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RELEVANT AND PETITE K-TYPES FOR SPLIT GROUPS

DAN BARBASCH

1. Introduction

This paper discusses the role of the relevant and petite K-types in the classification of the spherical unitary dual for split real groups.

Let G be the rational points of a connected linear reductive group over a local field of characteristic zero, more precisely $\mathbb{F} = \mathbb{R}$ or a p-adic field. Assume G is split, and denote a maximal abelian torus by $\mathbb{A} \cong (\mathbb{F}^{\times})^n$. Let K be a maximal compact subgroup as in section 2. A representation is called spherical if it has nontrivial K-fixed vectors. Any irreducible spherical representation is realized in a canonical way as a subquotient, denoted $L_{\mathbb{F}}(\chi)$, of a principal series $X_{\mathbb{F}}(\chi)$ for a character $\chi \in \widehat{\mathbb{A}}$. Two such representations $L_{\mathbb{F}}(\chi)$ and $L_{\mathbb{F}}(\chi')$ are equivalent if and only if there is an element w of the Weyl group W such that $w\chi = \chi'$. The module $L_{\mathbb{F}}(\chi)$ admits a nondegenerate hermitian form if and only if there is $w \in W$ such that $w\chi = \overline{\chi}^{-1}$. The hermitian form is obtained from a hermitian form on $X_{\mathbb{F}}(\chi)$; an $L_{\mathbb{F}}(\chi)$ is unitary if and only if the form on $X_{\mathbb{F}}(\chi)$ is positive semidefinite. For each K-type (E, μ), there is a hermitian form $A_{\mathbb{F}}(\mu, \chi)$ which depends meromorphically on χ. Thus $L_{\mathbb{F}}(\chi)$ is unitary if and only if $A_{\mathbb{F}}(\mu, \chi)$ is positive semidefinite for all μ.

We will use subscript R for the real case, and F for the p-adic case. When the context is clear, we will drop the subscript altogether.

In the p-adic case, the results in [BM1] and [BM2] reduce the problem to a finite one, namely to show that certain related forms $a_F(\sigma, \chi)$ obtained from the $A_F(\mu, \chi)$ are positive semidefinite for all $\sigma \in \widehat{W}$. These ideas were used in [BM3] to obtain the spherical unitary dual for split p-adic groups of classical type. The full unitary dual for $GL(n)$ was obtained by Tadić, and for G_2 by Muić by different methods.

Consider now the real case. Let M be the centralizer of $\mathbb{A}$ in K. Then $(E_{\mu}^{*})^{M}$ is a representation of the Weyl group, which we denote by σ. The operator $A_R(\mu, \chi)$ induces, by Frobenius reciprocity, a hermitian form $a_R(\mu, \chi)$ on $(E_{\mu}^{*})^{M}$. In sections 3 and 4 we define a set of K-types, which we call *petite*. They have the property that $a_R(\mu, \chi)$ only depends on σ, and in fact $a_R(\mu, \chi) = a_F(\sigma, \chi)$, wher the latter is defined in the context of p-adic groups as mentioned in the previous paragraph.

In [B5], for the classical cases, I introduced a subset of $\widehat{W}$ called **relevant**, which has the following properties:

(a) A representation $L_F(\chi)$ is unitary if and only if $a_F(\sigma, \chi)$ is positive semidefinite for all σ relevant.
(b) For every relevant σ there is a petite representation E_μ of the real group such that $(E_\mu^*)^M \cong \sigma$.

We will call the K-types in part (b) relevant as well. The existence of such a set of W-representations, implies the following result.

Theorem (1)**.** *The representation $L_R(\chi)$ is unitary **only if** $L_F(\chi)$ is unitary.*

For split p-adic F_4, the spherical unitary dual (in fact the full Iwahori spherical unitary dual) is obtained in [C]. For p-adic split type E_6, E_7, E_8, the spherical unitary dual is obtained in [BC]. In all these cases, a set satisfying part (a) is given. In this paper we show that these W-representations also satisfy part (b). As a consequence, theorem (1) holds in all cases.

It is natural to conjecture that the **only if** can be strengthened to **if and only if**, in the theorem. This is done for the classical groups in [B5].

The proof in this paper, of the matchup between petite K-types and relevant W-types is conceptually different from the one in [B5] for the classical groups. Instead of identifying the groups M and W explicitly and using restrictions to Levi components, I use properties of fine K-types, and tensor products. To decompose tensor products in the exceptional cases, I used the packages *GAP* and *LiE*. The results for $E_6 - E_8$ contain matchups for a larger class of petite K-types than needed for theorem (1). The definitions and background are in sections 2 and 3. The results are in section 4, and the calculations are in section 5.

Let us first consider the problem of restricting a K-type to M. The standard restriction formulas are designed to deal with restricting to a reductive subgroup $H \subset K$ which shares a Cartan subgroup with K. This is not the case for M. To get around this, one strategy would be to compute the restriction of the fundamental representations, and set up an induction by realizing K-types as composition factors of tensor products of fundamental representations. We can then exploit the fact that there are good formulas for decomposing tensor products.

It is certainly feasible to obtain the restrictions of fundamental representations to M. Combined with the induction sketched above, this gives an efficient way to restrict a K-type to M. However the restrictions of the fundamental representations to M are not very nice. On the other hand, the fine K-types (section 3) have very nice restrictions. Precisely, each fine K-type decomposes into the Weyl orbit of a single M-type, and every M-type occurs in at least one fine K-type. So the general aim is to replace the fundamental representations by the fine K-types. This works very well in the case of $GL(n, \mathbb{R})$ and $SO(n, n)$, $Sp(n, \mathbb{R})$ where all the fundamental representations are fine. To carry out the inductive strategy, it turns out that it is very useful to consider fine representations of the covers $\widetilde{K}$ and $\widetilde{M}$, arising in the context of nonlinear covers of the real group G.

For $SO(n+1, n)$, G_2, F_4, and the type E groups, the fine K-types do not generate the Grothendieck group of representations of K. But since the aim is only to determine the M-fixed vectors, a number of simplifications are available.

For $SO(n+1, n)$ we establish a weaker form of part (b) which is sufficient for theorem (1). This is done by decomposing a tensor product of fine K-types of a cover of $SO(n+1, n)$. In this case it is not difficult to restrict any fundamental representation of K to M (this is implicit in [B5]). But we need to use more explicit knowledge of the structure of M.

For the real group of type G_2, we use the representation of K on $\mathfrak{s}$, from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. Its restriction to M is easy to ascertain. In this case the full unitary dual is contained in the work of [V1]. So the *if and only if* analogue of theorem (1) is known to hold for spherical representations.

In the case of F_4, we also use the representation of $\mathfrak{s}$. We compute the restrictions to M of a larger number of petite representations than is needede for theorem (1).

For type E, we again use $\mathfrak{s}$ in addition to the fine K-types. The calculations are simplified considerably in the cases of E_6 and E_8 by the fact that $\widetilde{M}$ has a unique genuine representation. This fact implies that a genuine representation of $\widetilde{K}$ restricts to a multiple of the genuine $\widetilde{M}$-type, and the multiplicity is determined from the dimension.

To establish part (b), we also need to determine the Weyl group representation on the M-fixed vectors. This is done by using theorem 3.3 together with sections 5.1-5.2.

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2. Notation and background

2.1. Let $\mathbb{F}$ be a local field, either $\mathbb{R}$ or a p-adic field. In the latter case, let $\mathcal{R} \supset \mathcal{P}$ be the ring of integers and its maximal prime ideal respectively. Let $G := \mathbf{G}(\mathbb{F})$ be the rational points of a connected reductive group, $\mathbb{F}$ real or p-adic. Assume G is split, and fix a rational Borel subgroup $B = AN$. In the real case, fix a Cartan involution θ, and let K be the maximal compact group which is its fixed points. In the p-adic case, let $K := \mathbf{G}(\mathcal{R})$. A character $\chi \in \widehat{A}$ is called *unramified*, if $\chi\big|_{A\cap K}$ is trivial. Since G is split, $A \cong (\mathbb{F}^\times)^n$ where n is the rank of G. Thus an unramified character is of the form

$$\chi = (\chi_1, \dots, \chi_n) \tag{2.1.1}$$

where $\chi_i(t) = |t|^{\nu_i}$. Let $\mathcal{L} := (0)$ if $\mathbb{F} = \mathbb{R}$, $\mathcal{L} := (\frac{2i\pi}{\log q}\mathbb{Z})$ in the p-adic case. We can therefore identify χ with an element in $(\mathbb{C}/\mathcal{L})^n$. As is well known, any irreducible spherical admissible representation of G is the unique spherical subquotient of a *standard* induced module

$$X_B(\chi) := Ind_B^G(\chi \otimes \delta_B^{-1/2}), \tag{2.1.2}$$

where χ is an unramified character, and δ_B is the modulus function of B. Denote this subquotient by $L(\chi)$. Two such quotients $L(\chi)$ and $L(\chi')$ are isomorphic if and only if there is an element $w \in W := N_K(A)/C_K(A)$ such that $w\chi = \chi'$.

