SMALL ORBITS IN SP(N,R)

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1. Nilpotent Orbits

The nilpotent orbits are parametrized by partitions/tableaus with alternating signs on each row. Rows of the same size are interchangeable. As usual, for types B, D, even rows appear an even number of times. They have to occur in pairs one starting with $-$ one starting with $+$. The same rules apply for type C and odd sized rows.

Let $\mathfrak{l} = \mathfrak{l}_0 + \mathfrak{l}_1$ be a Levi component of a maximal parabolic subgroup $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, $\mathfrak{l}$ stable under conjugation. Here $\mathfrak{l}_0$ refers to a subalgebra of the same type as $\mathfrak{g}$ and $\mathfrak{l}_1$ is type A. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{l}_0 \cap s$. We are interested in the nilpotent orbit which is induced from $\mathcal{O} \times triv$ on $\mathfrak{l}_1$. If $\mathfrak{q}$ is θ–stable, then $\mathfrak{l}_1 = u(p,q)$. In types B, D, the induced nilpotent is obtained by adding p pluses and q minuses one to the beginning (right) of each row and the same to the end (left), as high as possible so that the result is again a signed tableau. If $\mathfrak{q}$ is real, then $\mathfrak{l}_1 = gl(m)$. in this case, add 2 to each row (as high as possible) to form a signed tableau. In this case we get a union of nilpotent orbits.

In type C the same rules apply, but for θ–stable parabolics, add p pluses and q minuses to the beginning of the rows but the opposite *i.e.* q pluses and p minuses to the end of the rows.

The notation is as follows. Real roots are $2\epsilon_i$ and $\epsilon_i + \epsilon_j$. A number $\underline{a}^{\pm}$ underlined means that this is a real part of the parameter and the real root is long. The $\pm$ refers to the parity, which is the product $-$(parity of the number)($sign$). A pair underlined $\underline{a,b}$ means that the corresponding $\epsilon - i + \epsilon_j$ is real while $\epsilon_i - \epsilon_j$ is imaginary. The real part of the parameter is $(a+b)/2$ and the imaginary part is $(a-b)/2$. There is no parity. The other numbers refer to the imaginary part of the parameter. The positive chamber for K is the usual one, $\epsilon_1 \geq \cdots \geq \epsilon_n$.

2. Some Examples

Sp(2).

Parameter	LKT	$\mathcal{V}$	$\check{\mathcal{V}}$
$\pm(1,0)\ (\underline{1}^-, \underline{0}^-)$	$\pm(2,2),\ \pm(1,1)$	$\pm(2+)(2+)$	$(3+)(1+)(1-)$
$\pm(\underline{1}^+, \underline{0}^-)$	$\pm(1,0)$	$(2+)(2-)$	$(3+)(1+)(1+)$
$(\underline{1}^+, \underline{0}^+)\ (\underline{1,0})$	$(0,0),\ (1,-1)$	$(2+)(2-)$	$(3-)(1+)(1+)$

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Sp(3).

Parameter	LKT	$\mathcal{V}$	$\check{\mathcal{V}}$
$\pm(1,\underline{1}^-,0)$	$\pm(2,2,2)$	$\pm(2+)(2+)(2+)$	$(3+)(3-)(1+)\cup$ $(3+)(3+)(1-)$
$\pm(\underline{1}^+,\underline{1}^-,0)$	$\pm(1,1,0)$	$\pm(2+)(2+)(2-)$	$(3+)(3-)(1-)$
$(\underline{1}^+,\underline{1}^+,\underline{0}^+)$	$(0,0,0)$	$(2+)(2-)(2+)\cup(2+)(2-)(2-)$	$(3+)(3+)(1-)$
$(\underline{1}^-,\underline{1}^-,0)$	$\pm(1,1,1)$	$\pm(2+)(2+)(2+)\cup(2+)(2+)(2-)$	$(3+)(3+)(1+)$

Sp(4).

Parameter	LKT	$\mathcal{V}$	$\check{\mathcal{V}}$
$\pm(2,1,\underline{1}^-,0)$ $\pm(\underline{2}^-,1,\underline{1}^-,0)$	$\pm(3,3,3,3)$ $\pm(2,2,2,2)$	$\pm(2+)(2+)(2+)(2+)$	$(5+)(3-)(1+)\cup$ $(5+)(3+)(1-)$
$\pm(\underline{2}^+,\underline{1}^-,1,0)$ $\pm(\underline{2}^-,\underline{1}^-,\underline{1}^+,0)$	$\pm(2,2,2,1)$ $\pm(1,1,1,0)$	$\pm(2+)(2+)(2+)(2-)$	$(5+)(3+)(1+)$
$(\underline{2}^+,\underline{1}^+,\underline{1}^+,\underline{0}^+)$ $(\underline{2,-1},\underline{1,0})$	$(0,0,0,0)$ $(1,1,-1,-1)$	$(2+)(2+)(2-)(2-)$	$(5+)(3-)(1+)$
$(\underline{2}^+\underline{1}^+,\underline{1}^-,0)$	$\pm(1,1,0,0)$	$\pm(2+)(2+)(2-)(2-)$	$(5+)(3+)(1-)$
$\pm(\underline{2}^-\underline{1}^-,\underline{1}^-,0)$ $\pm(\underline{2}^-,1,\underline{10})$	$\pm(1,1,1,1)$ $\pm(2,2,2,0)$	$\pm(2+)(2+)(2+)(2-)$	$(5-)(3+)(1+)$

