Dirac Cohomology and Unitary Representations

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Based on,

- Earlier results with Pavle Pandžić.
- Joint with Chao-Ping-Dong, and Daniel Wong for complex classical groups and spin groups.
- More recent joint with Daniel Wong on E_8

Still very much in progress. See the two sets of references at the end

Original Dirac Operator I

One of the simplest versions of the Dirac operator is

$$D = \sum \partial_i \epsilon_i$$

with the property that it is a *formal square root* of the Laplacian, i.e.

$$D^2 = \Delta = \sum (\epsilon_i \epsilon_j + \epsilon_j \epsilon_i) \partial_{ij} = 2 \sum \partial_i^2 .$$

This forces $\epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 2\delta_{ij}$, which makes sense in the Clifford algebra. According to Wikipedia, the original version of the Dirac equation, which he found staring into the fireplace, is

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}\partial_t \right) \psi = \kappa \psi$$

A, B, C, D are 4×4 matrices, formed out of the 2×2 Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac operator from Physics I

Look for D such that $D^2 = -\sum \partial_i^2$. (Or $D^2 = \sum \pm\partial_i^2$.)

If $D = \sum e_i \partial_i$, get

$$e_i^2 = -1; \qquad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

The original Dirac equation was motivated by trying to give a relativistic version of the Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{c^2}\right)\psi = \frac{m^2 c^2}{h^2}\psi.$$

Dirac replaced $\nabla^2 - \frac{1}{c^2}$ by $\left(A\partial_x + B\partial_y + C\partial_x + \frac{i}{c}D\partial_t\right)^2$. The requirements are

$$A^2 = B^2 = C^2 = D^2 = 1,$$
$$AB + BA = 0 \ldots$$

This leads to the Clifford algebra.

Dirac operator from Physics II

Similarly, the relativistic version of Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2m}\Delta\Psi \qquad h := 1$$

requires a square root of Δ. This also leads to the Dirac equation.

("Anonymous Quote")

In particle physics, the Dirac equation is a relativistic wave equation derived by British physicist Paul Dirac in 1928. In its free form, or including electromagnetic interactions, it describes all $spin - 1/2$ massive particles such as electrons and quarks for which parity is a symmetry. It is consistent with both the principles of quantum mechanics and the theory of special relativity, [D], and was the first theory to account fully for special relativity in the context of quantum mechanics. It was validated by accounting for

Dirac operator from Physics III

the fine details of the hydrogen spectrum in a completely rigorous way.
The equation also implied the existence of a new form of matter, antimatter, previously unsuspected and unobserved and which was experimentally confirmed several years later. It also provided a theoretical justification for the introduction of several component wave functions in Pauli's phenomenological theory of spin.

There are many sources, particularly courses in Physics departments.

Background I

If a Lie group G acts on a manifold X, then it also induces a representation on functions on X, via

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

Typically there is a G-invariant measure dx on X.
For example:

$$C^\infty(X) \text{ is a smooth representation of } G$$

$$L^2(X) \text{ is a unitary representation of } G$$

A representation of G is a complex topological vector space V, typically complete, with a continuous G-action by linear operators.
Harmonic analysis: *"decompose such representations into irreducible representations."*

Irreducible Representations: those with no closed invariant subspace.

Background II

Example: $G = \mathbb{T}$, the circle group. The irreducible modules are 1-dimensional, spanned by functions $f_n : e^{it} \mapsto e^{int}$ on $\mathbb{T}$, $n \in \mathbb{Z}$, and

$$L^2(\mathbb{T}) = \widehat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}\, f_n$$

(Fourier series).

Similarly, for $G = \mathbb{R}$, the irreducible unitary representations are 1-dimensional, spanned by the functions $f_n : t \mapsto e^{ixt}$ on $\mathbb{R}$, $x \in \mathbb{R}$, and

$$L^2(\mathbb{R}) = \int_{x\in\mathbb{R}}^{\oplus} \mathbb{C}\, f_x \, d\chi$$

(Fourier transformation).

Connection with differential equations

Let Δ be a $G-$invariant differential operator on X.
Then any eigenspace of Δ is $G-$invariant.

Conversely, (by some version of Schur's Lemma) Δ acts by scalars on irreducible G-subspaces.

So in the presence of such an operator, decomposing the representation is related to finding Δ-eigenspaces.