2.2. Suppose $Im\chi \neq 0$. Then $Im\chi$ defines a proper standard parabolic subgroup $P = MU$, where

$$\Delta(M) = \{\alpha \ : \ (\alpha, Im\chi) = 0\}, \qquad \Delta(U) = \{\alpha \ : \ (\alpha, Im\chi) > 0\}. \tag{2.2.1}$$

Let $L_M(\chi)$ be the spherical irreducible module of M corresponding to χ. Then $L_M(\chi) = L_M(\chi_r) \otimes \chi_i$ where χ_r is such that $Im\chi_r = 0$ and χ_i is unitary.

Theorem.

$$L(\chi) = Ind_M^G[L_M(\chi_r) \otimes \chi_i],$$

and $L(\chi)$ is unitary if and only if $L_M(\chi_r)$ is unitary.

Proof. For the real case, this is theorem 16.10 in [K]. For the p-adic case and I-spherical representations, it follows from [BM1] and [BM2]. □

Because of this theorem, we assume that χ is **real**, *i.e.* $Im\chi = 0$. In this situation, the lattice $\mathcal{L}$ plays no role, and we can treat the real and p-adic case in the same way.

2.3. Let $(\chi, \alpha) \geq 0$ for all roots $\alpha \in \Delta(\mathfrak{n}, \mathfrak{a})$. then $X(\chi)$ has a unique irreducible quotient, namely $L(\chi)$. We call such a χ *dominant.* If on the other hand $(Re\chi, \alpha) \leq 0$ for all $\alpha \in \Delta(\mathfrak{n}, \mathfrak{a})$, then $X(\nu)$ has a unique irreducible submodule which is again $L(\chi)$. We call such a χ *antidominant.*

Let $B' = AN'$ be another Borel subgroup. Then there is an intertwining operator

$$A(B, B', \chi) : X_B(\chi) \longrightarrow X_{B'}(\chi) \tag{2.3.1}$$

given by

$$A(B, B', \chi)f(g) := \int_{N'/N\cap N'} f(gn')\, dn'. \tag{2.3.2}$$

The integral is convergent for χ very dominant, and has a meromorphic extension to all χ. It has no poles if χ is dominant. Let $w \in W$, and write $B' := w(B)$. There is an isomorphism $R_w : X_{B'}(\chi) \longrightarrow X_B(w\chi)$ given by

$$R_w f(g) := f(gw^{-1}). \tag{2.3.3}$$

We write

$$A(w, \chi) := R_w \circ A(B, B', \chi) : \; X_B(\chi) \longrightarrow X_B(w\chi). \tag{2.3.4}$$

Recall that $G = KB$, and let (E, μ) be a K-type. Then the intertwining operator (2.3.1) gives rise to a map

$$A(\mu, w, \chi) : \mathrm{Hom}_K[E, X_B(\chi)] \longrightarrow \mathrm{Hom}_K[E, X_B(w\chi)]. \tag{2.3.5}$$

By Frobenius reciprocity, we get a map

$$a(\mu, w, \chi) : (E^*)^M \longrightarrow (E^*)^M. \tag{2.3.6}$$

We normalize the intertwining operator (2.3.1) so that $a(triv, w, \chi) = 1$ by multiplying it with the appropriate meromorphic function in χ.

The hermitian dual of $X_B(\chi)$ is $X_B(\chi^h)$, where $\chi^h := (\overline{\chi})^{-1}$ (χ not necessarily rea). Indeed, if $v \in X_B(\chi)$ and $w \in X_B(\chi^h)$, then $v\overline{w}$ transforms according to δ_B under the right action of B. Thus the integral of $v\overline{w}$ over G/B makes sense, and the hermitian pairing is

$$\big(v, w\big) := \int_{G/B} v(g)\overline{w(g)}\, dg. \tag{2.3.7}$$

Thus the hermitian dual of $L(\chi)$ is $L(\chi^h)$, and $L(\chi)$ is hermitian if and only if there is $w \in W$ such that $w\chi = \chi^h$. Suppose this is the case and $B' := w(B)$. If χ is dominant, then χ^h is antidominant. So the image of $A(w, \chi)$ is exactly $L(\chi)$. The hermitian form is given by

$$\langle v_1, v_2\rangle := \big(v_1, A(w, \chi)(v_2)\big), \tag{2.3.8}$$

where $\big(\;,\;\big)$ is the pairing in (2.3.7). Fix a positive definite K-invariant hermitian form for each K-type (E, μ). Then as in (2.3.3)-(2.3.4), we get hermitian symmetric maps $a(\mu, \chi)$. Then $L(\chi)$ is unitary if and only if $a(\mu, \chi)$ is positive semidefinite for all μ.

3. Fine K-types

3.1. A reference for the results in this section is chapter 4 in [V3].

Let (μ_a, V_a) and (μ_b, V_b) be representations of K. Then $\mathrm{Hom}_M[V_a, V_b]$ is endowed with a representation of $N_K(M)$ via

$$n \cdot f(v) := \mu_a(n) f(\mu_b(n^{-1})v).$$

Under this action, $M \subset N_K(M)$ acts trivially, so we get a representation of W. Note that

$$\mathrm{Hom}_M[V_a, V_b] \cong \mathrm{Hom}_M[V_a \otimes V_b^*, Triv].$$

In particular, if $V_a = E_\mu$, and $V_b = Triv$, we get the more familiar action of W on $(E_\mu^*)^M$.

3.2. Let α be a simple root and $P_\alpha = M_\alpha N$ be the standard parabolic subgroup so that M_α is θ-stable, and the Lie algebra of M_α is isomorphic to the $sl(2,\mathbb{R})$ generated by the root vectors $E_{\pm\alpha}$.

The map $\psi_\alpha : sl(2,\mathbb{R}) \longrightarrow \mathfrak{g}$ determined by

$$\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_\alpha, \qquad \psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{-\alpha},$$

gives a map

$$\Psi_\alpha \ : \ SL(2,\mathbb{R}) \longrightarrow G \tag{3.2.1}$$

with image $G_\alpha \subset M_\alpha$, a connected group with Lie algebra isomorphic to $sl(2,\mathbb{R})$. We assume that $\theta E_\alpha = -E_{-\alpha}$. Let $D_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$, and $s_\alpha = e^{\sqrt{-1}\pi D_\alpha/2}$. Then $s_\alpha^2 = m_\alpha$ is in $M \cap M_\alpha$, while s_α itself gives rise to the corresponding nontrivial element in the Weyl group.

Definition. *A K-type is called **fine** (Vogan), if $\mu(iD_\alpha) = 0, \pm 1$.*

More generally, a K-type is said to have level less than or equal to r, if $|\mu(iD_\alpha)| \le r$.

The fine K-types are the lowest K-types of principal series. For every δ, there is a fine K-type μ_δ whose restriction to M contains δ. The restriction of every fine K-type to M is the sum over the W-orbit of a single $\delta \in \widehat{M}$.

Fix a representative δ for each W-orbit, and a fine K-type μ_δ. In the case of a linear group, M is abelian, so $\widehat{M}$ is formed of characters.

Then

$$\mu_\delta \otimes \mu_{\delta'}^* \text{ is formed of level} \le 2 \text{ K-types only.}$$

3.3. We will use the ideas in section 3.2 to determine the Weyl group representation on $\mu_\delta \otimes \mu_{\delta'}^*$. The following properties hold.

- ${}^\vee\Delta^\delta := \{\check{\alpha} \mid \delta(m_\alpha) = 1\}$ is a roots system.
- The Weyl group generated by the roots in ${}^\vee\Delta^\delta$ is a normal subgroup of W_δ.
- The quotient $R_\delta := W_\delta / W_\delta^0$ is is a product of $\mathbb{Z}_2$'s.
- $\widehat{R_\delta}$ acts simply transitively on the fine K-types containing δ.