3. The Main Result

Step 1. Consider the parameters

$$
\begin{aligned}
&(\underline{r+k}^-,\dots,\underline{k+1}^-,\underline{k}^-,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0),\\
&(\underline{r+k}^-,\dots,\underline{k+1}^-,k,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0),
\end{aligned}
\tag{1}
$$

They have LKT's $(k+1,\dots,k+1)$ and $k,\dots,k)$ respectively; they are the unique parameters with the given infinitesimal character. These two parameters are also in the same cell. They satisfy

$$\begin{array}{lll} & \mathcal{V} & \check{\mathcal{V}} \\ (2) & (2+)^{2k}(1+)^r(1-)^r & [(2k+2r+1)+][(2k-1)-](1+)\cup \\ & & [(2k+2r+1)+][(2k+1)+](1-) \end{array}$$

For the fact that they are in the same cell it is enough to consider the case $r=0$. The two parameters are obtained by $\mathcal{R}_{\mathfrak{q}}$ from a parabolic with Levi component $\mathfrak{l}=u(2k)$. Since the have 1-dimensional lowest K-types, the parameter is determined by this and the infinitesimal character. For $r>0$ consider the parameters in SO obtained by ICiv duality; they are obtained from the case $r=0$ by $\mathcal{R}_{\mathfrak{q}}$ from a parabolic with Levi component $\check{\mathfrak{l}}=u(1)^r\times so(k)$.

The reason for $\mathcal{V}$ is that the second representation in (1) is also $\mathcal{R}_{\mathfrak{q}}$ from a character on an $\mathfrak{q}$ with $\mathfrak{l}\cong u(k)\times sp(r+k)$. This is a highest weight module and the parameter turns out to be the given one.

For $\check{\mathcal{V}}$, we argue as follows. The results hold for rank 2 by direct computation. Consider the parameter

$$(\underline{k}^+,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0). \tag{3}$$

It is *equivalent* to

$$(\underline{k}^+,\underline{k}^+,k-1,k-2,\underline{k-2}^-,\dots,1,\underline{1}^-,0)$$

under $T_{\alpha,\beta}$'s where the roots α,β are both short. By induction, $\mathcal{V}$ and $\check{\mathcal{V}}$ for $(k-1,k-2,\dots)$ are as in (2). The dual parameter to (3) is obtained by applying an $\mathcal{R}_{\check{\mathfrak{q}}}$ with $\check{\mathfrak{l}}=u(2)\times so(2k-3)$. We conclude that $\mathcal{V}$ is a union of real forms of $(2)^{2k}$ and $\check{\mathcal{V}}$ a union of real forms of $(2k+1)(2k-1)(1)$. Since both

$$(\underline{k}^{\pm},k-1,k-2,\underline{k-2}^-,\dots,1,\underline{1}^-,0)$$

are attached to $(2)^{2k}$ and the dual parameters are obtained by $\mathcal{R}_{\check{\mathfrak{q}}}$ from the dual of $(k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0)$, it follows that this parameter satisfies

$$\begin{array}{ll} \mathcal{V} & \check{\mathcal{V}} \\ (2+)^{2k-1} & [(2k-1)+][(2k-1)-](1+)\cup[(2k+1)+][(2k+1)+](1-) \end{array}$$

The fact about $\check{\mathcal{V}}$ now follows, because it is obtained from the above by applying $\mathcal{R}_{\check{\mathfrak{q}}}$. We also record the fact that the parameter $(\underline{k}^+,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0)$ satisfies

$$\begin{array}{lll} & \mathcal{V} & \check{\mathcal{V}} \\ (4) & (2+)^{2k-1}(2-) & [(2k+1)+][(2k-1)+](1+). \end{array}$$

In fact we can conclude the more general fact that the parameters

$$\begin{array}{ll} (5) & (\underline{r+k}^+,\dots,\underline{k+1}^+,\underline{k}^+,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0), \\ & (\underline{r+k}^-,\dots,\underline{k+1}^-,\underline{k}^-,\underline{k-1}^+,\underline{k-1}^-,k-2,\underline{k-2}^-,\dots,1,\underline{1}^-,0), \end{array}$$

belong to the same cell and satisfy

$$\begin{array}{lll} & \mathcal{V} & \check{\mathcal{V}} \\ (6) & (2+)^{2k-2}(2+)(2-)(1+)^r(1-)^r & [(2k+r+1)-][(2k-1)+](1+) \text{ or} \\ & & [(2k+r+1)+][(2k-1)+](1+) \end{array}$$

depending whether r is odd or even.