The representation of G gives extra structure to the eigenspace.

Real reductive groups

G: a real reductive Lie group (often assumed connected).

Main examples: closed (Lie) subgroups of $GL(n, \mathbb{C})$, stable under the Cartan involution $\Theta(g) = {}^t\bar{g}^{-1}$.

E.g., $SL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, $O(p, q)$.

$K = G^\Theta$: maximal compact subgroup

E.g., $SO(n) \subset SL(n, \mathbb{R})$; $U(p) \times U(q) \subset U(p, q)$;
$U(n) \subset Sp(2n, \mathbb{R})$, $O(p) \times O(q) \subset O(p, q)$.

The newer results are for the exceptional groups.

Cohomology of Discrete Groups I

- G the real points of a linear algebraic reductive connected group.
- $\mathfrak{g}_0 := Lie(G)$, θ a Cartan involution, $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}}$, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, K the maximal compact subgroup, $\mathfrak{k}_0 := Lie(K)$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$.
- A $(\mathfrak{g}, K)$ module $(\pi, \mathcal{H})$ is called unitary, if $\mathcal{H}$ admits a $\mathfrak{g}-$invariant positive hermitian form.
- $\Gamma \subset G$ a discrete cocompact subgroup. The theory of automorphic forms deals with the decomposition $L^2(\Gamma \backslash G) = \bigoplus m_\pi \pi$. Let $X := \Gamma \backslash G / K$. Then

$$H^i(\Gamma) = H^i(X) = \bigoplus m_\pi H^i(\mathfrak{g}, K, \pi).$$

Cohomology of Discrete Groups II

The multiplicities m_π are very hard to compute. In order to get information about Γ, one approach is to study $H^i(\mathfrak{g}, K, \pi)$ for π unitary.

Problem: Classify all unitary representations with nontrivial $(\mathfrak{g}, K)-$cohomology.

For complex groups, this was solved by Enright, and then generalized to real groups by Vogan-Zuckerman.

In the real case the answer is that $\pi = \mathcal{R}^s_{\mathfrak{q}}(\mathbb{C}_\lambda)$, where

- $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a $\theta-$stable parabolic subalgebra, $s = \dim \mathfrak{u} \cap \mathfrak{s}$,
- $\mathcal{R}^i_{\mathfrak{q}}$ is cohomological induction introduced by Parthasarathy and Zuckerman.
- $\mathbb{C}_\lambda$ is a unitary character such that $\mathcal{R}^s_{\mathfrak{q}}(\mathbb{C}_\lambda)$ has infinitesimal character the same as the trivial representation.

Cohomology of Discrete Groups III

$H^i(\mathfrak{g}, K, \pi) = \mathrm{Hom}_K[\wedge^i \mathfrak{s}, \pi]$ is computable explicitly for such modules.
Dirac cohomology is a generalization of $(\mathfrak{g}, K)-$cohomology. The underlying reason is that (essentially) $Spin \otimes Spin = \wedge^* \mathfrak{s}$.

While the calculations proceed case-by-case, one of the goals is to produce a proof of the B-Pandžić conjecture in a uniform way. So far ...

From High Hopes, Sinatra

Just what makes that little old ant
Think he'll move that rubber tree plant
Anyone knows an ant, can't
Move a rubber tree plant
But he's got high hopes, he's got high hopes . . .

Just remember that ant
Oops, there goes another rubber tree plant . . .

The Dirac and Casimir operators for G

Let b_i be any basis of $\mathfrak{p}$; let d_i be the dual basis with respect to B.
Dirac operator:

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

D is independent of b_i and K-invariant.
The Casimir operator, $\mathrm{Cas}_{\mathfrak{g}}$ is an element in the center of the enveloping algebra $U(\mathfrak{g})$:
Take dual bases e_i, f_i of $\mathfrak{g}$ with respect to B.

Write

$$\mathrm{Cas}_{\mathfrak{g}} = \sum e_i f_i.$$

D^2 is the spin Laplacian (Parthasarathy):

$$D^2 = -\operatorname{Cas}_{\mathfrak{g}} \otimes 1 + \operatorname{Cas}_{\mathfrak{k}_\Delta} + \text{constant}.$$

Here $\operatorname{Cas}_{\mathfrak{g}}$, $\operatorname{Cas}_{\mathfrak{k}_\Delta}$ are the Casimir elements of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$;

$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \quad \text{and} \quad \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p}).$$

The constant is explicitly computed as $-||\rho||^2 + ||\rho_{\mathfrak{k}}||^2$.