Since R_δ is a quotient of W_δ, we can inflate any $\tau \in \widehat{R_\delta}$ to a representation of W_δ, trivial on W_δ^0. Having fixed a μ_δ, associate it to the trivial representation of W_δ, and set up a 1-1 correspondence

$$\{\tau \in \widehat{W_\delta} \mid \tau|_{W_\delta^0} = triv\} \longleftrightarrow \{\mu_{\delta,\tau}\}, \qquad triv \longleftrightarrow \mu_\delta.$$

Theorem. *As a W-module,*

$$\operatorname{Hom}_M[\mu_{\delta,1}, \mu_{\delta,\tau}] \cong Ind_{W_\delta}^W[\tau].$$

Proof. Recall that a fine K-type has the property that its restriction to M is multiplicity free, and forms a single orbit under the action of M. The proof follows from the properties listed before the theorem, and standard properties of representations of finite groups. □

4. Relevant and Petite K-types

4.1. Recall the notation from section 3.2. Since the square of any element in M is in the center and M normalizes the root vectors, $\operatorname{Ad} m(D_\alpha) = \pm D_\alpha$. Let (E, μ) be a K-type, and grade $E^* = \oplus E_i^*$ according to the absolute values of the eigenvalues of D_α (which are integers). Then M preserves this grading, and

$$(E^*)^M = \bigoplus_{i\ even} (E_i^*)^M.$$

Let A_α be the maps (2.3.4) for M_α. Recall that for $SL(2, \mathbb{R})$, K-types are parametrized by integers, and only K-types parametrized by even integers occur in the spherical principal series.

Proposition. *On* $(E_{2m}^* + E_{-2m}^*)^M$,

$$a(2m, s_\alpha, \nu) = \begin{cases} Id & \text{if } m = 0, \\ \prod_{0<j\leq m} \frac{2j-1-<\nu,\check{\alpha}>}{2j-1+<\nu,\check{\alpha}>}\ Id & \text{if } m \neq 0. \end{cases}$$

Proof. The formula is well known for $SL(2, \mathbb{R})$. The general assertion follows from this and the listed properties of intertwining operators. □

Let $w \in W$. Then any reduced decomposition $w = s_1 \cdots s_k$ gives rise to a factoring

$$A(w, \chi) = \prod A(s_i, s_{i+1} \cdot\ \dots\ \cdot s_k\chi). \tag{4.1.1}$$

The resulting operator $A(w, \chi)$ does not depend on the particular reduced decomposition of w, and each $A(s_i, s_{i+1} \cdot \dots \cdot s_k\chi)$ is induced from the corresponding A_{α_i}.

Corollary. *In particular,* $A(w, \chi)$ *is an isomorphism unless* $\langle \nu, \check{\alpha} \rangle \in \mathbb{N}$ *for some root* α.

Proof. This follows from the properties of the intertwining operator listed above. □

Definition. *A K-type μ is called petite if each A_{α_i} satisfies*

$$a(\mu, s_i, \nu) = \begin{cases} Id & \text{on the +1 eigenspace of } s_{\alpha_i}, \\ \frac{1-<\nu,\check{\alpha}>}{1+<\nu,\check{\alpha}>}\ Id & \text{on the -1 eigenspace of } s_{\alpha_i}. \end{cases}$$

In other words, the restriction of μ to $K_\alpha := M_\alpha \cap K$ consists of K-types with $2m = 0, \pm 2$ only. The hermitian symmetric matrix $a(\mu, \chi)$ depends only on the W-structure of $(E^*_\mu)^M$.

Theorem. *Level ≤ 3 K-types are petite.*

Proof. This follows from the above discussion. □

4.2. Affine Hecke algebra. We now show that the formulas in the previous section coincide with corresponding ones in the p-adic case. Recall from [BM3] that the induced module is $X(\chi) := \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\chi$ where $\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}$ is the graded affine Hecke algebra. The abelian subalgebra $\mathbb{A}$ is $S(\mathfrak{a})$ ($\mathfrak{a} = Lie(A)_c$), and $\mathbb{C}[W]$ is generated by $\{t_\alpha\}_{\alpha\ simple}$ satisfying $t_\alpha^2 = 1$. They are subject to the relations

$$\omega t_\alpha = s_\alpha(\omega) t_\alpha + c_\alpha < \omega, \alpha >, \qquad \omega \in \mathfrak{a}. \tag{4.2.1}$$

Because the group is assumed to be split, $c_\alpha = 1$. The intertwining operator $A(w, \nu)$ is a product of operators A_{α_i} according to a reduced decomposition of $w = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_k}$. If α is a simple root, let

$$r_\alpha := (t_\alpha \check{\alpha} - c_\alpha) \frac{1}{\check{\alpha} - c_\alpha}, \qquad A_\alpha \ : \ x \otimes \mathbb{1}_\nu \mapsto x r_\alpha \otimes \mathbb{1}_{s_\alpha \nu}. \tag{4.2.2}$$

The r_α are being multiplied on the right, so we can replace $\check{\alpha}$ with $\langle s_\alpha \chi, \check{\alpha} \rangle$ in the formulas. The $A(w, \chi)$ have the same properties as their counterparts in the real case.

Furthermore,

$$\mathbb{C}[W] = \sum_{\sigma \in \widehat{W}} V_\sigma \otimes V_\sigma^*.$$

Since r_α acts as multiplication on the right, it gives rise to an operator

$$r_\sigma(s_\alpha, \nu) : V_\sigma^* \longrightarrow V_\sigma^*.$$

Theorem. *Let (E, μ) be a petite K-type such that $V_\sigma^* \cong (E^*_\mu)^M$. Then $a(\mu, w, \chi)$ on $(E^*_\mu)^M$ coincide with $r_\sigma(w, \chi)$ on V_σ^*.*

Proof. The assertion is clear from definition (4.1) and formula (4.2.2). □

The algebra $\mathbb{H}$ has a $*$, coming from the corresponding p-adic group, given by

$$\omega^* = -\overline{\omega} + \sum_{\alpha \in \Delta(\mathfrak{n})} (\omega, \check{\alpha}) t_\alpha. \tag{4.2.3}$$

So it makes sense to talk about its unitary dual.

We remark that in the work of Lusztig and Kazhdan-Lusztig, the Hecke algebra is defined in terms of the dual root system so that there is no discrepancy between α and $\check{\alpha}$ in the formulas. For p-adic groups the determination of the Iwahori spherical unitary dual is equivalent to the determination of the unitary dual of $\mathbb{H}$. As explained in [BM1] and [BM2], an infinitesimal character is given by the conjugacy class of a semisimple element $s \in \check{G}$. In turn this element has a decomposition $s = s_e s_h$, where s_e is elliptic and s_h is hyperbolic. The Iwahori spherical dual consisting of representations with infinitesimal character s is in 1-1 correspondence with the unitary dual of an affine graded Hecke algebra $\mathbb{H}(s_e)$ with real infinitesimal character. A representation is called *spherical* if it contains the trivial W-type. The classification of spherical (and hermitian) representations with real infinitesimal character is the same as for the real and p-adic cases.

4.3. Relevant K-types. A set of K-types is called **relevant**, if the $(E^*)^M$ form a relevant set, *i.e.* they satisfy part (a) of the introduction. We list them for the simple root systems. For the classical groups they are already in [B5].

The representations of $W(A_{n-1}) = S_n$ are parametrized by partitions of n, written as $(a_1, \dots, a_k)$, $a_i \le a_{i+1}$. The representations of $W(B_n) \cong W(C_n)$ are parametrized as in [L] by pairs of partitions

$$(a_1, \dots, a_k) \times (b_1, \dots, b_l),$$
$$a_i \le a_{i+1}, \quad b_j \le b_{j+1}, \quad \sum a_i + \sum b_j = n. \tag{4.3.1}$$

Precisely, the representation σ parametrized by (4.3.1) is as follows. Let $k = \sum a_i$, $l = \sum b_j$. Recall that $W \cong S_n \rtimes \mathbb{Z}_2^n$. Let χ be the character of $\mathbb{Z}_2^n$ which is trivial on the first k $\mathbb{Z}_2$'s, sign on the rest. Its centralizer in S_n is $S_k \times S_l$. Let σ_1 and σ_2 be the representations of S_k, S_l corresponding to the partitions (a) and (b). Then σ, is

$$Ind_{(S_k \times S_l)\times \mathbb{Z}_2^n}^{W} \; [(\sigma_1 \otimes \sigma_2) \otimes \chi].$$

For $W(D_n)$, the representations are parametrized as in (4.3.1) except that $(a) \times (b)$ and $(b) \times (a)$ parametrize the same representation, and when $(a) = (b)$, there are two of them $(a) \times (a)_{I,\ II}$. This is because the restriction of $(a) \times (b)$ to $W(D_n)$ is irreducible when $(a) \neq (b)$ and equal to the restriction of $(b) \times (a)$, while the restriction of $(a) \times (a)$ consists of two nonisomorphic irreducible representations labelled $(a) \times (a)_{I,\ II}$. These are usually easy to deal with.

