, since $Id \in B$ implies $v \in Bv$.

The proof is completed by applying the observation to

$$v = (x, y = Tx), \qquad Zx = x, \quad ZTx = x.$$

Then

$$(Z, Z)(x, Tx) = (x, Tx) \in Cl[(Ux, UTx)].$$

REFERENCES Wallach

6.8. The Enveloping Algebra Case. The analogous theorem for $(\mathfrak{g}, K)-$modules is much easier. We are **not** assuming that the representation comes from an irreducible unitary representation on a Hilbert space.

Theorem (Dixmier)**.** *Let (π, V) be an irreducible module for $U(\mathfrak{g})$ on a space with countable base. Then* $\mathrm{Hom}_{U(\mathfrak{g})}[V, V] = \mathbb{C}Id$.

Proof. Suppose not. The kernel and image of $T - \lambda I$ are 0 and V respectively; $T - \lambda I$ is invertible. The same holds for any $P(T)$ where P is a polynomial, and therefore for any $f(T)$ where f is a rational function. Then $\{f(T)\}$ is a field which does not have a countable base as a vector space, mapping into V which does; a contradiction. So $T - \lambda Id$ has a nontrivial kernel for some λ, and the kernel must be all of V.

Precisely, let $0 \neq v \in V$. Define an inclusion $\mathbb{C}(x) \longrightarrow V$ by

$$\frac{P}{Q} \mapsto P(T)Q(T)^{-1}v.$$

$\mathbb{C}(x)$ does not have a countable base. □

For an irreducible representation (π, V) on a Banach space, we know that the $(\mathfrak{g}, K)-$module V^K is irreducible. Since $U(\mathfrak{g})$ has a countable base, and $V = U(\mathfrak{g})v_0$ for any $0 \neq v_0 \in V^K$, the theorem applies.

6.9. Square Integrable Representations. Let $(\pi, \mathcal{H})$ be an irreducible Hilbert space representation. View $L^2(G, d_r g)$ as a representation of G under R_g. Abbreviate it as L^2 or $L^2(G)$.

6.10.

Definition. *A matrix coefficient is a function of the form*

$$f_{v_1,v_2}(g) = \langle \pi(g)v_1, v_2\rangle$$

We say $(\pi, \mathcal{H})$ *is square integrable if there are* $v_1, v_2 \in \mathcal{H}$ *such that* $f_{v_1,v_2} \in L^2(G, d_rg)$.

Lemma. *If* $(\pi, \mathcal{H})$ *is square integrable, then all matrix entries are in* L^2.

Proof. Fix w', v' so that $f_{w',v'} \in L^2$. Let

$$D' := \{v \in \mathcal{H} \ : \ f_{v,v'} \in L^2\}.$$

Since $R_x f_{v,v'} = f_{\pi(x)v,v'}$, $span\{\pi(g)w'\} \subset D'$, so D' is dense in $\mathcal{H}$. Similarly D' is invariant under G. In summary we can define a $G-$equivariant linear map $T : D' \longrightarrow L^2(G, d_rg)$ by

$$Tv := f_{v,v'}.$$

D' admits an inner product,

$$(v_1, v_2) := \langle v_1, v_2\rangle + \langle Tv_1, Tv_2\rangle,$$

which is continuous. Then D' is complete with respect to (,). Indeed if $v_j \to v$ is a Cauchy sequence in this topology, v_j and Tv_j are Cauchy sequences. So $v_j \to v$ and $Tv_j \to f$. But

$$|f_{v_j,v'}(g) - f_{v,v'}(g)| \le ||\pi(g)(v_j - v)|| \cdot ||v'|| \le ||v_j - v|| \cdot ||v'||,$$

so $f_{v_j,v'} \to f_{v,v'}$ uniformly. Since also $Tv_j \to f$ *a.e*,, we conclude $f \in D'$.

The inclusion $i : D' \longrightarrow \mathcal{H}$ is a bounded linear map. Let i^* be the adjoint. By the infinitesimal version of Schur's lemma, we find that $i^* = aId$. Then $D' = \mathcal{H}$. Furthermore, there is $b > 0$ such that

$$\langle Tv, Tw\rangle = b\langle v, w\rangle$$

Then we can renormalize T by $b^{1/2}$ to make it unitary. □

Corollary. *If* $(\pi, \mathcal{H})$ *is square integrable, there is a unitary operator* $T \in \mathrm{Hom}_G[\mathcal{H}, L^2(G, d_rg)]$ *with closed range and such that* $T(\mathcal{H})$ *is a subspace of* $C(G) \cap L^2(G)$. *Furthermore there is a* $d(\pi) > 0$ *such that*

$$\int_G f_{v_1,v_2}(g)\overline{f_{w_1,w_2}}(g)\, dg = d(\pi)^{-1}\langle v_1, w_1\rangle \cdot \overline{\langle v_2, w_2\rangle}.$$

Matrix entries from inequivalent square integrable representations are orthogonal.

REFERENCES Wallach, Varadarajan

7. An Example

Let $(\mathcal{X}, dx)$ be a compact space with a Borel measure, and a continuous action of a group G (locally compact, Hausdorff etc) such that dx is invariant. Then G acts on $L^2(\mathcal{X})$ by unitary transformations.

Example.