Atiyah-Schmid, Schmid, and Parthasarthy use these notions to construct Discrete Series.

$(\mathfrak{g}, K)$-modules

It is always easier to study representations of the Lie algebra, and then derive properties of the representations of the Lie group.

For real reductive groups, these are the $(\mathfrak{g}, K)-$modules.

Following Harish-Chandra, one associates a $(\mathfrak{g}, K)-$module to each representation of the group. Let V be an admissible representation V of G, i.e., $\dim \mathrm{Hom}(V_\delta, V) < \infty$ for all irreducible K-representations V_δ.

Let V_K be the space of K-finite vectors in V. These vectors are smooth i.e. one can differentiate the group action to get an action of the Lie algebra. $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$, the complexification of the real Lie algebra acts automatically.

Definition

A $(\mathfrak{g}, K)-$module is a vector space V, with a Lie algebra action of $\mathfrak{g}$ and a locally finite action of K, which are compatible, i.e., induce the same action of $\mathfrak{k}_0 := Lie(K)$. (If K is disconnected, one requires also that the action $\mathfrak{g} \otimes V \to V$ is $K-$equivariant). Such a V can be decomposed under K as

$$V = \bigoplus_{\delta \in \hat{K}} m_\delta V_\delta.$$

V is called a Harish-Chandra module if it is finitely generated and all $m_\delta < \infty$.

Casimir element

The Casimir Element, $\mathrm{Cas}_{\mathfrak{g}}$, in the center of the enveloping algebra $U(\mathfrak{g})$ is defined as follows:

Fix a nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$ (e.g. $\mathrm{tr}\, XY$ for $gl(n)$).

Take dual bases b_i, d_i of $\mathfrak{g}$ with respect to B.

Write

$$\mathrm{Cas}_{\mathfrak{g}} = \sum b_i d_i.$$

Infinitesimal character

The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a polynomial algebra; one of the generators is $\mathrm{Cas}_{\mathfrak{g}}$.

All elements of $Z(\mathfrak{g})$ act as scalars on irreducible modules.

This defines the infinitesimal character of a module M, $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$.

Harish-Chandra proved that $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)^W$, so infinitesimal characters correspond to $\mathfrak{h}^*/W$.

($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$; in examples, the diagonal matrices. W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$; it is a finite reflection group.)

The Clifford algebra for G

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

($\mathfrak{k}$ and $\mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution;

$\mathfrak{k}$ is the complexified Lie algebra of the maximal compact subgroup K of G.)

Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B:

the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx + 2B(x, y) = 0.$$

The Dirac operator for G

Let b_i be any basis of $\mathfrak{p}$; let d_i be the dual basis with respect to B.
Dirac operator:

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

D is independent of b_i and K-invariant.

D^2 is the spin Laplacian:

$$D^2 = -\operatorname{Cas}_{\mathfrak{g}} \otimes 1 + \operatorname{Cas}_{\mathfrak{k}_\Delta} + \text{constant}.$$

Here $\operatorname{Cas}_{\mathfrak{g}}$, $\operatorname{Cas}_{\mathfrak{k}_\Delta}$ are the Casimir elements of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$;

$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \quad \text{and} \quad \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p}).$$

The constant is explicitly computed as $-||\rho||^2 + ||\rho_{\mathfrak{k}}||^2$.

The formula for D^2 was first establieshed by Parthasarathy and Schmid.
Atiyah-Schmid, Schmid, and Parthasarthy use these ideas to construct Discrete Series.

Dirac cohomology

Motivated by the Dirac inequality (see below) and its uses to compute spectral gaps, Vogan introduced the notion of Dirac Cohomology.

Let M be an admissible $(\mathfrak{g}, K)$-module. Let S be a Spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ maximal isotropic.

Then D acts on $M \otimes S$.

Dirac cohomology of M:

$$H_D(M) = \operatorname{Ker} D / (\operatorname{Im} D \cap \operatorname{Ker} D)$$

$H_D(M)$ is a module for the spin double cover $\widetilde{K}$ of K. It is finite-dimensional if M is of finite length.