Step 2. Consider the parameters

$$\begin{gathered}(\underline{r+k}^-,\underline{r+k}^-,\dots,\underline{k+1}^-,\underline{k+1}^-,\underline{k}^-,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0)\\(\underline{r+k}^-,\underline{r+k}^+,\dots,\underline{k+1}^-,\underline{k+1}^+,\underline{k}^-,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0)\\(\underline{r+k}^-,\underline{r+k}^+,\dots,\underline{k+1}^-,\underline{k+1}^+,\underline{k}^+,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0).\end{gathered}$$

The first one satisfies

$$\begin{array}{ll} \mathcal{V} & \check{\mathcal{V}} \\ [(2+)]^{2k}[(2+)(2-)]^r & [(2k+2r+1)+][(2k+2r-1)+](1+) \text{ or} \\ & \text{or } [(2k+2r+1)+][(2k+2r-1)+](1-). \end{array}$$

The second one satisfies

$$\begin{array}{ll} \mathcal{V} & \check{\mathcal{V}} \\ [(2+)]^{2k}[(2+)(2-)]^r & [(2k+2r+1)-][(2k+2r-1)+](1-) \text{ or} \\ & [(2k+2r+1)+][(2k+2r-1)-](1+) \end{array}$$

The third one satisfies

$$\begin{array}{ll} \mathcal{V} & \check{\mathcal{V}} \\ [(2+)]^{2k}[(2+)(2-)]^r & [(2k+2r+1)+][(2k+2r-1)+](1-) \text{ or} \\ & [(2k+2r+1)+][(2k+2r-1)+](1+). \end{array}$$

The first case for $\check{\mathcal{V}}$ occurs when r is odd, the second one when r is even.

The facts about $\check{\mathcal{V}}$ come from the fact that all these parameters are $\mathcal{R}_{\check{\mathfrak{q}}}$'s from parameters as in Step1. For $\mathcal{V}$ view them as irreducible subquotients of modules which are real induced from parameters as in Step1. The Levi component is $\mathfrak{m} = gl(r) \times sp(k+r)$.

Each parameter has a companion in the same cell and is equivalent to a unipotent one at infinitesimal character

$$(k+r, k+r-1, k+r-1, \dots, 1, 1, 0)$$

via $T_{\alpha\beta}$'s with α, β short roots.

Step 3. Consider the parameter $(\underline{k+r}^+,\underline{k+r}^+,\dots,\underline{k}^+,\underline{k}^+,\dots,\underline{1}^+,\underline{1}^+,\underline{0}^+)$. This is contained in $Ind_{gl(2k+2r+1)}^{sp(2k+2r+1)}(triv)$ so it has $\mathcal{V}$ contained in $(2)^{k+r+1}$. On the other hand it is also contained in an induced $Ind_{gl(k+r+1)\times sp(k+r)}^{sp(2k+2r+1)}(\chi \otimes triv)$ where χ is a character. Its $\mathcal{V}$ is $[(3+)(3-)]^{\frac{k+r}{2}}(2+)(2-)[(1+)(1-)]^{\frac{k+r}{2}}$ if $k+r$ is even, $[(3+)(3-)]^{\frac{k+r+1}{2}}[(1+)(1-)]^{\frac{k+r+1}{2}}$ if $k+r$ is odd. In any case, this implies that $\mathcal{V}$ of this parameter is contained in $(2+)^{k+r+1}(2-)^{k+r} \cup (2+)^{k+r}(2-)^{k+r+1}$.

Step 4. Consider the parameters

$$(7) \qquad \begin{array}{l}(\underline{k+r}^-,\underline{k+r}^-,\dots,\underline{k}^-,\underline{k}^-,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0)\\ (\underline{k+r}^-,\underline{k+r}^+,\dots,\underline{k}^-,\underline{k}^+,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0)\end{array}$$

The first one satisfies

$\mathcal{V}$	$\check{\mathcal{V}}$
$[(2+)]^{2k-1}[(2+)(2-)]^r(2+)(2+)\cup$	$[(2k+2r+1)+][(2k+2r-1)+](1+)$ or
$[(2+)]^{2k-1}[(2+)(2-)]^r(2+)(2-)$	$[(2k+2r+1)+][(2k+2r-1)+](1-)$

The first case of $\check{\mathcal{V}}$ occurs when r is even, the second one when r is odd.