(1) *$\mathcal{X} = K$, where K is a compact group acting on $\mathcal{X}$ by left or right translations.*
(2) *$G = Sl(2, \mathbb{R})$ and $\mathcal{X} = G/\Gamma$ where Γ is a discrete subgroup. Discrete subgroups such that G/Γ is compact exist.*

7.1.

Theorem (Gelfand, Graev, Piatetskii-Shapiro)**.**

$$L^2(\mathcal{X}) = \widehat{\bigoplus} m_\pi V_\pi$$

with $0 \leq m_\pi < \infty$.

Proof. It is enough to prove the following:

Assume $(\pi, \mathcal{H})$ is a unitary representation of G on a separable Hilbert space $\mathcal{H}$ such that $\pi(\phi)$ is compact for any $\phi \in C_c(G)$. Then $(\pi, \mathcal{H})$ splits into a direct sum of irreducible unitary representations with finite multiplicities.

$C(G)$ admits a $\dagger$ given by $\phi^\dagger(g) := \overline{\phi(g^{-1})}$. As before,

$$\pi(\phi)v := \int_G \phi(g)\pi(g)v \, dg.$$

Assume $\phi^\dagger = \phi$. Then $\pi(\phi)$ is bounded selfadjoint (and compact by assumption).

Recall that an operator is compact if it maps bounded sets into precompact sets. The basic property of a compact selfadjoint operator T is that

$$\mathcal{H} = \mathcal{H}_0 + \bigoplus \mathcal{H}_k$$

where

$$\mathcal{H}_k = \{v \in \mathcal{H} \ : \ Tv = \lambda_k v\}$$

are finite dimensional ($\lambda_k \neq 0$). Furthermore, if $\mathcal{H}' \subset \mathcal{H}$ is a closed $T-$invariant subspace, then the analogous decomposition holds.

Let $\mathcal{H}_1$ be the minimal subspace which contains all the $\mathcal{H}_{k,\phi}$.
CLAIM: $\mathcal{H}_1 = \mathcal{H}$.

Clearly $\mathcal{H}_1$ is invariant under all the $\pi(\phi)$, and so is $\mathcal{H}_1^\perp$. Suppose $\mathcal{H}_1 \neq \mathcal{H}$. Let $0 \neq f \in \mathcal{H}_1^\perp$. Then $\pi(\phi)f = 0 \ \forall \phi \in C_c(G)$. Use an approximate identity to show that $f = 0$.

CONCLUSION: Every (closed) invariant $V \subset \mathcal{H}$ must satisfy

$$V \cap \mathcal{H}_{k,\phi} \neq (0)$$

for some ϕ and some $\lambda_k \neq 0$.

Fix a $\mathcal{H}_{\phi,k}$ and let $0 \neq \mathcal{H}_1 \subset \mathcal{H}_{\phi,k}$ be such that

(1) there is $V \subset \mathcal{H}$ invariant satsifying $\mathcal{H}_1 = V \cap \mathcal{H}_{\phi,k}$,
(2) $\dim[\mathcal{H}_1]$ is minimal subject to (1).

Consider the set

$$\{\mathcal{H}' \subset \mathcal{H} \ \text{ invariant } \ : \ \mathcal{H}' \cap \mathcal{H}_{\phi,k} = \mathcal{H}_1\}.$$

There is a minimal $\mathcal{H}_m$ in this set, the intersection of all the subspaces.
CLAIM: $\mathcal{H}_m$ is irreducible.

If not, $\mathcal{H}_m = V + V^{\perp}$ both invariant proper nonzero subspaces. One of them must intersect $\mathcal{H}_{\phi,k}$ in $\mathcal{H}_1$, a contradiction to the minimality of $\mathcal{H}_m$.
CONCLUSION: $\mathcal{H} = \bigoplus \mathcal{H}_\pi$ countable sum of irreducible unitary representations.

This follows by doing a transfinite induction.

Remains to show that any irreducible $(\pi, \mathcal{H}_\pi)$ occurs only a finite number of times. Fix a $\mathcal{H}_\pi$ irreducible. There is $\phi = \phi^\dagger$ such that $\mathcal{H}_\pi \cap \mathcal{H}_{k,\phi} \neq (0)$. For any other $\mathcal{H}_\ell$, $\mathcal{H}_\ell \cap \mathcal{H}_{k,\phi}$ is orthogonal to $\mathcal{H}_\pi \cap \mathcal{H}_{k,\phi}$. So there are only finitely many $\mathcal{H}_\ell$ that intersect $\mathcal{H}_{\phi,k}$ nontrivially. But if $\mathcal{H}_\pi \simeq \mathcal{H}_\ell$, then $\pi(\phi)$ and $\ell(\phi)$ must be equivalent, so they have nontrivial intersection with $\mathcal{H}_{k,\phi}$.

□

REFERENCE Gelfand, Graev, Piatetskii-Shapiro.

This is a good introduction to the beginings of automorphic forms. Concentrates on $GL(2)$ in detail.

8. Summary

Assume (π, V) is a continuous Banach space representation of a real reductive group G with maximal compact subgroup K.

8.1. Banach Space.

Definition. *A* $(\mathfrak{g}, K)-$*module is an algebraic module of* $\mathfrak{g}$ *and therefore* $U(\mathfrak{g})$*, such that*

(1) *the actions of* $\mathfrak{g}$ *and* K *are* ***compatible*** *ie,*

$$\pi(k)\pi(X) = \pi \mathrm{Ad} k X)\pi(k), \qquad d\pi_K = \pi_{\mathfrak{g}}.$$

(2) $V = \bigoplus V_\xi$.