If M is unitary, then D is self adjoint *w.r.t.* an inner product. So

$$H_D(M) = \operatorname{Ker} D = \operatorname{Ker} D^2,$$

and $D^2 \geq 0$ (Dirac inequality).

Vogan's Conjecture

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. View $\mathfrak{t}^* \subset \mathfrak{h}^*$ via extension by 0 over $\mathfrak{a}$.

The following was conjectured by Vogan in 1997, and proved by Huang-Pandžić in 2002.

Theorem

Assume M has an infinitesimal character, and $H_D(M)$ contains a $\widetilde{K}$-type E_τ of highest weight $\tau \in \mathfrak{t}^*$. Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be a fundamental $\theta-$stable Cartan subalgebra. The infinitesimal character is a $W-$orbit of a semisimple element $\lambda \in \mathfrak{h}^*$.

Then there is $w \in W$ such that $w\lambda \mid_{\mathfrak{t}} = \tau + \rho_{\mathfrak{k}}$, and $w\lambda \mid_{\mathfrak{a}} = 0$.

Motivation

- unitarity: Dirac inequality and its improvements.
- irreducible unitary M with $H_D \neq 0$ are interesting (discrete series, $A_{\mathfrak{q}}(\lambda)$ modules, unitary highest weight modules, some unipotent representations...) They should form a nice part of the unitary dual.
- H_D is related to classical topics like generalized Weyl character formula, generalized Bott-Borel-Weil Theorem, construction of discrete series, multiplicities of automorphic forms
- There are nice constructions of representations with $H_D \neq 0$; e.g., Parthasarthy and Atiyah-Schmid constructed the discrete series representations using spin bundles on G/K.

Complex Groups

Let G be a complex reductive group viewed as a real group. Let K be a maximal compact subgroup of G. Let Θ be the corresponding Cartan involution, and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G. Let $H = TA$ be a $\theta-$stable Cartan subgroup of G, with Lie algebra $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$, a $\theta-$stable Cartan subalgebra of $\mathfrak{g}_0$. We assume that $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{s}_0$.

Let $B = HN$ be a Borel subgroup of G. Let $(\lambda_L, \lambda_R) \in \mathfrak{h}_0 \times \mathfrak{h}_0$ be such that $\mu := \lambda_L + \lambda_R$ is integral. Write $\nu := \lambda_L - \lambda_R$. We can view μ as a weight of T and ν as a character of A. Let

$$X(\lambda_L, \lambda_R) := \mathrm{Ind}_B^G[\mathbb{C}_\mu \otimes \mathbb{C}_\nu \otimes \mathbb{1}]_{K-finite}.$$

Then the $K-$type with extremal weight μ occurs in $X(\lambda_L, \lambda_R)$ with multiplicity 1. Let $L(\lambda_L, \lambda_R)$ be the unique irreducible subquotient containing this $K-$type.

Admissible Representations

Theorem ([Zh], [PRV])

1. *Every irreducible admissible* $(\mathfrak{g}, K)$ *module is of the form* $L(\lambda_L, \lambda_R)$.
2. *Two such modules* $L(\lambda_L, \lambda_R)$ *and* $L(\lambda'_L, \lambda'_R)$ *are equivalent if and only if the parameters are conjugate by* $\Delta(W) \subset W_c \cong W \times W$. *In other words, there is* $w \in W$ *such that* $w\mu = \mu'$ *and* $w\nu = \nu'$.
3. $L(\lambda_L, \lambda_R)$ *admits a nondegenerate hermitian form if and only if there is* $w \in W$ *such that* $w\mu = \mu$, $w\nu = -\overline{\nu}$.

This result is a special case of the more general Langlands classification, which can be found for example in [Kn], Theorem 8.54.

Spin Representation I

We next describe the spin representation of the group $\widetilde{K}$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{b},\mathfrak{h})}\alpha$. Let r denote the rank of $\mathfrak{g}$.

Lemma

The spinor representation Spin viewed as a $\widetilde{K}$-module is a direct sum of $2^{[\frac{r}{2}]}$ copies of the irreducible representation $E(\rho)$ of $\widetilde{K}$ with highest weight ρ.

Lemma 4 implies that in calculating $H_D(\pi)$ for unitary π, one can replace *Spin* by $E(\rho)$ and then in the end simply multiply the result by the multiplicity $2^{[\frac{r}{2}]}$.