4.4. Orthogonal groups. Because we are dealing with the spherical case, we can use the orthogonal group instead of its connected component. We follow Weyl's conventions to parametrize the representations of $O(n)$. Embed $O(a) \subset U(a)$ in the standard way. An irreducible representation of $O(n)$ is parametrized by

$$(a_1, \dots, a_k, 0, \dots, 0; \epsilon), \qquad a_i \ge a_{i+1},\ \epsilon = \pm 1. \tag{4.4.1}$$

The ϵ is (sometimes) abbreviated as $\pm$. The parameter in (4.4.1) is the irreducible representation generated by the highest weight vector of the irreducible representation of $U(a)$ with highest weight

$$(a_1, \dots, a_k, \underbrace{1, \dots, 1}_{n-(1-\epsilon)k}, 0, \dots, 0). \tag{4.4.2}$$

For $O(n,n)$ we have $K = O(n) \times O(n)$. The fine K-types (in the sense of [V4]), and their restrictions to M are

$$\begin{array}{ll} K-type & M-type \\ (\underbrace{1,\dots,1}_{k},0,\dots,0)\otimes(0,\dots,0) & \\ & \binom{n}{k}\ characters\ \delta_k \\ (0,\dots,0)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0), & \end{array} \tag{4.4.3}$$

and we suppress the $\pm$. The meaning of (4.4.3) is that both representations of K in the left column restrict to an orbit of characters of M. The size of the orbit is $\binom{n}{k}$, and we fix a representative denoted δ_k.

The K-types

$$(0, \dots, 0; +) \otimes (\underbrace{2, \dots, 2}_{r}, 0, \dots, 0; +) \tag{4.4.4}$$

$$(\underbrace{1, \dots, 1}_{k}, 0, \dots, 0; \epsilon) \otimes (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0; \epsilon) \qquad \epsilon = \pm \tag{4.4.5}$$

form a relevant set. The representation of W on V^M is

$$(r, n-r) \times (0) \qquad \longleftrightarrow (4.4.4) \tag{4.4.6}$$

$$(n-k) \otimes (k), \quad k \le [n/2] \qquad \longleftrightarrow (4.4.5) \text{ with } +, \tag{4.4.7}$$

$$(n-k) \otimes (k), \quad k > [n/2] \qquad \longleftrightarrow (4.4.5) \text{ with } -. \tag{4.4.8}$$

For $O(n+1,n)$, we have $K = O(n+1) \times O(n)$. The fine K-types ([V4]), and their restriction to M are

$$\begin{array}{ll} K-type & M-type \\ (0,\dots,0)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0) & \binom{n}{k}\ characters\ \delta_k \end{array} \tag{4.4.9}$$

The K-types

$$(0,\dots,0;+)\otimes(\underbrace{2,\dots,2}_{r},0,\dots,0;+) \tag{4.4.10}$$

$$(\underbrace{1,\dots,1}_{k},0,\dots,0;+)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0;+) \qquad \text{for } k\le[n/2] \tag{4.4.11}$$

$$(\underbrace{1,\dots,1}_{n+1-k},0,\dots,0;-)\otimes(\underbrace{1,\dots,1}_{n-k},0,\dots,0;-) \qquad \text{for } k>[n/2] \tag{4.4.12}$$

form a relevant set, and the corresponding representations of $W(B_n)$ on V^M are

$$(r,n-r)\times(0),\quad r\le n/2 \qquad \longleftrightarrow (4.4.11) \tag{4.4.13}$$

$$(n-k)\otimes(k),\quad k\le[n/2] \qquad \longleftrightarrow (4.4.12) \text{ with } +, \tag{4.4.14}$$

$$(n-k)\otimes(k),\quad k>[n/2] \qquad \longleftrightarrow (4.4.12) \text{ with } -. \tag{4.4.15}$$

Theorem ([B5])**.** *A spherical irreducible representation for $\mathbb{H}$ is unitary if and only if the hermitian form is positive definite on the W-types*

$$(r,n-r)\times(0), \qquad (n-k)\times(k).$$

Corollary ([B5])**.** *A spherical representation $L_R(\chi)$ for a real split orthogonal group is unitary only if the corresponding $L_F(\chi)$ for $\mathbb{H}$ is unitary.*

In other words, the set of spherical unitary parameters for the real split group is contained in the set of spherical unitary parameters for $\mathbb{H}$. In fact in [B5] the stronger result is proved that the unitary duals of $G(\mathbb{R})$ and $\mathbb{H}$ coincide.

4.5. Symplectic groups. The maximal compact subgroup of $Sp(n,\mathbb{R})$ is $U(n)$. Its irreducible representations are parametrized by

$$(a_1,\dots,a_n), \qquad a_i\ge a_{i+1},\ a_i\in\mathbb{Z}. \tag{4.5.1}$$

The fine K-types and their restrictions to M are

$$\begin{array}{ll} K-type & M-type \\ \mu_+(k):=(\underbrace{1,\dots,1}_{k},0,\dots,0) & \\ & \binom{n}{k}\ characters\ \delta_k \\ \mu_-(k):=(0,\dots,0,\underbrace{-1,\dots,-1}_{k}). & \end{array} \tag{4.5.2}$$

The K-types

$$(\underbrace{2,\dots,2}_{k},0,\dots,0)\longleftrightarrow(n-k)\times(k), \tag{4.5.3}$$

$$(0,\dots,0,\underbrace{-2,\dots,-2}_{k})\longleftrightarrow(k)\times(n-k), \tag{4.5.4}$$

$$(\underbrace{1,\dots,1}_{r},0,\dots,0,\underbrace{-1,\dots,-1}_{r})\longleftrightarrow(r,n-r), \tag{4.5.5}$$

form a **relevant set**.

Theorem ([B5]). *A spherical representation for $\mathbb{H}$ is unitary if and only if the hermitian form is positive definite on the W-types*

$$(r,n-r)\times(0),\qquad (n-k)\times(k).$$

Corollary. *A spherical representation $L_R(\chi)$ for a real split orthogonal group is unitary only if the corresponding $L_F(\chi)$ for $\mathbb{H}$ is unitary.*

In fact in [B5] the stronger result is proved that the unitary duals of $G(\mathbb{R})$ and $\mathbb{H}$ coincide.

4.6. General linear group. The maximal compact subgroup of $GL(n,\mathbb{R})$ is $O(n)$. The fine K-types and their restrictions to M are

$$\begin{array}{ll} K-type & M-type \\ (\underbrace{1,\dots,1}_{k},0,\dots,0) & \binom{n}{k}\ characters\ \delta_k \end{array} \tag{4.6.1}$$

According to [V2], a set of relevant K-types is given by

$$(\underbrace{2,\dots,2}_{k},0,\dots,0). \tag{4.6.2}$$

The corresponding representation of S_n on V^M is, according to [B5],

$$(\underbrace{2,\dots,2}_{k},0,\dots,0)\longleftrightarrow(n-k,k) \tag{4.6.3}$$

In the case n even, we have

$$(2,\dots,2,\pm 2)\longleftrightarrow(n/2,n/2)_{I,II}. \tag{4.6.4}$$

The theorems and corollaries analogous to the ones in sections 4.4 and 4.5 follow from [V2] and [T]. For a construction of the relevant K-types for $SL(n)$ starting from the Weyl group module on E^M, the interested reader may consult [P].

4.7. G2. We refer to [V1] for more details about the structure of the split group of type G_2. The split real form which consists of the real points of the complex simply connected linear group of type G_2 has

$$K := [SU(2)_s \times SU(2)_l]/\{\pm Id\}. \tag{4.7.1}$$

It is more convenient to work with the (nonlinear) double cover for which

$$\widetilde{K} = SU(2)_s \times SU(2)_l. \tag{4.7.2}$$

The Borel subgroup is $\widetilde{B} = \widetilde{M}AN$ where $\widetilde{M}$ is the group of order 8 which contains $\{\pm Id\}$ in (4.7.1) as a central sugroup, with quotient $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let T be the maximal compact subgroup. We identify the Lie algebra $\mathfrak{t}$ with $\{(a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0\}$, and use the standard positive system of roots

$$\begin{aligned}\{(1,-1,0),\ (-1,2,-1),\ (0,1,-1),\ (1,0,-1),\\ (1,1,-2),\ (2,-1,-1)\}.\end{aligned} \tag{4.7.3}$$

A K-type is determined by its highest weight which we denote

$$(p \mid q) \longleftrightarrow (q, \frac{p-q}{2}, \frac{-p-q}{2}), \qquad p,\ q \in \mathbb{N}. \tag{4.7.4}$$

The positive roots in $\mathfrak{k}$ are $(0, 1, -1)$ and $(2, -1, -1)$. The fine K-types are

$$\begin{array}{ll} \widetilde{K}-type & \widetilde{M}-type \\ (0 \mid 0) & \delta_0,\ \text{the trivial character}, \\ (1 \mid 0) & \delta_2,\ \text{a representation of dimension 2}, \\ (2 \mid 0) & \delta_3,\ \text{three characters}. \end{array} \tag{4.7.5}$$

The one dimensional representations all factor to M, while the two dimensional one is a genuine representation of $\widetilde{M}$. The Weyl group is generated by s_s, the reflection about the simple short root, and s_l, the reflection about the simple long root. The representations of W are parametrized as in [L],

$$\begin{array}{ll} 1_1 & trivial \\ 1_2 & s_s = 1,\ s_l = -1, \\ 1_3 & s_s = -1,\ s_l = +1, \\ 1_4 & s_s = -1,\ s_l = -1, \\ 2_1 & reflection, \\ 2_2 & 2_1 \otimes 1_2. \end{array} \tag{4.7.6}$$

Theorem ([C]). *A spherical representation of $\mathbb{H}$ is unitary if and only if the r_σ are positive definite on the W-types*

$$1_1,\ 2_1,\ 2_2.$$

Proposition. *The following list consists of petite K-types.*

$K-type$	$W-type$
$(0 \mid 0)$	$1_1,$
$(3 \mid 1)$	$2_1,$
$(4 \mid 0)$	$2_2.$

Thus the W-types in theorem 4.7 form a **relevant set**. More detailed results along these lines can be found in [C].