The module is called ***admissible*** *if* $\dim V_\xi < \infty$.

Theorem (Harish-Chandra)**.**

$$V^K := \{K-\textit{finite analytic vectors } \subset V\}$$

is dense in V*.*

$V^K = \bigoplus_{\xi\in\widehat{K}} V_\xi$*, and the Fourier series of any* $C^\infty-$*vector converges to the vector absolutely.*

For admissible modules, the correspondence $\mathcal{W} \subset V \longleftrightarrow \mathcal{W}^K$ *from closed invariant subspaces of* V *to* $U(\mathfrak{g})-$*submodules of* V^K *is a bijection. The inverse is* $\mathcal{W} \mapsto Cl(\mathcal{W})$*.*

For an admissible module, we can drop the *analytic* part in the definition of V^K.

For examples/counterexamples see REFERENCE Soergel.

8.2. Hilbert Space.

Theorem (Harish-Chandra)**.** *Any irreducible unitary representation is admissible. In general,*

$$\dim \mathcal{H}_\xi \leq C(\dim \xi)^2$$

for some $C > 0$*. Any bounded operator* T *which commutes with* $\pi(g)$ $\forall g \in G$ *is a multiple of a scalar.*

There is a $1-1$ *correspondence*

$$\{\textit{ Unitary irreducible } (\mathfrak{g}, K)-\textit{modules}\} \longleftrightarrow \{\textit{ Unitary irreducible } G-\textit{modules}\}$$

For any irreducible $(\mathfrak{g}, K)-$*module* (π, V)*, there is a unique unitary* $G-$*module* $\pi, \mathcal{H})$ *such that* $\mathcal{H}^K \cong V$*.*

Any unitary admissible module is a direct sum of irreducible modules.

Two modules $(\pi_i, \mathcal{H}_i)$ are called infinitesimally equivalent if $\mathcal{H}_1^K \cong \mathcal{H}_2^K$. infinitesimally equivalent unitary modules are equivalent.

NOT TRUE for Banach space representations.

8.3. Example $Sl(2,\mathbb{R})$**.** Recall $G = KAN$ with

$$K = \{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\} \qquad A = \{\begin{bmatrix}e^t & 0 \\ 0 & e^{-t}\end{bmatrix}\} \qquad N = \{\begin{bmatrix}1 & x \\ 0 & 1\end{bmatrix}\}$$

Write $B = \{\begin{bmatrix}a & x \\ 0 & a^{-1}\end{bmatrix}\}$ with $a \in \mathbb{R}^\times$. Gven any $\nu \in \mathbb{C}$, we can define two characters of B, by the formulas

$$\chi_{+,\nu}(b) = e^{(\nu)\log|a|}, \qquad \chi_{-,\nu}(a) = sgn(a)e^{(\nu)\log|a|}$$

The Principal Series $X(B,\chi)$ are defined as the representations by left translation on the spaces

$$\{f : G \longrightarrow \mathbb{C} \ : \ f(gb) = \chi(a^{-1})\delta(a)^{-1}f(g)\}.$$

A function in such a space is determined by its values on K. We assume that the space is formed of functions which are L^2 when restricted to K. So this is a Hilbert space, though the representation is not necessarily unitary. The $K-$finite vectors are $v_n(r(\theta)) = e^{in\theta}$, so these are admissible representations. These vectors are orhtogonal with respect to the inner product (with appropriate Haar measure on K).

8.4. We compute the *infinitesimal action* of $U(\mathfrak{g})$ on $X(B,\chi)$. Two bases of the complexified Lie algebra are

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$e = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \qquad h = -i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad f = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 0 \end{bmatrix}.$$

It is an exercise to verify that

$$h \cdot v_n = nv_n, \qquad e \cdot v_n = \frac{\nu + 1 + n}{2} v_{n+2}, \qquad f \cdot v_n = \frac{\nu + 1 - n}{2} v_{n-2}$$

Then $X(B,\chi_+) = \bigoplus V_{2n}$ and $X(B,\chi_-) = \bigoplus V_{2n+1}$. The first one is irreducible except for $\nu = 2n+1$, the second one for $\nu = 2n$. The representations are unitary for $\nu \in i\mathbb{R}$ with the given inner product. At the reducibility points there are either 2 or three irreducible factors.

Theorem. *The factors of the principal series are* ***all*** *the admissible irreducible* $(\mathfrak{g}, K)-$*modules.*

The main point of this example is to highlight that for a general real reductive group, irreducible admissble representations are not much more complicated.

8.5. $(\mathfrak{g}, K)-$**modules.** Let $U(\mathfrak{g})^K$ be the ring of operators invariant under K; if K is connected, we can replace it by $\mathfrak{k}$. if (π, V) is a $(\mathfrak{g}, K)-$module, then $U(\mathfrak{g})^K$ acts on V_ξ. In particular $\mathcal{Z} := U(\mathfrak{g})^G \subset U(\mathfrak{g})^K$ acts on V_ξ.

Theorem (1)**.** *If* V *is finitely generated, then* V_ξ *is a finitely generated* $\mathcal{Z}-$*module.*

Corollary (1)**.** *Any finitely generated* $(\mathfrak{g}, K)-$*module is admissible.*

Theorem (2)**.** *There is* $C > 0$ *depending on* $\mathfrak{g}$ *and the number of generators such that* $\dim V_\xi \leq C(\dim(\xi)^2$.