So a unitary representation $L(\lambda_L, \lambda_R)$ has nonzero Dirac cohomology if and only if there is $(w_1, w_2) \in W \times W$ such that

$$w_1\lambda_L - w_2\lambda_R = 0, \qquad w_1\lambda_L + w_2\lambda_R = \tau + \rho. \tag{1}$$

We get $w_1\lambda = w_2\lambda$, and $2w_1\lambda = 2w_2\lambda = \tau + \rho$. Since τ is the highest weight of a $\widetilde{K}-$type (which occurs in $L(\lambda_L, \lambda_R) \otimes E(\rho)$),

Spin Representation II

we conclude that 2λ is regular integral. Thus $w_1 = w_2$ as well. We write the representation as $L(\lambda, s\lambda)$. with $s \in W$. More precisely

$$[H_D(\pi) : E(\tau)] = 2^{[\frac{r}{2}]} \sum_{\mu} \, [\pi : E(\mu)] \, [E(\mu) \otimes E(\rho) : E(\tau)], \quad (2)$$

where the sum is over all K-types $E(\mu)$ of π.

Dirac Cohomology for Unitary Representations

Since $L(\lambda, s\lambda)$ is assumed unitary, it is hermitian. So there is $w \in W$ such that

$$w(\lambda + s\lambda) = \lambda + s\lambda, \quad w(\lambda - s\lambda) = -\lambda + s\lambda. \tag{3}$$

This implies that $w\lambda = s\lambda$, so $w = s$ since λ is regular, and $ws\lambda = s^2\lambda = \lambda$. So s must be an involution.
Thus to compute $H_D(\pi)$ for π that are unitary, we need to know

1. $L(\lambda, s\lambda)$ that are unitary with

$$2\lambda = \tau + \rho, \tag{4}$$

   in particular 2λ is regular integral,
2. The multiplicity

$$\left[\ L(\lambda, s\lambda) \otimes E(\rho)\ :\ E(\tau)\ \right]. \tag{5}$$

Unitary Dual

Conjecture

A representation $L(\lambda, s\lambda)$ is unitary if and only if it is unitarily induced from a unipotent representation on a Levi subgroup. (assuming 2λ regular integral).

Theorem (Classical Groups, [B])

A hermitian module with infinitesimal character (λ, λ) with 2λ integral is unitary if and only if it is unitarily induced from a unipotent representation. For the classical groups, (aside from the trivial representation) they are

- *Type A* $\lambda = (a, \dots, -a, b - 1/2, \dots, -b + 1/2)$, $a, b \in \mathbb{N}$,
- *Type B* $\Theta-$*lifts of the trivial representation of an Sp in the stable range,*
  $\lambda = (-K_0 + 1/2, \dots, -1/2, -N_0, \dots, -1)$ $K_0 \geq N_0$
- *Type C* *The components of the metaplectic representation,*
  $\lambda = (-K_0 + 1/2, \dots, -1/2)$,
- *Type D* $\Theta-$*lifts of the metaplectic representation,*

Remarks

- $(Sp(2n,\mathbb{C}), O(m,\mathbb{C}))$ are dual pairs; $Sp(2n,\mathbb{C})$ is simply connected.
- [BP] computes some of the Dirac cohomology of the representations; the unipotent cases, and some of the unitarily induced modules. An elegant complete calculation is in [BDW].
- [BDW] completes the classification of unitary modules with nontrivial Dirac cohomology for the classical cases (including the Spin groups).
- The relevant unipotent representations for all complex groups are listed in [BP]. The next slide gives E_6, E_7 and E_8.

Type E6

We use the Bourbaki realization for the root system. There are two integral systems, A_5A_1 which gives the nilpotent $3A_1$, and D_5T_1 which gives $2A_1$. The parameters are

$$\begin{aligned} \lambda &= (-5/2, -3/2, -1/2, 1/2, 3/4, -3/4, -3/4, 3/4) \longleftrightarrow 3A_1 \\ \lambda &= (-9/4, -5/4, -1/4, 3/4, 7/4, -7/4, -7/4, 7/4) \longleftrightarrow 2A_1. \end{aligned} \tag{6}$$

The representations are factors in $\mathrm{Ind}_{A_5}^{E_6}[\mathbb{C}_\nu]$. The parameter is

$$\begin{aligned} &(-11/4, -7/4, -3/4, 1/4, 5/4, -5/4, -5/4, 5/4)+ \\ &+\nu(1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2). \end{aligned} \tag{7}$$

The two points above are $\nu = 1/2$ and $\nu = 1$. The representations are unitary because the induced module has multiplicity 1 $K-$structure.