For $\check{\mathcal{V}}$, the reason for this is just Step1 and the algorithm for computing the variety of an $\mathcal{R}_{\mathfrak{q}}$. In particular it follows that $\mathcal{V}$ is a union of real forms of $(2)^{k+r+1}$. Consider the parameter

$$(8) \qquad (\underline{k+r}^-,\underline{k+r-1}^-,\underline{k+r-1}^-,\dots,\underline{k}^-,\underline{k}^-,k-1,\underline{k-1}^-,\dots,1,\underline{1}^-,0).$$

By the previous steps, it has $\mathcal{V}$ equal to $(2+)^{2k}[(2+)(2-)]^r$. Adding the $\underline{k+r}^-$ makes the irreducible module sit inside an induced from a real parabolic subalgebra with Levi component $gl(1)\times sp(2k+2r)$. The variety of this induced module is $(4+)(2+)^{2k-1+r}(2-)^r\cup(4-)(2+)^{2k+r}(2-)^{r-1}$. Thus the variety is contained in the union indicated in (1).

The second type are all $\mathcal{R}_{\mathfrak{q}}$'s so

$\mathcal{V}$	$\check{\mathcal{V}}$
$[(2+)]^{2k-1}[(2+)(2-)]^r$	$[(2k+2r+1)+][(2k+2r-1)+](1-)$

Step 5. Consider the two modules $Ind_{gl(2k+2r+1)}^{sp(2k+2r+1)}[triv]$ and $Ind_{gl(2k+2r+1)}^{sp(2k+2r+1)}[sgn]$. The only factors of the first one are from Step3. For the second induced module, the only possible factors are the second ones in Step4. The inclusions that we proved, imply that the $\mathcal{V}$'s of the possible factors in a given induced module are disjoint (except of course for strictly smaller orbits). Counting the nilpotent orbits that must occur in $\mathcal{V}$ ofr the induced module, we get the equality in the claimend answer.

4. Dual Pairs Correspondence

4.1. **The Characteristic Variety and the Theta Correspondence.** Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}_0\subset\mathfrak{g}_0$ be real forms of the algebras. We assume that $\mathfrak{g}$ is reductive and choose a Cartan involution which stabilizes $\mathfrak{h}$. The Cartan decomposition is $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$.

Let G be a real connected group corresponding to $\mathfrak{g}_0$ and H the subgroup corresponding to $\mathfrak{h}_0$. Let K be the maximal compact group with Lie algebra $\mathfrak{k}_0$. Let Ω be a $(\mathfrak{g},K)$ module, and Π an irreducible $(\mathfrak{h},K\cap H)$ quotient of Ω. In other words there is an exact sequence

$$(9) \qquad \Omega\xrightarrow{\rho}\Pi\longrightarrow 0.$$

Now take compatible good filtrations on Ω and Π. Indeed take a finite set σ of generators of Π. Then $\Sigma:=\rho^{-1}(\sigma)$ is a set of generators Ω. Then the filtrations

$\{\mathcal{U}(\mathfrak{g})_n\Sigma\}$ and $\{\mathcal{U}(\mathfrak{h})_n\sigma\}$ are good compatible. We get an exact sequence of the corresponding graded $S(\mathfrak{h})$ modules

$$\omega \to \pi \to 0. \tag{10}$$

Let $\mathbb{I} := Ann_{S(\mathfrak{g})}\omega$ and $\mathbb{J} := Ann_{S(\mathfrak{h})}\pi$. Then $\mathbb{I} \cap S(\mathfrak{h}) \subset \mathbb{J}$ and so

$$\mathcal{V}(\pi) \subset \mathcal{V}(\mathbb{I} \cap S(\mathfrak{h})). \tag{11}$$

4.2. We are going to specialize (11) to the case of $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_2 = sp(2m) \times o(p,q)$ (or any other dual pair) and $\mathfrak{g} = sp(2mn)$ with $n = p + q$. The main point is to compute $\mathcal{V}(\mathbb{I} \cap S(\mathfrak{h}))$. We use the usual coordinates and realization of sp.

