Iwahori-Hecke algebra C3

This corresponds to the unitary dual of the group of type B_3. The parameters are all real.

Diagram: $-\beta - \alpha_1 - \alpha_2 = \alpha_3$

The reflection s_3 corresponds to the long simple root.

Simple Roots and Co-weights:

$$\alpha_1 = \epsilon_1 - \epsilon_2 \qquad \omega_1 = \epsilon_1$$

$$\alpha_2 = \epsilon_2 - \epsilon_3 \qquad \omega_2 = (\epsilon_1 + \epsilon_2)$$

$$\alpha_3 = 2\epsilon_3 \qquad \omega_3 = \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

Positive Roots: $\{\epsilon_i \pm \epsilon_j,\ i < j \leq 3,\ 2\epsilon_1,\ 2\epsilon_2,\ 2\epsilon_3\}$.

Longest Root: $-\beta = 2\epsilon_1$.

Rho : $\rho = (3, 2, 1)$.

Order of $W(B_3)$: $3!\ 2^3 = 48$.

Ordering of nilpotents:
$$\begin{tikzcd}[column sep=small, row sep=small] & & (33) \arrow[dr] & & & & \\ (6) \arrow[r, dash] & (42) \arrow[ur] \arrow[dr] & & (222) \arrow[r, dash] & (2211) \arrow[r, dash] & (21111) \arrow[r, dash] & (11111) \\ & & (411) \arrow[ur] & & & & \end{tikzcd}$$

(6)	$2 - 2 = 2$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$
(42)	$2 - 0 = 2$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
(411)	$2 - 1 = 0$	$(\frac{3}{2}, \frac{1}{2}, 0)$
(33)	$0 - 2 = 0$	$(1, 1, 0)$
(222)	$0 - 0 = 2$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
(2211)	$0 - 1 = 0$	$(\frac{1}{2}, \frac{1}{2}, 0)$
(21111)	$1 - 0 = 0$	$(\frac{1}{2}, 0, 0)$
(111111)	$0 - 0 = 0$	$(0, 0, 0)$

Green's polynomials:

(6)	$0 \times 1^3_*$
(42)	$(0 \times 1^3 \ + \ 1 \times 11_*) \ + \ (1^3 \times 0_*)$
(411)	$0 \times 12_* \ + \ 0 \times 1^3 \ + \ 1 \times 11$
(33)	$11 \times 1_* + 0 \times 1^3 + 1 \times 11 + 1^3 \times 0$
(2^3)	$1 \times 2_* + 0 \times 12 + 0 \times 1^3 + 11 \times 1 + 1 \times 11$
$(2^2 1^2)$	$(2 \times 1_* + 2 \cdot 1 \times 11 + 1^3 \times 0 + 0 \times 1^3 + 0 \times 12 + 1 \times 2 + 11 \times 1) + (12 \times 0_* + 1^3 \times 0 + 11 \times 1)$
(21^4)	$0 \times 3_* + 2 \cdot 0 \times 12 + 2 \times 1 + 11 \times 1 + 2 \cdot 1 \times 2 + 2 \cdot 1 \times 11$
(1^6)	$3 \times 0_* + 2 \cdot 12 \times 0 + 1^3 \times 0 + 3 \cdot 2 \times 1 + 3 \cdot 11 \times 1 + 3 \cdot 1 \times 2 + 3 \cdot 1 \times 11 + 0 \times 3 + 2 \cdot 0 \times 12 + 0 \times 1^3$

(6): This is the Steinberg representation.

(42): There are two discrete series.

(411): The standard module is induced from the Steinberg representation on the Levi component $GL(1)Sp(2)$. The parameter is $(\nu_1; \frac{3}{2}, \frac{1}{2})$. Reducibility occurs at $\nu_1 = \frac{1}{2}, \ \frac{5}{2}$, corresponding to $(511), (7)$. The signs are

ν_1	0×12	0×1^3	1×11
	$+$	$+$	$+$
$\frac{1}{2}$	$+$	0	0
	$+$	$-$	$-$
$\frac{5}{2}$	$+$	0	$-$
	$+$	$+$	$-$

The complementary series is the interval $0 \le \nu_1 < \frac{1}{2}$.

(33): The standard module is induced from the Steinberg representation on the Levi component $GL(3)$. The parameter is $(1 + \nu_2, \nu_2, -1 + \nu_2)$.

Reducibility occurs at $\nu_1 = \frac{1}{2}, \ \frac{5}{2}$, corresponding to $(42), (6)$. The signs are

ν_1	11×1	1×11	0×1^3	$1^3 \times 0$
	$+$	$+$	$+$	$+$
$\frac{1}{2}$	$+$	0	0	0
	$+$	$-$	$-$	$-$
$\frac{5}{2}$	$+$	$-$	0	$-$
	$+$	$-$	$+$	$-$

The complementary series is $0 \leq \nu_2 < \frac{1}{2}$.

(222): The standard module is induced from the Steinberg representation on the Levi component $GL(2)SO(3)$. The parameter is $(\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \frac{1}{2})$. The complementary series is $0 \leq \nu_1 < 1$.

(2211): The standard module is induced from the trivial tensored with the Steinberg representation on the Levi component $GL(1)GL(2)$. The parameter is $(\nu_1; \frac{1}{2}+\nu_2, -\frac{1}{2}+\nu_2)$. The only unitary representations attached to this nilpotent are in the region

$$\{\ \nu_2 = 0,\ 0 \leq \nu_1 < \frac{1}{2}\ \}.$$

The main reason for this is that the standard representation for $SO(5)$ corresponding to the nilpotent orbit 22 does not have a complementary series. When $\nu_2 = 0$ there are two irreducible representations for each parameter, one corresponding to 2×1 the other to 12×0.

(21111): The standard module is induced from the trivial tensored with the Steinberg representation on the Levi component $GL(1)GL(1)SO(3)$. The parameter is $(\nu_1; \nu_2; \frac{1}{2})$. The representations in the region

$$\{0 \leq \nu_2 \leq \nu_1 < \frac{1}{2}\ \}$$

form the complementary series attached to this nilpotent.
Note: There are other unitary representations with a similar looking parameter attached to (222).

(111111): The standard module is induced from the trivial on the Levi component $GL(1)GL(1)GL(1)$. The parameter is $(\nu_1; \nu_2; \nu_3)$ with $0 \leq \nu_3 \leq \nu_2 \leq \nu_1$. The complementary series is

$$\{\ 0 \leq \nu_3 \leq \nu_2 \leq \nu_1 < \frac{1}{2}\ \}.$$