4.8. F4. The split real form which is the rational points of the complex linear group of type F_4 has maximal compact subgroup

$$K := [Sp(1) \times Sp(3)]/\{\pm Id\}. \tag{4.8.1}$$

It is more convenient to use the double cover which has

$$\widetilde{K} = [Sp(1) \times Sp(3)] \tag{4.8.2}$$

The Borel subgroup is $\widetilde{B} = \widetilde{M}AN$, where $\widetilde{M}$ is a finite nonabelian group of order 2^5. It has a central subgroup of order 2 (the $\pm Id$ in (4.8.1)) such that the quotient is $\mathbb{Z}_2^4$.

Let T be a maximal compact Cartan subgroup. We use the standard positive system and roots,

$$\{2\epsilon_1,\ 2\epsilon_k,\ \epsilon_k \pm \epsilon_\ell\}_{2 \le k \le \ell \le 4}. \tag{4.8.3}$$

The highest weight of a $\widetilde{K}$-type will be denoted

$$(a_1 \mid a_2, a_3, a_4, a_5), \qquad a_i \in \mathbb{N}, \quad a_2 \ge a_3 \ge a_4 \ge 0. \tag{4.8.4}$$

The *fine* K-types (in the sense of [V4]), and their restrictions to $\widetilde{M}$ are

$\widetilde{K}-type$	$\widetilde{M}-type$	W_δ
$(0 \mid 0,0,0)$	δ_0, one character	$F_4,$
$(1 \mid 0,0,0)$	δ_2, a representation of dimension 2	F_4
$(2 \mid 0,0,0)$	δ_3, three characters	$C_4,$
$(0 \mid 1,0,0)$	δ_6, three representations of dimension 2	$B_4,$
$(1 \mid 1,0,0)$	δ_{12}, twelve characters	B_3A_1

(4.8.5)

The 1-dimensional representations all factor to M. The remaining ones are genuine for $\widetilde{M}$.

The representations of the Weyl group of type F_4 are parametrized as in [L].

Theorem ([C])**.** *A spherical representation of* $\mathbb{H}$ *of type* F_4 *is unitary if and only if the* r_σ *are positive semidefinite on the W-types*

$$1_1,\ 2_3,\ 8_1,\ 4_2,\ 9_1.$$

The next result was obtained joint with D. Vogan.

Proposition. *The following list consists of petite K-types.*

$K-type$	$W-type$
$(0 \mid 0,0,0)$	$1_1,$
$(0 \mid 1,1,0)$	$2_1,$
$(4 \mid 0,0,0)$	$2_3,$
$(1 \mid 2,1,0)$	$8_1,$
$(1 \mid 1,1,1)$	$4_2,$
$(2 \mid 2,0,0)$	$9_1.$

Thus the W-types in theorem 4.8 form a **relevant set**.

Corollary. *A spherical representation for the real split group G of type F_4 is unitary only if the corresponding $L_F(\chi)$ for $\mathbb{H}$ is unitary.*

The results in [C] provide an explicit list of the unitary dual of $\mathbb{H}$. The unitarity of the unipotent representations and the relevant irreducibility results for the real case are not (yet) available.

4.9. E6. The maximal compact subgroup of the rational points of the simply connected complex group of type E_6 is

$$K = Sp(4)/\{\pm Id\}. \tag{4.9.1}$$

Again it is more convenient to use the double cover which has

$$\widetilde{K} = Sp(4). \tag{4.9.2}$$

We use the same conventions as in section 4.8 for the positive roots and highest weights of representations of $\widetilde{K}$. The finite group $\widetilde{M}$ has size 2^7, and its quotient by the central group $\{\pm Id\}$ is a $\mathbb{Z}_2^6$. The fine K-types and their restrictions to $\widetilde{M}$ are

$\widetilde{K}-type$	$\widetilde{M}-type$	W_δ^0	
$(0,0,0,0)$	δ_1, the trivial character,	$E_6,$	
$(1,0,0,0)$	δ_8, a representation of dimension 8,	$E_6,$	(4.9.3)
$(1,1,0,0)$	δ_{27}, twenty seven characters,	$D_5,$	
$(2,0,0,0)$	δ_{36}, thirty six characters,	$A_5A_1.$	

The second representation is genuine for $\widetilde{M}$, the others all factor to M.

The W-types are parametrized as in [L].

Theorem ([BC])**.** *A spherical representation of $\mathbb{H}$ of type E_6 is unitary if and only if the r_σ are positive semidefinite on the W-types*

$$1_p,\ 6_p,\ 20_p,\ 30_p,\ 15_q.$$

We denote by ω_i the fundamental weights of $sp(4)$. In coordinates they are $\omega_i = \sum_{j\leq i}\epsilon_j$.

Proposition. *The following list consists of petite K-types.*

$K-type$	$W-type$
$(0)=(0,0,0,0)$	$1_p,$
$\omega_4=(1,1,1,1)$	6_p
$2\omega_2=(2,2,0,0)$	$20_p,$
$4\omega_1=(4,0,0,0)$	$15_q,$
$2\omega_1+\omega_4=(3,1,1,1)$	$30_p,$
$\omega_1+\omega_2+\omega_3=(3,2,1,0)$	$64_p,$
$3\omega_1+\omega_3=(4,1,1,0)$	$60_p,$
$2\omega_3=(2,2,2,0)$	$15_p,$
$2\omega_1+2\omega_2=(4,2,0,0)$	$81_p,$
$3\omega_2=(3,3,0,0)$	$24_p,$
$6\omega_1=(6,0,0,0)$	$24'_p.$

Corollary. *A spherical representation $L_R(\chi)$ for the real group G of type E_6 is unitary only if the corresponding $L_F(\chi)$ for $\mathbb{H}$ is unitary.*

Thus the W-types in theorem 4.9 form a **relevant set**.

4.10. E7. The maximal compact subgroup of the split real form of the simply connected group of type E_7 is

$$K=SU(8)/\{\pm Id\}. \tag{4.10.1}$$

We work with the double cover for which

$$\widetilde{K}=SU(8). \tag{4.10.2}$$

The finite group $\widetilde{M}$ has size 2^8, and its quotient by the center in (4.10.1) is a $\mathbb{Z}_2^7$. The fine K-types are

$\widetilde{K}-type$	$\widetilde{M}-type$	W^0_δ	
(0)	δ_1, trivial representation,	E_7	
ω_1	δ_8, eight dimensional representation,	E_7	
ω_7	δ_8^*, eight dimensional representation,	E_7	(4.10.3)
$\omega_2,\ \omega_6$	δ_{28}, twenty eight characters,	E_6	
$2\omega_1,\ 2\omega_7$	δ_{36}, thirty six characters,	A_7	
$\omega_1+\omega_7$	δ_{63}, sixty three characters,	D_6A_1	

In (4.10.3), the ω refer to the fundamental weights of $\widetilde{K}$, usual labelling.

The second and third are genuine representations of $\widetilde{M}$, the others are single orbits under the action of W.

The Weyl group representations are parametrized as in [L].