Corollary (2)**.** *If* (π, V) *is irreducible, it is admissible.*

Theorem. *Any admissible irreducible* (π, V) *is a subsquotient of a* ***principal series*** $\mathrm{Ind}_P^G[\sigma \otimes \nu]$.

In this theorem, $P = MAN$ is a minimal parabolic subgroup, and $\sigma \otimes \nu$ a representation of MA.

8.6. Characters. Recall $\Omega_K = 1 - \sum X_i^2$ with X_i an orthonormal basis of $\mathfrak{k}$ with respect to a nondegenerate invariant form of $\mathfrak{g}$ negative on $\mathfrak{k}$. This operator acts by a scalar $c(\xi) \geq 1$ on any irreducible $K-$module; so also on any V_ξ. It satisfies

(1) There exist $r, C > 0$ such that $\dim \xi \leq Cc(\xi)^r$.
(2) For $m >> 0$, $\sum c(\xi)^{-m} < \infty$.

Definition. *An operator* $A : \mathcal{H} \longrightarrow \mathcal{H}$ *is called* ***trace class*** *if for any o.n. basis* $\{e_i\}$,

$$\sum \langle Ae_i, e_i\rangle < \infty.$$

A is ***summable*** *with respect to* $\{e_i\}$ *if*

$$\sum_{i,j} |\langle Ae_i, e_j\rangle| < \infty$$

In this case, $\mathrm{tr}(A) := \sum \langle Ae_i, e_i\rangle$ exists, and is independent of the choice of o.n. basis. It satisfies

$$\mathrm{tr}(SAT) = \mathrm{tr}(ATS) = \mathrm{tr}(TSA)$$

for any bounded operators S, T.

Summable implies trace class.

Theorem (1, Harish-Chandra)**.** *Assume* $\dim \mathcal{H}_\xi < c \dim(\xi)^2$. *Then* $\pi(f)$ *is trace class for any* $f \in C_c^\infty(G)$. *In particular*

$$\Theta_\pi(f) := \mathrm{tr}(\pi(f)$$

defines a $G-$invariant distribution. If $\mathcal{Z}$ acts by a character, then Θ_π is an eigendistribution for $\mathcal{Z}$.

Sketch. A matrix entry of π is a function of the form $f_{v,w}(g) := \langle \pi(g)v, w\rangle$. Choose an o.n. basis $\{e_{i,\xi}\}$ so that the $e_{i,\xi}$ form a basis of V_ξ. Any matrix entry can be differentiated by Ω_K on the right and on the left:

$$\partial^r_{L,\Omega_K} \partial^r_{R,\Omega_K} f_{e_{i,\xi}, e_{j,\eta}} = c(\xi)^r c(\eta)^r f_{e_{i,\xi}, e_{j,\eta}}.$$

Then

$$|\langle \pi(f) e_{i,\xi}, e_{j,\eta}\rangle| \leq c(\xi)^{-r} c(\eta)^{-r} \int_G ||\pi(x)|| \cdot \left|\partial^r_{L,\Omega_K} \partial^r_{R,\Omega_K} f\right| \, dx.$$

□

Theorem (2, Harish-Chandra)**.** *Two irreducible representations (on Hilbert spaces) are infinitesimally equivalent if and only if their characters coincide.*

8.7. Example. We apply some of this theory to the case of $L^2(G/\Gamma)$ with $\Gamma \subset G$ discrete cocompact. First some more facts about compact and trace class operators.

REFERENCE Reed-Simon, chapter VI.

Lemma (1)**.** *Let $K(\mathcal{H})$ be the space of compact operators on the separable Hilbert space $\mathcal{H}$. Then $K(\mathcal{H})$ is a closed ideal.*

Lemma (2)**.** *Any trace class operator is compact.*

Definition. *We say that T is Hilbert-Schmidt (H-S)if $\sum ||Tv_j|| < \infty$ for some o.n basis $\{v_j\}$.*

(1) If A is H-S, then so is A^*.
(2) If A is H-S, then so are AB, BA for any bounded B.
(3) T is of trace class if and only if it can be written as $T = AB$ with A, B H-S.

Theorem (Dixmier-Malliavin)**.** *Fort a Lie group G, any $f \in C_c^\infty(G)$ can be written as a sum $\phi_i \star \psi_i$ with $\phi_i, \psi_i \in C_c^\infty(G)$.*

We apply this to the case of $\mathcal{X} = G/\Gamma$ and $(\Pi, L^2(\mathcal{X}))$. Any $\pi(\phi) \in C_c^\infty(G)$ is compact because it is a kernel operator, $K_\phi(x, y) = \sum_{\gamma\in\Gamma} \phi(x\gamma y^{-1}$. Kernel operators are H-S. By the summary above, $\pi(\phi)$ is therefore trace class.