Let $(V_1, <\ ,\ >)$ be a real symplectic space and $(V_2, (\ ,\ ))$ be a real quadratic space. Let $W = V_1 \otimes V_2$. Then $sp = sp(W)$ and $\mathfrak{h}_0 = sp(V_1) \times o(V_2)$. Decompose $V_2 = V_2^+ \oplus V_2^-$ according to the signature of the form $(,)$. Then let $\mathbb{W},\ \mathbb{V}_1,\ \mathbb{V}_2^{\pm}$ be the corresponding complexifications. We view $\mathbb{V}_1$ as a symplectic space and $\mathbb{V}_2^{\pm}$ as quadratic spaces with nondegenerate forms. Then the K-finite vectors of ω can be identified with

$$V_\omega = P(\mathbb{L}) \tag{12}$$

where $\mathbb{L}$ is a Lagrangian subspace of $\mathbb{W}$. Choose a Lagrangian space $\mathbb{L}_1 \subset \mathbb{V}_1$. We will use $\mathbb{L} = \mathbb{L}_1 \otimes \mathbb{V}_2^+ \oplus \mathbb{L}_1^\perp \otimes \mathbb{V}_2^-$. Write $\mathfrak{h}_1 = sp(\mathbb{V}_1)$ and $\mathfrak{h}_2 = so(V_2)$. The Lie algebra $\mathfrak{h}_1$ is identified with $S^2(V_1)$ and the Lie algebra of $\mathfrak{h}_2$ is identified with $\bigwedge^2(V_2)$. Choose a basis $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ of V_1 such that the dual basis is $\{e_{m+1}, \dots, e_{2m}, e_1, \dots, e_m\}$ and a basis $\{f_1, \dots, f_p, f_{p+1}, \dots, f_n\}$ such that the dual basis is $\{f_1, \dots, f_p, -f_{p+1}, \dots, -f_n\}$. The Cartan involutions are $-S^2(\theta_1)$ and $-\bigwedge^2(\theta_2)$, where

$$\begin{aligned} &\theta_1(e_i) = e_i, &&\theta_1(e_j) = -e_j, &&i \le m < j,\\ &\theta_2(f_k) = f_k, &&\theta_2(f_l) = -f_l, &&k \le p < k. \end{aligned}$$

The Cartan involution of $sp(2mn)$ is $\theta = -S^2(\theta_1) \otimes \bigwedge^2(\theta_2)$. Furthermore, $\mathfrak{h}_1,\ \mathfrak{h}_2$ embed in $\mathfrak{g}$ via the formulas

$$\begin{aligned} e_i \cdot e_j &\mapsto \sum_s [e_i \otimes f_s][e_j \otimes f_s^*],\\ f_k \wedge f_l &\mapsto \sum_r [e_r \otimes f_k][e_r^* \otimes f_l]. \end{aligned} \tag{13}$$

This is the setup from section 4.1. Then $\mathbb{L} \subset \mathbb{W}$ has as basis

$$\mathbb{L} := span\ \{e_i \otimes f_k, e_j \otimes f_l\} \qquad \text{where } i \le m,\ j > m,\ k \le p,\ l > p. \tag{14}$$

Denote by z_{rs} the variable corresponding to $e_r \otimes f_s$ in (14). Then ω is given by

$$\omega([e_i \otimes f_k] \cdot [e_j \otimes f_l]) = \begin{cases} m_{z_{ik} \cdot z_{jl}} & \text{if } e_i \otimes f_k,\ e_j \otimes f_l \in \mathbb{L},\\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Composing (13) with (15) we get a homomorphism

$$\phi : P(\mathfrak{h}) \longrightarrow P(\mathbb{L}). \tag{16}$$

Its kernel is exactly $\mathbb{I} \cap P(\mathfrak{h})$. But every such algebra homomorphism is composition with a map

$$\iota : \mathbb{L} \longrightarrow \mathfrak{h}. \tag{17}$$

The closure of its image is precisely $\mathbb{V}(\mathbb{I} \cap S(\mathfrak{h}))$. This map $\iota = (\iota_1, \iota_2)$ sends $x = \sum v_i \otimes w_i \in \mathbb{L}$ to

$$\begin{aligned} \iota_1(x) &= \sum (w_i, w_j) v_i \cdot v_j, \\ \iota_2(x) &= \sum < v_i, v_j > w_i \wedge w_j. \end{aligned}$$

4.3. **Nilpotent orbits in Sp.** Let $\mathbb{L} := \mathit{span}\ \{e_1, \dots, e_n\}$, and write e_i^* for the duals. Then there are two version of the principal nilpotent corresponding to the partition $(2n)$,

$$\begin{aligned} &e_1e_2 + e_2^*e_3^* + \cdots + \epsilon_n^2 &&\longleftrightarrow &&e_1^* \mapsto e_2 \mapsto e_3^* \cdots \mapsto e_1 \mapsto 0, \\ &e_1^*e_2^* + e_2e_3 + \cdots + \epsilon_n^2 &&\longleftrightarrow &&e_1 \mapsto e_2^* \mapsto e_3 \cdots \mapsto e_1^* \mapsto 0 \end{aligned} \tag{18}$$

where $\epsilon = e$ or e^* depending on the parity of n. The first is labelled $- + \dots$, the second $+ - \dots$. (???)