Theorem ([BC]). *A spherical representation of* $\mathbb{H}$ *of type* E_7 *is unitary if and only if* r_σ *is positive semidefinite for*

$$1_a,\ 7'_a,\ 27_a,\ 56'_a,\ 21'_b,\ 35_b,\ 105_b.$$

Proposition. *The following list gives the petite* K*-types with* M*-spherical vectors, and the corresponding Weyl group representations:*

$K-type$	$W-type$
(0)	$1_a,$
ω_4	$7'_a$
$2\omega_2,\ 2\omega_6$	$21'_b,$
$\omega_2+\omega_6$	$27_a,$
$2\omega_1+2\omega_7$	$35_b,$

$K-type$	$W-type$
$4\omega_1,\ 4\omega_7$	$15'_a,$
$\omega_2+\omega_3+\omega_7,\ \omega_1+\omega_5+\omega_6$	$105'_a,$
$\omega_1+\omega_4+\omega_7$	$56'_a,$
$2\omega_1+\omega_3+\omega_7,\ \omega_1+\omega_5+2\omega_7$	$189'_b,$
$\omega_1+\omega_3+\omega_6,\ \omega_2+\omega_5+\omega_7$	$168_a,$
$3\omega_1+\omega_5,\ \omega_3+3\omega_7$	$105_b,$
$\omega_3+\omega_5$	$21_a,$
$\omega_1+\omega_2+\omega_5,\ \omega_3+\omega_6+\omega_7$	$120_a,$
$\omega_1+\omega_2+\omega_6+\omega_7,$	$168_a+210_a,$
$3\omega_1+3\omega_7,$	$84_a+105_c,$
$\omega_1+2\omega_2+\omega_7, \omega_1+\omega_6+\omega_7,$	$189'_c$
$3\omega_1+\omega_2+\omega_7, \omega_1+\omega_6+3\omega_7,$	$216'_a,$
$3\omega_1+\omega_6+\omega_7, \omega_1+\omega_2+3\omega_7,$	$280_b,$
$5\omega_1+\omega_7, \omega_1+5\omega_7,$	$84'_a.$

Corollary. *A spherical representation* $L_R(\chi)$ *for the real group* G *of type* E_7 *is unitary only if the corresponding* $L_F(\chi)$ *for* $\mathbb{H}$ *is unitary.*

Thus the W-types in theorem 4.10 form a **relevant set**.

4.11. E8. The maximal compact subgroup of the split real form of the simply connected complex group of type E_8 is

$$Spin(16)/\{Id,\omega\}, \tag{4.11.1}$$

for ω the appropriate element of order two in the center (the quotient is **not** SO(16)). As before we work with the double cover which has

$$\widetilde{K}=Spin(16). \tag{4.11.2}$$

The group $\widetilde{M}$ has size 2^9, and its quotient by the center in (4.11.1) is $\mathbb{Z}_2^8$. The fine K-types are

$$\begin{array}{lr} \widetilde{K}-type & \widetilde{M}-type \\ (0) & \delta_1,\quad \text{trivial representation}, \\ \omega_1 & \delta_{16},\quad \text{sixteen dimensional representation}, \\ \omega_2 & \delta_{120},\quad \text{one hundred and twenty characters}, \\ 2\omega_1 & \delta_{135},\quad \text{one hundred thirty five characters}, \end{array} \tag{4.11.3}$$

Only the second representation is genuine.

The representations of the Weyl group are parametrized as in [L].

Theorem. *A spherical representation of* $\mathbb{H}$ *of type* E_8 *is unitary if and only if* r_σ *is positive semidefinite for*

$$1_x,\ 8_z,\ 35_x,\ 50_x,\ 84_x,\ 112_z,\ 400_z,\ 300_x,\ 210_x.$$

Proposition. *The following list gives petite* K*-types and the corresponding Weyl group representations on* $(V^*)^M$ *:*

$K-type$	W-type on $(V^*)^M$
(0)	$1_x,$
ω_8	8_z
ω_4	$35_x,$
$2\omega_2$	$84_x,$
$\omega_2+\omega_8$	$112_z,$
$4\omega_1$	$50_x,$
$3\omega_1+\omega_7$	$400_z,$
$\omega_3+\omega_7$	$160_z,$
ω_6	$28_x,$
$\omega_1+\omega_5$	$210_x,$
$\omega_1+\omega_2+\omega_7$	$560_x,$
$\omega_2+\omega_4$	$567_x,$
$2\omega_3$	$300_x,$
$2\omega_1+\omega_4$	$700_x,$
$3\omega_1+\omega_3$	$1050_x,$
$\omega_1+\omega_2+\omega_3$	$1344_x,$
$3\omega_2$	$525_x,$
$2\omega_1+2\omega_2$	$972_x,$
$4\omega_1+\omega_2$	$700_{xx},$
$6\omega_1$	$168_y.$

Corollary. *A spherical representation* $L_R(\chi)$ *for the real group* G *of type* E_8 *is unitary only if the corresponding* $L_F(\chi)$ *for* $\mathbb{H}$ *is unitary.*

5. Some proofs

5.1. Let $\widetilde{K}$ be a compact group, $\widetilde{M} \subset \widetilde{N}$ finite subgroups such that $\widetilde{M}$ is normal in $\widetilde{N}$. Denote by W the quotient $\widetilde{N}/\widetilde{M}$. In the applications, $\widetilde{N}$ is the normalizer of $\mathfrak{a}$ in $\widetilde{K}$, which is simply connected.

Let $V_a,\ V_b$ be representations of $\widetilde{K}$. In the nicer cases, the restrictions of $V_a,\ V_b$ to $\widetilde{M}$ are multiples of the same representation V_δ, which in turn extends to a representation of $\widetilde{K}$. This is the case for E_8. The standard 16-dimensional representation is genuine, and restricts to a single $\widetilde{M}$-type denoted δ_{16}. We already defined the action of W in the next proposition in section 3.2.

Proposition. *There is a natural action of* W *on* $\mathrm{Hom}_M[V_a, V_b]$.

Proof. The action on $f \in \mathrm{Hom}[V_a, V_b]$ is given by

$$(n \cdot f)(v) = \mu_a(n) f(\mu_b(n^{-1})v). \tag{5.1.1}$$

This is clearly an action of $\widetilde{N}$. The fact that the action does not depend on the right $\widetilde{M}$-coset is straightforward. □

This action is compatible with the canonical isomorphism

$$\mathrm{Hom}_{\widetilde{M}}[V_a, V_b] \cong [V_a^* \otimes V_b]^{\widetilde{M}}, \tag{5.1.2}$$

and is also compatible with the isomorphism

$$\mathrm{Hom}_{\widetilde{M}}[V_a, V_b]^* \cong \mathrm{Hom}_{\widetilde{M}}[V_b^*, V_a^*]. \tag{5.1.3}$$

This action is a generalization of the usual one on $V^{\widetilde{M}}$.

5.2. Now consider two genuine modules $V_a,\ V_b$. Then there is a homomorphism

$$\mathrm{Hom}_{\widetilde{M}}[V_a, V_\delta] \otimes \mathrm{Hom}_{\widetilde{M}}[V_\delta, V_a] \longrightarrow \mathrm{Hom}_{\widetilde{M}}[V_a, V_b]. \tag{5.2.1}$$

This is compatible with the action of W. When $V_a,\ V_b$ are multiples of a single orbit of V_δ, this is an isomorphism.

5.3. Orthogonal groups. In the case of $SO(n+1, n)$, the Weyl group is $W(B_n)$. The centralizer $W_\delta^0 = W_\delta$ corresponding to $(0) \otimes (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0)$ is isomorphic to $W(C_k \times C_{n-k})$. Then

$$Ind_{W(C_k \times C_{n-k})}^{W(C_n}[triv] = \sum (n-k+\ell, k-\ell) \times (0). \tag{5.3.1}$$

The corresponding tensor product is

$$\sum (0) \otimes (\underbrace{2, \dots, 2}_{a}, 1, \dots, 1, \underbrace{0, \dots, 0}_{b}). \tag{5.3.2}$$

These K-types are automatically level ≤ 2. The formulas imply that these W-types are realized in the M-spherical vectors of petite K-types. This is sufficient for the purpose of corollary 4.4. For the more precise results in (4.4.13-4.4.15) one needs to show first that the factors occuring in (5.3.2) with 1's do not have any M fixed vectors. This can be done by showing that such factors occur in tensor products of distinct fine K-types which cannot have any M-spherical vectors. To sort out the matchup of the remaining K-types with the W-representations, one needs to compare the formulas (5.3.1) and (5.4.2) for various values of k.