The following is a formal computation which can be justified with appropriate estimates.

let $\{e_i\}$ be an o.n. basis of $\mathcal{H} := L^2(\mathcal{X})$. We can write $K = K_\phi$ as

$$K(x, y) = \sum_{k,l} \lambda_{k,l} e_k(x)\overline{e_l(y)},$$

with $\lambda_{k,l} = \displaystyle\int_{\mathcal{X}\times\mathcal{X}} K(x, y)\overline{e_k(x)}e_l(y)\, dxdy$. Then $Ke_i(x) = \sum \lambda_{k,i} e_k(x)$. So

$$\sum_i \langle Ke_i, e_i\rangle = \sum \lambda_{i,i}.$$

On the other hand,

$$\sum_i \lambda_{i,i} = \int_{\mathcal{X}} K(x, x)\, dx.$$

In our case, we conclude

$$\mathrm{tr}\Pi(\phi) = \int_{\mathcal{X}} \sum_{\gamma\in\Gamma} \phi(x\gamma x^{-1})\, dx.$$

On the left, we can write $\sum m_\pi \mathrm{tr}\pi(\phi)$. On the right we can express the sum as follows. Let $\{\Gamma\}$ be a set of representatives of $G-$conjugacy classes of elements in Γ. If γ_1 and γ_2 are conjugate, then the terms in the sum are the same. Collecting such terms we get

$$\int_{G/\Gamma} \sum_{\gamma\in\Gamma} \phi(x\gamma x^{-1})\, d\gamma = \int_{G/\Gamma} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\Gamma/\Gamma_\gamma} \phi(x\delta\gamma\delta^{-1}x^{-1})\, d\gamma =$$

$$= \sum_{\gamma\in\{\Gamma\}} \int_{G/\Gamma_\gamma} \phi(x\gamma x^{-1})\, d\gamma =$$

$$= \sum_{\gamma\in\{\Gamma\}} \int_{G/G_\gamma} \int_{G_\Gamma/\Gamma_\gamma} \phi(xu\gamma u^{-1}x^{-1})\, d\gamma = \sum_{\gamma\in\{\Gamma\}} vol(G_\gamma/\Gamma_\gamma) \int_{G/G_\gamma} \phi(x\gamma x^{-1})\, dx.$$

The simplest example of this formula is the Poisson formula, $G = (\mathbb{R}, +)$ and $\Gamma = \mathbb{Z}$. On the left hand side, we have the decomposition of $\mathbb{R}/\mathbb{Z} = S^1$. The space is the direct sum of the representations $\{e^{in\theta}\}$. Taking traces we get $\sum \widehat{\phi}(n)$. On the other side we get $\sum f(n)$.

9. Center of the Enveloping Algebra

9.1. Universal Enveloping Algebra. The universal enveloping algebra of a real Lie group can be identified with the left or right invariant differential operators. The algebraic definition comes from the following theorem.

Theorem. *There is a unique (up to isomorphism) universal object* $(U(\mathfrak{g}), \iota$ *where* $U(\mathfrak{g})$ *is an associative algebra with unit, and iota* : $\mathfrak{g} \longrightarrow U(\mathfrak{g})$ *is a Lie algebra homomorphism such that any Lie algebra map* $\phi : \mathfrak{g} \longrightarrow A$ *into an associative algebra extends to a commutative diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & A \\ \downarrow \iota & \nearrow \Phi & \\ U(\mathfrak{g}) & & \end{array}$$

Properties:

(1) ι is an inclusion,
(2) if $X_1, \dots X_n$ is an ordered basis of $\mathfrak{g}$, then $X_1^{a-1} \dots X_n^{a_n}$ is a basis of $U(\mathfrak{g})$.
(3) Equivalently, $U(\mathfrak{g})$ has a filtration by $U(\mathfrak{g})_m$ which are generated by the monomials $Y_1 \cdot, \cdots \cdot Y_m$. The graded $Gr(U(\mathfrak{g})$ is avbelian and isomorphic to $S(\mathfrak{g})$.
(4) ad : $\mathfrak{g} \longrightarrow \mathrm{Hom}(\mathfrak{g})$ given by $\mathrm{ad}X(Y) := [X, Y]$ extends to a representation of $\mathfrak{g}$ on $S(\mathfrak{g})$ and $U(\mathfrak{g})$. The canonical map $gr : U(\mathfrak{g}) \longrightarrow Gr(U(\mathfrak{g})) \cong S(\mathfrak{g})$ is a Lie algebra isomorphism.
(5) The map

$$sym : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}), \qquad sym(Y_1 \cdot \dots \cdot Y_m) := \frac{1}{m!} \sum_{\sigma \in S_m} Y_{\sigma(1)} \cdot \dots \cdot Y_{\sigma(m)}$$

is a $\mathfrak{g}$-isomorphism, inverse to gr.

Corollary. *If* G *is connected,*

$$S(\mathfrak{g})^G = S(\mathfrak{g})^G \cong U(\mathfrak{g})^G = U(\mathfrak{g})^{\mathfrak{g}}.$$

9.2. Restricted Cartan Subalgebras. Let $\mathfrak{g}$ be real reductive, corresponding to the real reductive group G. We may assume connected semisimple for simplicity.

Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subalgebra.

Proposition. *Any two such subalgebras in* $\mathfrak{s}$ *are conjugate under* K.

Let $\mathfrak{m} := C_{\mathfrak{k}}(\mathfrak{a})$; this is a reductive subalgebra. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{m}$, then $\mathfrak{h} := \mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra . The group $M := C_K(\mathfrak{a})$ has Lie algebra $\mathfrak{m}$. The restricted Weyl group is $W := N_K(\mathfrak{a})/C_K(\mathfrak{a})$.