Next consider the nilpotent orbit with partition (n, n) and n odd corresponding to $\begin{matrix} + & - & \cdots \\ - & + & \cdots \end{matrix}$. This corresponds to

$$e_1e_2 + e_2^*e_3^* + e_3e_4 + \cdots + \epsilon_{n-1}^*\epsilon_n^*. \tag{19}$$

A general nilpotent orbit has a representative formed of a direct sum of expressions as in (18) and (19).

4.4. **Nilpotents in So(2m).** Write $p = 2r + \epsilon_1$ and $q = 2s + \epsilon_2$ and assume $p \geq q$. Choose bases $\{f_1, \dots f_r, f_1^*, \dots f_r^*, f_{r+\epsilon_1}\}$ and $\{g_1, \dots g_s, g_1^*, \dots g_s^*, g_{s+\epsilon_2}\}$. There are two versions of the principal nilpotent corresponding to the partition $(2m-1, 1)$. Suppose $\epsilon_1 = \epsilon_2 = 0$. Then there are

$$\begin{aligned} &f_1^* \wedge g_1 + g_1^* \wedge f_2 + f_2^* \wedge g_2 + \dots &&\leftrightarrow &&f_1 \mapsto g_1 \mapsto f_2 \mapsto g_2 \mapsto \cdots \mapsto f_1^* \mapsto 0 \\ &g_1^* \wedge f_1 + f_1^* \wedge g_2 + g_2^* \wedge f_2 + \dots &&\leftrightarrow &&g_1 \mapsto f_1 \mapsto g_2 \mapsto f_2 \mapsto \cdots \mapsto g_1^* \mapsto 0. \end{aligned}$$

The term in the middle is $f_s \wedge g_s + g_s^*$ or a $f_s + f_s^* \wedge g_s$ respectively.

5. A special case

We are looking at the pairs $O(p,q) \times Sp(2m)$, where $p + q = 2n$ and m equals either $2n$ or $2n-1$. We are interested in the case when the representation on the $O(p,q)$ side is 1–dimensional. This representation is determined by its *lowest K-type* which is obviously also its *harmonic K-type* . We only consider the case $p \geq q$, the other one is obtained by an outer automorphism. For the case $m = 2n$, the correspondence is as follows. Denote $r := \frac{p-q}{2}$. Then

$$
\begin{array}{lll}
O(p,q) & Sp(2m) & Parameter \\
++ & (\underbrace{r}_{p},\underbrace{r}_{q}) & (\underline{n}^-,\underline{n-1}^-,\underline{n-1}^-,\dots,\underline{r}^-,r-1,\underline{r-1}^-,\dots,1,\underline{1},0) \\
 & & (\underline{n}^+,\underline{n-1}^+,\underline{n-1}^+,\dots,\underline{1}^+,\underline{1}^+,\underline{0}^+) \ \text{ if } \ r=0, \\
+- & (\underbrace{r}_{p},\underbrace{r-1}_{q}) & (\underline{n}^-,\underline{n-1}^-,\underline{n-1}^+,\dots,\underline{r}^-,\underline{r}^+,r-1,\underline{r-1}^-,\dots,1,\underline{1}^-,0) \\
-+ & (\underbrace{r+1}_{p},\underbrace{r}_{q}) & (\underline{n}^+,\underline{n-1}^-,\underline{n-1}^+,\dots,\underline{r+1}^-,\underline{r+1}^+,r,\underline{r}^-,\dots,1,\underline{1}^-,0) \\
-- & (\underbrace{r+1}_{p},\underbrace{r-1}_{q}) & (\underline{n,n-1},\dots,\underline{r+1,r},r,r-1,\underline{r-1}^-,\dots,1,\underline{1}^-,0)
\end{array}
\tag{20}
$$

The first and fourth parameter are $\mathcal{R}_{\mathfrak{q}}$'s. The parameters corresponding to (0+)(0+), (0−)(0−) and (0+)(0−), (0−)(0+) form cells.

For the case $m = 2n-1$ the correspondence is

$$
\begin{array}{lll}
O(p,q) & Sp(2m) & Parameter \\
(+)(+) & (\underbrace{r,\dots,r}_{p+q-1}) & (\underline{n-1}^-,\underline{n-1}^-,\dots,\underline{r}^-,r-1,\underline{r-1}^-,\dots,1,\underline{1}^-,0) \\
 & & (\underline{n}^+,\underline{n-1}^+,\underline{n-1}^+,\dots,\underline{1}^+,\underline{1}^+,\underline{0}^+) \ \text{ if } \ r=0, \\
(+)(-) & (\underbrace{r}_{p-1},\underbrace{r-1}_{q}) & (\underline{n-1}^-,\underline{n-1}^+,\dots,\underline{r}^-,\underline{r}^+,r-1,\underline{r-1}^-,\dots,1,\underline{1}^-,0) \\
(-)(+) & (\underbrace{r+1}_{p},\underbrace{r}_{q-1}) & (\underline{n-1}^-,\underline{n-1}^+,\dots,\underline{r+1}^-,\underline{r+1}^+,r,\underline{r}^-,\dots,1,\underline{1}^-,0).
\end{array}
\tag{21}
$$

The case (0−)(0−) does not occur in the correspondence. The second parameters are $\mathcal{R}_{\mathfrak{q}}$'s.