Type E7 I

We use the Bourbaki realization. There are three integral systems, $D_6 A_1$ which gives the nilpotent $(3A_1)'$, $E_6 T_1$ which gives $2A_1$, and A_7 which gives $4A_1$. The parameters for the first two are

$$\begin{aligned} \lambda &= (0,1,2,3,4,5,-1,1) \longleftrightarrow (3A_1)' \\ \lambda &= (0,1,2,3,4,-7/2,-17/4,17/4) \longleftrightarrow 2A_1. \end{aligned} \tag{8}$$

The first representation is a factor in $\mathrm{Ind}_{D_6}^{E_6}[\mathbb{C}_\nu]$. The parameter is

$$(0,1,2,3,4,5,0,0) + \nu(0,0,0,0,0,0,-1,1). \tag{9}$$

The point above is $\nu = 1$, an end point of a complementary series. In any case the representation is multiplicity free, so the representation is unitary. The second representation is a factor in $\mathrm{Ind}_{E_6}^{E_7}[\mathbb{C}_\nu]$. The parameter is

$$(0,1,2,3,4-4,-4,4) + \nu(0,0,0,0,0,1,-1/2,1/2) \tag{10}$$

Type E7 II

with $\nu = 1/2$. The representation is unitary because it is at an end point complementary series; also the induced module is multiplicity free.

The third representation has parameter

$$(-9/4, -5/4, -1/4, 3/4, 7/4, 11/4, -4, 4). \tag{11}$$

This is the minimal length parameter which gives the integral system A_7. By [AHV], the $K-$structure is multiplicity free and a full lattice in E_7. It does not occur in a multiplicity free induced module, and is not an end point of a complementary series. I know of no "elementary" proof.

Type E8

Same here, the Bourbaki realization. There are two integral systems, D_8 which gives the nilpotent $4A_1$, and E_7A_1 which gives $3A_1$. The parameters are

$$\begin{aligned} \lambda &= (0, 1, 2, 3, 4, 5, 6, 8) \longleftrightarrow 4A_1 \\ \lambda &= (0, 1, 2, 3, 4, 5, -8, 9) \longleftrightarrow 3A_1. \end{aligned} \tag{12}$$

These are the minimal length parameters which gives the integral systems D_8 and E_7A_1. Again by [AHV], the $K-$structure of the first one is multiplicity free and a full lattice in E_8. It does not occur in a multiplicity free induced module, and is not an end point of a complementary series. The second one is also multiplicity free, and occurs at an endpoint of a complementary series. It is conjectured that all these representations are unitary.

Most Recent Results, complex E_8. I

See [BW]. The strategy applies to all cases of complex groups.

- Realize the representation in the [PRV], equivalently [Zh] classification. The Levi component is formed of Type A factors, and possibly one more of the same Type as $\mathfrak{g}$.
- In all cases, starting with [V1] and [B], there is a small set of $K-$types that detect the lack of unitarity (subsequently called certificates of unitarity). In [B] these $K-$types are called *level less than or equal to 4*.
- Such $K-$types are considered for all the split real and complex groups in [BC] called *petite* ("single petaled" by Oda). The complex case requires *level* ≤ 4. They are fundamental representations with label $\leq 1, 2$ and tensor products of two fundamental representations with label ≤ 2. The calculations rely heavily on knowledge of intertwining operators.

Most Recent Results, complex E_8. II

- A fundamental notion that allows one to conclude that the signature on a $K-$type coincides with one on a Levi component is *"bottom layers"*. This allows one to rule out the bulk of the representations not satisfying the conjecture.
- The condition 2λ regular integral rules out practically all remaining ones, except for a very few cases, which turn out to be unitary, and satisfy the conjecture.

This gives an outline of the proof of the conjecture in [BP]

THANK YOU FOR LISTENING SO FAR

and to the organizers for their work to make this meeting a success,

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