9.3. Iwasawa Decomposition. We can define $\Delta(\mathfrak{g}, \mathfrak{a})$ as in the case of a complex algebra. These are called the **restricted roots**. We can choose a positive system P. If α is a root, so is $-\alpha$, and at most $\pm 2\alpha$ or $\pm\frac{1}{2}\alpha$ are also roots. The restricted Weyl group acts transitively on the positive systems chambers etc. Let $\mathfrak{n}^{\pm}$ be the sums of roots spaces for $\pm P$. Let A, N be the connected subgroups corresponding to $\mathfrak{a}, \mathfrak{n}$. Note that $\mathfrak{n}^{-} = \theta(\mathfrak{n})$.

Proposition. $\mathfrak{g} = \mathfrak{n} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}^{-}$, *and* $G \cong KAN$.

REFERENCE Helgason.

9.4. We first determine $S(\mathfrak{s})^K$. We will often identify $S(V)$ for a vector subspace of $\mathfrak{g}$ with $P(V)$; this is because there is a $K-$ invariant inner product on $\mathfrak{g}$.

Proposition.

$$S(\mathfrak{s})^K \cong S(\mathfrak{a})^W.$$

Proof. We will do it in several steps.

I. The retriction map $\phi : P(\mathfrak{s}) \longrightarrow P(\mathfrak{a})$ is injective.

This follows from the fact that any $X \in \mathfrak{s}$ can be embedded into a restricted Cartan subalgebra , and any two are conjugate by K.

So we need to show that the map is onto.

II. $P(\mathfrak{s})^K$ separates points.

if $K \cdot h_1 \cap K \cdot h_2 = \emptyset$, there existws a contunuous function f which is 1 on $K \cdot h_1$ and 0 on $K \cdot h_2$. By the Stone-Weierstrass theorem, there is a polynomial p which is within less than $1/2$ of f on the compact union of these sets. Then $\int_K p(k \cdot X)\, dk$ is the desired element.

III. Let $S = P(\mathfrak{a}) \supset I = P(\mathfrak{a})^W \supset J = Im\phi \cong P(\mathfrak{s})^K$, and let $C \supset C(I) \supset C(J)$ be the quotient fields. Then $S, P(\mathfrak{s})$ are integrally closed. This follows from the fact that they satisfy unique factorization: if $r := p/q$ is a reduced expression and satisfies an equation

$$r^m + a_{m-1} r^{m-1} + \dots a_0 = 0,$$

then any prime $s|p$ must also divide q. But then $r = p/q$ is not reduced unless q is a unit.

IV. S is integral over I.

$\prod_{w\in W}(t - w\cdot p)$ is a polynomial in t with coefficients in I, and has p as a root.

V. If $p_1, p_2 \in P(\mathfrak{s})^K$ and $p_2 = p_1 \cdot q$, then $q \in P(\mathfrak{s})^K$. Similar for I.

Integrate $p - 2 = p_1 \cdot q$ over K or W respectively.

VI. $J \cong P(\mathfrak{s})^K$ is integrally closed. If $x \in C(J)$ is integral, then it is integral in $C(P(\mathfrak{s}))$. By the previous, $x = \frac{p_2}{p_1}$ with $p_1, q_1 \in J$; so $x \in P(\mathfrak{s})$, and then the previous point shows that it is in J.

VII. $P(\mathfrak{a})$ is integral over J.

Consider the polynomial

$$\det\left(\lambda - (\mathrm{ad}X)^2\right) = \lambda^r + p_{r-1}\lambda^{r-1} + \dots p_\ell \lambda^\ell.$$

It has coefficients in J. Restict this polynomial to $\mathfrak{a}$.

IX. let $h_1, h_2 \in \mathfrak{a}$ be such that $p(h_1) = p(h_2)$ for any $p \in J$. Then $h_2 \in W \cdot h_1$.

From (III), there is $k \in K$ such that $k \cdot h_2 = \cdot h_1$. Consider $C_{\mathfrak{g}}(h_1)$. This is a $\theta-$stable reductive subalgebra, and both $\mathfrak{a}$ and $k \cdot \mathfrak{a}$ are restricted Cartan subalgebra 's in $C_{\mathfrak{g}}(h_1)$. So there is an element $k' \in K \cap C_G(h_1)$ such that $k'k\mathfrak{a} = \mathfrak{a}$. Thus $k'k \in N_K(\mathfrak{a})$ and $k'k)h_2 = h_1$ as claimed.

X. $C/C(J)$ is a normal extension. Let

$$F(\lambda) := \lambda^{2r} + p_{2r-2}\lambda^{r-1} + \dots p_\ell \lambda^{2\ell}.$$

Its roots are the restricted roots $\pm\alpha \in \Delta$. C is obtained from adjoining these roots.

XI. Any $\sigma \in Aut(C)/C(J)$ must fix I.

The map

$$h_0 \in \mathfrak{a} \mapsto (p \mapsto p^\sigma(h_0))$$

is an algebra map of C, so of the form $p \mapsto p(h_1)$ for some $h_1 \in \mathfrak{a}$. This restricts to a $\lambda\mathfrak{a}^*$ of the form $\lambda(h) = B(h, h_1)$. ($\mathfrak{g}$ assumed semisimple). Then

$$p(h_1) = \lambda(p) = p^\sigma(h_0) = p(h_0) \qquad \forall p \in J.$$

By (IX), there is $w \in W$ such that $h_1 = w \cdot h_0$.