Tensor products of fine K-types of $SO(n+1,n)$ are not sufficient to realize the remaining relevant Weyl group representations in the M-spherical vectors of petite K-types. To achieve this, we observe that $Spin(n+1,n)$, the rational points of $Spin(2n+1,\mathbb{C})$ has another pair of fine K-types

$$spin\otimes spin^{\pm}. \tag{5.3.3}$$

In this case $W^0_\delta = S_n$, while $W_\delta = S_n\times\mathbb{Z}_2$. Then

$$\begin{aligned} Ind_{S_n\times\mathbb{Z}_2}[triv\otimes triv] &= \sum (n-k)\times(k), \qquad k\ even,\\ Ind_{S_n\times\mathbb{Z}_2}[triv\otimes sgn] &= \sum (n-k)\times(k), \qquad k\ odd. \end{aligned} \tag{5.3.4}$$

We combine these formulas with

$$\begin{aligned} spin\otimes spin &= \sum(\underbrace{1,\dots,1}_{k},0,\dots,0), \qquad \text{for Spin(2n+1)},\\ spin^+\otimes spin^- &= \sum_{k\ even}(\underbrace{1,\dots,1}_{n-k-1},0,\dots,0),\\ spin^+\otimes spin^+ &= (1,\dots,1)+\sum_{k>0\ even}(\underbrace{1,\dots,1}_{n-k},0,\dots,0),\\ spin^-\otimes spin^- &= (1,\dots 1,-1)+\sum_{k>0\ even}(\underbrace{1,\dots,1}_{n-k},0,\dots,0). \end{aligned} \tag{5.3.5}$$

This proves that the sum of petite representations in formulas (4.4.9) contains the sum of all the relevant W-representations. This is sufficient for corollary 4.4.

In the case $O(n,n)$, we do not need to use the covers. We can also derive the matchup of the petite K-types with the relevant W-representations in (4.4.6-4.4.8) in a manner similar to the symplectic group case, where we give a few more details.

5.4. Symplectic groups. The basic tool is the tensor product formulas

$$\begin{aligned} \mu_+(k)\otimes\mu_-(k) &= \sum(\underbrace{1,\dots,1}_{a},\underbrace{0,\dots,0}_{b},-1,\dots,-1),\\ \mu_+(k)\otimes\mu_+(k) &= \sum(\underbrace{2,\dots,2}_{a},\underbrace{1,\dots,1}_{b},0,\dots,0), \qquad 2k\leq n \end{aligned} \tag{5.4.1}$$

with $2a+b=2k$. These K-types are automatically level ≤ 2, so petite. The stabilizer W^0_δ for $\mu_\pm(k)$ is $W(D_k)\times W(C_{n-k})$, while $W_\delta = W(C_k)\times W(C_{n-k})$. Then

$$Ind^{W(C_k)}_{W(D_k)}[triv] = (k)\times(0) + (0)\times(k). \tag{5.4.2}$$

The induced modules corresponding to (5.4.1) are

$$\begin{aligned} Ind^{W(C_n)}_{W(C_k)\times W(C_{n-k})}[(k)\times(0)\otimes(n-k)\times(0)] &= \sum(n-k+\ell,k-\ell)\times(0),\\ Ind^{W(C_n)}_{W(C_k)\times W(C_{n-k})}[(0)\times(k)\otimes(n-k)\times(0)] &= (n-k)\times(k). \end{aligned} \tag{5.4.3}$$

These results imply that any relevant Weyl group representation is realized in $(E^+_\mu)^M$ for some petite K-type. To get the more precise result of (4.5.3-4.5.4), one needs to decompose $\mu_+(r)\otimes\mu_+(s)$ for all $r,\ s$. For $r\ne s$, none of the factors have M-fixed vectors. Then one makes a comparison of the remaining representations occuring in the tensor products for $r=s$ with the induced representations of the corresponding Weyl groups.

5.5. G2. The stabilizer of δ_3 is the subgroup $W(A_{1,s}\times A_{1,l})$. The induced module

$$Ind^{W(G2)}_{W(A_{1,s}\times A_{1,l}}[triv] = 1_1 + 2_2 \tag{5.5.1}$$

corresponds to the tensor product

$$(2\mid 0)\otimes(2\mid 0) = (4\mid 0)+(2\mid 0)+(0\mid 0). \tag{5.5.2}$$

On the other hand,

$$(1\mid 0)\otimes(1\mid 0) = (2\mid 0)+(0\mid 0). \tag{5.5.3}$$

Since the M-fixed vectors in the tensor product have dimension 1, it follows that $(2\mid 0)$ has no M-fixed vectors. Combining this with the fact that the representation of W on the M-fixed vectors of $\mathfrak{s}=(3\mid 1)$ is the reflection representation, we conclude

$$\begin{aligned} (0\mid 0) &\longleftrightarrow\ 1_1,\\ (4\mid 0) &\longleftrightarrow\ 2_2,\\ (3\mid 1) &\longleftrightarrow\ 2_1. \end{aligned} \tag{5.5.4}$$

5.6. F4. We first compute some tensor products of the δ's. In the process we get the restrictions of the $\widetilde{K}$-types in theorem 4.8 to $\widetilde{M}$.

- $\delta_2\otimes\delta_2 = \delta_0+\delta_3$ because it equals
$$(1\mid 0,0,0)\otimes(1\mid 0,0,0) = (2\mid 0,0,0)+(0\mid 0,0,0).$$
- $\delta_2\otimes\delta_3$ equals
$$(1\mid 0,0,0)\otimes(2\mid 0,0,0) = (3\mid 0,0,0)+(1\mid 0,0,0) = 2\delta_2+\delta_2,$$
because there is no room for a δ_6. So we also get $(3\mid 0,0,0)=2\delta_2$.

- $\delta_2 \otimes \delta_6$ equals
$$(1 \mid 0,0,0) \otimes (0 \mid 1,0,0) = (1 \mid 1,0,0) = \delta_{12}.$$
- $\delta_2 \otimes \delta_{12} = \delta_2 \otimes \delta_2 \otimes \delta_6$ equals
$$(\delta_1 + \delta_3) \otimes \delta_6 = \delta_6 + \delta_3 \otimes \delta_6.$$
It also equals
$$(1 \mid 0,0,0) \otimes (1 \mid 1,0,0) = (2 \mid 1,0,0) + (0 \mid 1,0,0) = \delta_3 \otimes \delta_6 + \delta_6 = 4\delta_6.$$
Then $\delta_3 \otimes \delta_6 = a\delta_2 + b\delta_6$. But
$$\delta_2 \otimes \delta_3 \otimes \delta_6 = 3\delta_2 \otimes \delta_6 = 3\delta_{12},$$
so $a = 0$. So also $(2 \mid 1,0,0) = \delta_3 \otimes \delta_6 = 3\delta_6$.
- $\delta_6 \otimes \delta_6 = 3\delta_1 + a\delta_3 + b\delta_{12}$. Because
$$\delta_3 \otimes \delta_6 \otimes \delta_6 = 3\delta_6 \otimes \delta_6,$$
$a = 3$ and $b = 2$. Since also
$$(0 \mid 1,0,0) \otimes (0 \mid 1,0,0) = (0 \mid 2,0,0) + (0 \mid 1,1,0) + (0 \mid 0,0,0),$$
we conclude
$$(0 \mid 2,0,0) = 3\delta_3 + \delta_{12},$$
$$(0 \mid 1,1,0) = 2\delta_1 + \delta_{12}.$$

We now compute the induced modules
$$\begin{aligned} &Ind_{W(B_4)}^{W(F_4)}[triv] = 1_1 + 2_1,\\ &Ind_{W(C_4)}^{W(F_4)}[triv] = 1_1 + 2_3,\\ &Ind_{W(B_3A_1)}^{W(F_4)}[triv] = 1_1 + 2_1 + 9_1, \end{aligned} \tag{5.6.1}$$
corresponding to the tensor products
$$\begin{aligned} &(0 \mid 1,0,0) \otimes (0 \mid 1,0,0) = (0 \mid 2,0,0) + (0 \mid 1,1,0) + (0 \mid 0,0,0),\\ &(2 \mid 0,0,0) \otimes (2 \mid 0,0,0) = (4 \mid 0,0,0) + (2 \mid 0,0,0) + (0 \mid 0,0,0),\\ &(1 \mid 1,0,0) \otimes (1 \mid 1,0,0) = (2 \mid 2,0,0) + (2 \mid 1,1,0) + (2 \mid 0,0,0)+\\ &+ (0 \mid 2,0,0) + (0 \mid 1,1,0) + (0 \mid 0,0,0). \end{aligned} \tag{5.6.2}$$
Noting also that
$$(2 \mid 2,0,0) = (2 \mid 0,0,0) \otimes (0 \mid 2,0,0) = \delta_3 \otimes (3\delta_3 + \delta_{12}), \tag{5.6.3}$$
we conclude that the M-fixed vectors of $(2 \mid 0,0,0)$ are nine dimensional. Therefore
$$\begin{aligned} &(0 \mid 1,1,0) \longleftrightarrow 2_1,\\ &(4 \mid 0,0,0) \longleftrightarrow 2_3,\\ &(2 \mid 2,0,0) \longleftrightarrow 9_1. \end{aligned} \tag{5.6.4}$$

Because $(1 \mid 1,1,1)$ is the representation of K on $\mathfrak{s}$, the Weyl group representation on the fixed vectors is the reflection representation 4_2.