Fix p, q. Consider an element $\sum \alpha_i \otimes \beta_i$ which maps to 0 under ι_2. Write the element as

$$\sum_{i=1}^{p} \alpha_i \otimes f_i + \sum_{j=p+1}^{n} \alpha_j \otimes f_j. \tag{22}$$

Its image under ι_1 is

$$\sum_{1\le i\le p} \alpha_i^2 - \sum_{p<j\le n} \alpha_j^2. \tag{23}$$

This is a nilpotent element of the form $(2+)^a(2-)^b[(1+)(1-)]^c$ where $a \le p$ and $b \le q$. If $2n = m$, the image is the closure of $(2+)^p(2-)^q$. If $2n = m+1$, the image is the union $(2+)^p(2-)^{q-1} \cup (2+)^{p-1}(2-)^q$. Equality follows as before.

Write

$$(a_1,\dots,a_k,0,\dots,0;\epsilon)\otimes(b_1,\dots,b_v,0,\dots,0;\eta)\quad a_k>0,\ b_v>0 \tag{24}$$

for a K-type of $O(p)\times O(q)$. Then (a small generalization of) condition ($\dagger'$) of Moeglin says that if

$$k+\frac{1-\epsilon}{2}(p-2k)+v+\frac{1-\eta}{2}(q-2v)\le\frac{n}{2} \tag{25}$$

and $2n\le m$, then the representations occur in the correspondence and the Langlands parameters correspond. This takes care of the cases $(0+)(0+)$ and $(0+)(0-)$ when $m=2n$ (recall $p\ge q$). The fact that the other cases occur at all comes from the formula

$$n(\pi)+n(\pi\otimes sgn)=2n, \tag{26}$$

$n(\pi)$ the smallest m such that π occurs in the correspondence. The basic fact is that $(0+)$ on $O(2r,0)$ occurs with $Sp(0)$ and $(0-)$ first occurs with $Sp(4r)$. This implies the claim for $(0-)(0-)$. The parameter $(0+)(0-)$ first occurs in the correspondence with $Sp(2q)$. The infinitesimal character of the parameter in $Sp(2q)$ is

$$(n-1,\dots,p). \tag{27}$$

The entries less than r form the parameter of a holomorphic discrete series. The entries larger than r form the real part of the parameter, all of them with a $-$, *i.e.* $(\underline{n-1}^-,\dots)$.

So the first occurence for $(0-)(0+)$ is with $Sp(2p)$. The parameter is

$$(\underline{n-1}^+,\dots,\underline{r+1}^+,r,\underline{r}^-,\dots,1,\underline{1}^-,0) \tag{28}$$

The results about $(0-)(0+)$ in (20) and (21) follow by the extended induction principle.

The remaining case is $(0-)(0-)$. This does not seem to be covered by the induction principle. We go all the way down to basics and find the K-structure of the modules as well.

6. Explicit Correspondence

Consider the pair $M\times M'=U(m)\times U(p,q)\subset Sp(2mn)$ where $n=p+q$. The metaplectic representations can be realized in the space

$$\mathbb{C}[z_{i,j}]\qquad 1\le i\le m,\ 1\le j\le n. \tag{29}$$

The restriction of the action to the subalgebra $\mathfrak{m}\times\mathfrak{m}'$ is given by the following formulas.
$U(m)$:

$$\sum_{k=1}^{p} z_{i,k}\partial_{j,k}-\sum_{k=p+1}^{p+q} z_{i,k}\partial_{j,k}+\frac{p-q}{2},\qquad 1\le i,j\le m. \tag{30}$$