If $q \in I$, then

$$q^\sigma(h_0) = q(h_1) = q(w \cdot h_0) = q(h_0).$$

So σ fixes I, and therefore $I = J$. $\square$

9.5. Complex Algebras. In this case, $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ where $\mathfrak{u}$ is a compact form. So $\mathfrak{k} = \mathfrak{u}$ and $\mathfrak{s} = i\mathfrak{u}$. Then $S(\mathfrak{s})^{\mathfrak{k}} = S(\mathfrak{g})^{\mathfrak{g}}$ by complexifying $\mathfrak{u}$.

9.6. The Harish-Chandra Homomorphism.

9.7. Let $\mathfrak{g}$ be complex semisimple (reductive, but for simplicity ...). Recall $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$. Then $U(\mathfrak{g}) = U(\mathfrak{h}) + \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$. Let $q : U(\mathfrak{g})^{\mathfrak{h}} \longrightarrow U(\mathfrak{h})$ be the projection according to the space $\big(\mathfrak{n}^- U(\mathfrak{g})\big)$.

Lemma. *$q : U(\mathfrak{g})^{\mathfrak{h}} \longrightarrow U(\mathfrak{h})$ is an algebra homomorphism.*

Proof. This follows from the identity

$$\big(\mathfrak{n}^- U(\mathfrak{g})\big)^{\mathfrak{h}} = \big(U(\mathfrak{g})\mathfrak{n}\big)^{\mathfrak{h}}.$$

If a monomial element in $U(\mathfrak{g})^{\mathfrak{h}}$ is of the form $X_\alpha u$ with $X_\alpha \in \mathfrak{n}$, then u itself must be of the form $u_H X_{-\beta}$ with $X_{-\beta} \in \mathfrak{n}^-$. $\square$

Note that in $sl(2)$, the element $h^2 + 2(ef + fe)$ projects onto $h^2 + 2h$. This is not in $S(\mathfrak{h})^W$; the Harish-Chandra homomorphism means to identify $U(\mathfrak{g})^{\mathfrak{g}}$ with $S(\mathfrak{h})$.

9.8. Recall $\rho := \frac{1}{2}\sum_{\alpha \in P} \alpha$. Let $\mu : S(\mathfrak{h}) \longrightarrow S(\mathfrak{h})$ be the map defined by $\mu(h) = h - \rho(h)$.

Theorem. *The map $\gamma = \mu \circ q : U(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W$ is an algebra isomorphism independent of the choice $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}^-$.*

We have identified $U(\mathfrak{h})$ with $S(\mathfrak{h})$ because $\mathfrak{h}$ is abelian.

Proof. We sketch the part that $\gamma(\mathcal{Z}) \subset S(\mathfrak{h})^W$. Recall from the general structure theory that $W = N_G(\mathfrak{h})/C_G(\mathfrak{h})$ is a finite group generated by the reflections $s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$. Here α is a simple root associated to $\Delta(\mathfrak{n}, \mathfrak{h})$.

Let $\lambda \in \mathfrak{h}^*$. We associate a $1-$dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}$ to λ by the formula $(H + X) \cdot v_\lambda = \lambda(H) v_\lambda$. The **Verma module** $M(\lambda)$ is defined as

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}.$$

As an $\mathfrak{n}^-{-}$module it is $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho}$. The action of $\mathfrak{h}$ is via *commutation*, and the action of $\mathfrak{n}^+$ is also via brackets.

FACT 1. $M(\lambda)$ has a unique irreducible quotient called $L(\lambda)$. Let $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. Then $s_\alpha(\lambda) = \lambda - (\lambda\alpha^\vee)\alpha$. Assume $(\lambda, \alpha^\vee) \in \mathbb{N}$ for all simple roots. We say λ is dominant integral; in this case the $L(\lambda)$ give all the irreducible finite dimensional representations of $\mathfrak{g}$.

FACT 2. Let $v_{\lambda-\rho} := 1 \otimes \mathbb{1}_{\lambda-\rho}$. If $u \in U(\mathfrak{g})^{\mathfrak{h}}$, then

$$u \cdot v_{\lambda-\rho} = q(u)(\lambda - \rho) v_{\lambda-\rho} = \gamma(u)(\lambda) v_{\lambda-\rho}.$$

In particular, this is the case for any $z \in \mathcal{Z}$; in fact it acts by this scalar on all of $M(\lambda)$.

FACT 3. When λ is dominant integral, there is an inclusion $M(w\lambda) \subset M(\lambda)$ for any $w \in W$. It follows that

$$\gamma(z)(\lambda) = \gamma(z)(w\lambda) \qquad \forall w \in W.$$

□

Corollary.

$$\widehat{Z(\mathfrak{g})} \overset{\sim}{\longleftrightarrow} \mathfrak{h}^*/W.$$

REFERENCES Dixmier, Wallach. The proofs in the two references are different.

The modules $M(\lambda)$ will play an essential role later. They are part of a category similar to $(\mathfrak{g}, K)-$modules, called $(\mathfrak{g}, B)-$modules.

10. Admissible $(\mathfrak{g}, K)-$modules

10.1. Graded Modules. Let (π, X) be a $(\mathfrak{g}, K)$-module. We say it is finitely generated if it is finitely generated as a $U(\mathfrak{g})-$module. Assume it is finitely generated by a $K-$invariant set X_0. Recall that $U(\mathfrak{g})$ has a filtration $U(\mathfrak{g})_n$ defines by

$$U_0 = \mathbb{C}, \qquad U(\mathfrak{g})_{n+1} = \mathfrak{g}U(\mathfrak{g})_n.$$

Then

$$grU(\mathfrak{g}) := \bigoplus U(\mathfrak{g})_n/U(\mathfrak{g})_{n-1} \cong \bigoplus S^n(\mathfrak{g})$$

by the PBW-theorem. Set

$$X_n := U(\mathfrak{g})_n X_0.$$

This is a filtration of X by $K-$invariant finite dimensional subspaces. We denote

$$gr(X) := \bigoplus X_n/X_{n-1}.$$

This is a module for $S(\mathfrak{g})$ such that $S(\mathfrak{k})$ acts trivially; so it is a $S(\mathfrak{s})-$module. The action of K is preserved however, and $gr(X)$ is a finitely generated $(S(\mathfrak{g}), K)-$module.

10.2. Recall than $X = \bigoplus X(\xi)$. Then $\mathcal{Z} \subset U(\mathfrak{g})^K$, so it acts on each $V(\xi)$.

Theorem. *If (π, X) is finitely generated, then $X(\xi)$ is finitely generated as a $\mathcal{Z}-$module.*

Proof. From the structure theorem of $\mathcal{Z}$ we know that $gr(\mathcal{Z})$ maps to $S(\mathfrak{g})^G$; identify $S(\mathfrak{g})$ with $P(\mathfrak{g})$. Then $P(\mathfrak{g})^G$ will act on $gr(X)$ by restricting the polynomials to $\mathfrak{s}$. We know from the earlier proof that $S(\mathfrak{s})^K$ is finitely generated as a $Res_{\mathfrak{s}}S(\mathfrak{g})^{\mathfrak{g}}-$module. So it is enough to show that $X(\xi)$ is finitely generated as a $S(\mathfrak{s})^{\mathfrak{k}}-$module. The module

$X(\xi)$ is the image of $\mathrm{Hom}_{\mathbb{C}}[V_\xi, X] \otimes V_\xi$ under the map $T \otimes v \mapsto T(v)$. The module $\mathrm{Hom}_{\mathbb{C}}[V_\xi, X]$ is a finitely generated $K \otimes S(\mathfrak{s})-$module, and because $S(\mathfrak{s})$ is Noetherian, so is the module $S(\mathfrak{s}) \cdot \mathrm{Hom}_K[V_\xi, X]$. So there are $T_1, \dots, T_d \in \mathrm{Hom}_K[V_\xi, X]$ which generate this module. So for any $T \in \mathrm{Hom}_K[V_\xi, X]$, there are $p \in S(\mathfrak{s})$ such that $T = \sum p_i T_i$. Integrating over K, we find that $T = \sum p_i^K T_i$ with $p_i^K \in S(\mathfrak{s})^K$ as claimed. It follows that $X(\xi)$ is finitely generated. $\square$

Corollary. *If* (π, X) *is finitely generated, and locally* $\mathcal{Z}-$*finite, then it is admissible. This is true for any irreducible* $(\mathfrak{g}, K)-$*module.*

References

[B] T. Bröcker, T. tomDieck, *Representations of Compact Lie Groups*
[BW] A. Borel, N. Wallach, *Continuous Cohomology* Princeton University Press
[Dx] J. Dixmier, *Enveloping Algebras*
[FW] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*
[NW] N. Wallach, *Real Reductive Groups* vol. I and II
[GGP] I. Gelfand, Graev, I. Piatetskii-Shapiro, *Generalized Functions,* volume 6
[GW] G. Warner, *Harmonic Analysis on Semi-Simple Lie groups* vol. I and II
[He] S. Helgason, *Differential geometry and Symmetric Spaces* 1962
[HC] Harish-Chandra, *Representation of Semi-Simple Lie Group on a Banach space I and II* TAMS 1953 and 1955
[HT] R. Howe, E.C. Tan *Non-Abelian harmonic Analysis; Applications to* $Sl(2,\mathbb{R})$
[J] N. Jacobson, *Lie Algebras*
[Kn1] A. Knapp, *Representation Theory of Semi-Simple Lie groups; an Overview based on Examples*
[Kn2] A. Knapp, *Lie groups beyond an Introduction*
[KnV] A. Knapp, D. Vogan *Cohomological Induction and Unitary Pepresentations* Princeton University Press
[L] S. Lang, *Analysis II*
[LV] G. Lion, M. Vergne *The Weil Representation, Maslov Index, and Theta series*
[N] E. Nelson, *Analytic Vectors*, Ann. of Math. 1959
[SR] B. Simon, M. Reed *Methods of Modern Mathematica Physics* volume I, Functional Analysis
[Soe] W. Soergel *An irreducible not admissible Banach space representation of* $Sl(2,\mathbb{R})$ Proc. AMS 104, 1988, 1322-1324
[Va] V.S. Varadarajan, *An Introduction to Harmonic Analysis on Semisimple Lie Groups*
[V] D. Vogan, *Representations of Real Reductive Lie Groups* Birkhäuser, 1981
[W] N. Wallach, *Real Reductive Groups*, vol I and II, Academic Press 132 and 133