$U(p,q)$:

$$
\begin{aligned}
\epsilon_i-\epsilon_j &\longleftrightarrow p_{ij}:=\sum_{k=1}^{m} z_{k,i}z_{k,j} \quad 1\le i\le p<j\le n\\
\epsilon_j-\epsilon_i &\longleftrightarrow \partial_{p_{ij}}:=\sum_{k=1}^{m} \partial_{k,i}\partial_{k,j} \quad 1\le i\le p<j\le n\\
h_i &\longleftrightarrow \sum_{k=1}^{m} z_{k,i}\partial_{k,i}+\tfrac{m}{2} \quad 1\le i\le p\\
\epsilon_i-\epsilon_j &\longleftrightarrow e_{ij}:=\sum_{k=1}^{m} z_{k,i}\partial_{k,j} \quad 1\le i<j\le p\\
h_j &\longleftrightarrow -\sum_{k=1}^{m} z_{k,j}\partial_{k,i}-\tfrac{m}{2} \quad p<j\le n\\
\epsilon_i-\epsilon_j &\longleftrightarrow f_{ij}:=\sum_{k=1}^{m} z_{k,j}\partial_{k,i} \quad p<i<j\le n
\end{aligned}
\tag{31}
$$

A basis for $\mathcal{W}$ is given by

$$x(P,\alpha_{ij}):=P(z_{ij})\prod_{1\le i\le p<j\le n} p_{ij}^{\alpha_{ij}} \tag{32}$$

where α_{ij} are integers and the P form a basis of harmonic polynomials. These form irreducible representations with highest weights $\delta_1^{a_1}\cdot\dots\cdot\delta_p^{a_p}$ and $\eta_1^{b_1}\cdot\dots\cdot\eta_q^{b_q}$ where $\delta_r,\ \eta_s$ are determinants

$$\delta_r:=det[z_{ij}]_{i,j\le r},\qquad \eta_s:=det[z_{ij}]_{n-s<i,j\le n} \tag{33}$$

Remark. The representation with highest weight δ_r is a fundamental representation and has a basis formed by all the $r\times r$ minors.

The linear span of the elements $x(P,\alpha_{ij})$ where P belongs to the representations determined by a fixed $(a_1,\dots,a_p,b_1,\dots,b_q)$ forms a generalized Verma module for $u(p,q)$ and parabolic subalgebra with Levi factor $u(p)\times u(q)$.

The subalgebra $\mathfrak{g}:=o(p,q)$ has a basis $e_{ij}-e_{ji}, f_{ij}-f_{ji}, p_{ij}-\partial_{p_{ij}}$. Consider the quotient

$$\Pi:=\mathcal{W}/U(\mathfrak{g})_+\mathcal{W}.$$

This is a module for $\mathfrak{g}\times\mathfrak{g}'$ where $\mathfrak{g}'\cong sp(2m)$. The images of $x(P,0)$ generate this module. This is proved by induction on $\sum\alpha_{ij}$. It is a general (and trivial) fact about Verma modules. If some $\alpha_{rs}>0$, then

$$Px(1,\alpha_{ij})-(p_{rs}-\partial_{p_{rs}})(Px(1,\alpha_{1,p+1},\dots,\alpha_{rs}-1,\dots,\alpha_{p,n}))$$

is a term of the same sort but with lower $\sum\alpha_{ij}$. This follows from the formula

$$\partial_{p_{ij}}(P\cdot Q)=\sum_k \partial_{ki}\partial_{kj}P\cdot Q+\partial_{kj}P\cdot\partial_{ki}Q+\partial_{ki}P\cdot\partial_{kj}Q+P\cdot\partial_{ki}\partial_{kj}Q. \tag{34}$$

When P is harmonic, the first term is zero. The third term is zero because $i\le p<j$. The fourth term lowers $\sum\alpha_{ij}$. Finally for the second term use the fact that $\sum_k \partial_{ki}det[z_{ab}]\cdot\partial_{kj}p_{a'b'}$ is either 0 or a new minor where a column is replaced by a different one.

Since $so(p,q)$ has to act trivially on π, only the P with

$$a_2 \equiv \cdots \equiv a_p \equiv 0(2) \quad \text{and} \quad b_2 \equiv \cdots \equiv b_q \equiv 0(2) \tag{35}$$

have nonzero images. For $p+q=m$, Π is a direct sum of the four modules of $sp(2m)$ corresponding to $++,\ +-,\ -+,\ --$. For $p+q=m-1$, only $x(P,\alpha_{ij})$ with $a_p=0$ or $b_q=0$ appear. Π is a direct sum of the modules corresponding to $++,\ +-,\ -+$.

The action of $sp(2m)$ is given by

$$\begin{aligned} \mathfrak{p}^+ : \qquad & \epsilon_i+\epsilon_j \longleftrightarrow \sum_{k=1}^{p} z_{ik}z_{jk} - \sum_{k=p+1}^{n} \partial_{ik}\partial_{jk}, \\ \mathfrak{p}^- : \qquad & -\epsilon_i-\epsilon_j \longleftrightarrow \sum_{k=1}^{p} \partial_{ik}\partial_{jk} - \sum_{k=p+1}^{n} z_{ik}z_{jk} \end{aligned} \tag{36}$$