Now consider

$$\mathrm{Hom}_M[(1 \mid 1,0,0),(0 \mid 1,1,0)] = \mathrm{Hom}_M[\delta_{12}, 2\delta_1 + \delta_{12}]. \qquad (5.6.5)$$

It is therefore 12-dimensional. Because δ_{12} corresponds to B_3A_1, in (4.8.5), the W-module is induced from a character of $W(B_3A_1)$. Thus the tensor product

$$(1 \mid 1,0,0) \otimes (0 \mid 1,1,0) = (1 \mid 2,1,0) + (1 \mid 1,1,1) + (1 \mid 1,0,0) \qquad (5.6.6)$$

has the property that the W-representation on the M-fixed vectors is induced from this 1-dimensional character. It follows that the M-fixed vectors of $(1 \mid 2,1,0)$ are 8-dimensional. To determine the character, we note that the restriction of 4_2 to $W(B_3A_1)$ is

$$4_2 \longrightarrow [(2\times 1)\otimes(2)] + [(3\times 0)\otimes(1^2)]. \qquad (5.6.7)$$

The induced module from the character which occurs, is

$$[(3\times 0)\otimes(1^2)] \longrightarrow 4_2 + 8_1. \qquad (5.6.8)$$

The proof of theorem 4.8 follows.

5.7. E6. The proof will occupy several sections. We will exploit the fact that genuine representations are all multiples of δ_8.

The following induced modules,

$$\begin{aligned} Ind_{W(D5)}^{W(E6)}[triv] &= 1_p + 6_p + 20_p,\\ Ind_{W(A5A1)}^{W(E6)}[triv] &= 1_p + 15_q + 20_p, \end{aligned} \qquad (5.7.1)$$

correspond to the tensor products

$$\begin{aligned} \omega_2 \otimes \omega_2 &= (2\omega_1) + (\omega_1+\omega_3) + (2\omega_2) + (\omega_4) + (\omega_2) + (0),\\ 2\omega_1 \otimes 2\omega_1 &= (4\omega_1) + (2\omega_1+\omega_2) + (2\omega_1) + (2\omega_2) + (\omega_2) + (0). \end{aligned} \qquad (5.7.2)$$

We know that $\omega_4 \longleftrightarrow 6_p$ because this is the representation of $\widetilde{K}$ on $\mathfrak{s}$ of the Cartan decomposition. We conclude that

$$\begin{aligned} (2\omega_1+\omega_2) + (2\omega_2) &\longleftrightarrow 20_p,\\ (\omega_1+\omega_3) + (2\omega_2) &\longleftrightarrow 20_p. \end{aligned} \qquad (5.7.3)$$

But $(2\omega_1)\otimes(\omega_2)$ does not have any M-fixed vectors, so $2\omega_1+\omega_2$ which is a factor does not have any either. We conclude that

$$\begin{aligned} (0) &\longleftrightarrow 1_p,\\ (\omega_4) &\longleftrightarrow 6_p,\\ (2\omega_2) &\longleftrightarrow 20_p,\\ (4\omega_1) &\longleftrightarrow 15_q. \end{aligned} \qquad (5.7.4)$$

The rest of the representations considered so far do not have any M-fixed vectors. Using tensoring with ω_1 we also find that

$$\begin{aligned}(\omega_1) &\longleftrightarrow \quad 1_p,\\ (\omega_3) &\longleftrightarrow \quad 6_p,\\ (\omega_1+\omega_2) &\longleftrightarrow \quad 20_p,\\ (3\omega_1) &\longleftrightarrow \quad 15_q.\end{aligned} \tag{5.7.5}$$

In these matchups, the W representations are on $\mathrm{Hom}_M[\omega_1, V]$.

5.8. The tensor product

$$(\omega_3)\otimes(\omega_3) = (2\omega_1)+(\omega_1+\omega_3)+(2\omega_2)+(\omega_2+\omega_4)+(\omega_2)+(2\omega_3)+(0) \tag{5.8.1}$$

corresponds to

$$6_p \otimes 6_p = 1_p + 20_p + 15_p. \tag{5.8.2}$$

Since $\omega_4 = 6\delta_1 + \delta_{36}$, $(\omega_2)\otimes(\omega_4)$ does not have any M-fixed vectors. Thus $(\omega_2+\omega_4)$ doesn't either. We conclude

$$(2\omega_3) \longleftrightarrow 15_p. \tag{5.8.3}$$

5.9. The representations $(3\omega_1)$ and ω_3 are genuine, and

$$(3\omega_1)\otimes(\omega_3) = (3\omega_1+\omega_3)+(2\omega_1+\omega_2)+(2\omega_1+\omega_4)+(\omega_1+\omega_3) \tag{5.9.1}$$

corresponds to

$$6_p \otimes 15_q = 30_p + 60_p. \tag{5.9.2}$$

Since

$$(\omega_1)\otimes(\omega_1+\omega_4) = (2\omega_1+\omega_4)+(\omega_1+\omega_3)+(\omega_2+\omega_4)+(\omega_4), \tag{5.9.3}$$

and the dimension of $(\omega_1+\omega_4)$ is $8\cdot 36$, we conclude that

$$\begin{aligned}(2\omega_1+\omega_4) &\longleftrightarrow \quad 30_p,\\ (3\omega_1+\omega_3) &\longleftrightarrow \quad 60_p.\end{aligned} \tag{5.9.4}$$

This completes the proof of corollary 4.9.

5.10. The tensor product

$$\begin{aligned}(\omega_1+\omega_2)\otimes(\omega_3) = &(\omega_1+\omega_2+\omega_3)+(2\omega_1+\omega_2)+(2\omega_1+\omega_4)+\\ &(2\omega_1)+2(\omega_1+\omega_3)+(2\omega_2)+(\omega_2+\omega_4)+(\omega_2)+(\omega_4)\end{aligned} \tag{5.10.1}$$

corresponds to

$$6_p \otimes 20_p = 6_p + 30_p + 20_p + 64_p. \tag{5.10.2}$$

We conclude that

$$(\omega_1+\omega_2+\omega_3) \longleftrightarrow 64_p. \tag{5.10.3}$$

5.11. The tensor product

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+\omega_2)=(4\omega_1+\omega_2)+(4\omega_1)+(3\omega_1+\omega_3)+\\&(2\omega_1+2\omega_2)+2(2\omega_1+\omega_2)+(2\omega_1)+\\&(\omega_1+\omega_2+\omega_3)+(\omega_1+\omega_3)+(2\omega_2)+(\omega_2)\end{aligned}\tag{5.11.1}$$

corresponds to

$$15_q\otimes 20_p=15_q+20_p+60_p+60_s+64_p+81_p.\tag{5.11.2}$$

We conclude that

$$(4\omega_1+\omega_2)+(2\omega_1+2\omega_2)\longleftrightarrow 81_p+60_s.\tag{5.11.3}$$

5.12. The tensor product

$$\begin{aligned}&(\omega_1+\omega_2)\otimes(\omega_1+\omega_2)=\\&(4\omega_1)+(3\omega_1+\omega_3)+(2\omega_1+2\omega_2)+3(2\omega_1+\omega_2)+\\&(2\omega_1+\omega_4)+2(2\omega_1)+2(\omega_1+\omega_2+\omega_3)+3(\omega_1+\omega_3)+\\&(3\omega_2)+2(2\omega_2)+(\omega_2+\omega_4)+(2\omega_2)+(2\omega_3)+(\omega_4)+(0)\end{aligned}\tag{5.12.1}$$

corresponds to

$$20_p\otimes 20_p=1_p+6_p+15_q+15_p+2\cdot 20_p+24_p+30_p+60_p+2\cdot 64_p+81_p.\tag{5.12.2}$$

We conclude that

$$(2\omega_1+2\omega_2)+(3\omega_2)\longleftrightarrow 81_p+24_p.\tag{5.12.3}$$

5.13. The tensor product

$$\begin{aligned}&(3\omega_1)\otimes(3\omega_1)=(6\omega_1)+(4\omega_1+\omega_2)+(4\omega_1)+\\&(2\omega_1+2\omega_2)+2(2\omega_1+\omega_2)+(2\omega_1)+\\&(3\omega_2)+(2\omega_2)+(\omega_1)+(0)\end{aligned}\tag{5.13.1}$$

corresponds to

$$15_q\otimes 15_q=1_p+15_q+20_p+24_p+24'_p+60_s+81_p.\tag{5.13.2}$$

We conclude that

$$(6\omega_1)+(4\omega_1+\omega_2)+(2\omega_1+2\omega_2)+(3\omega_2)\longleftrightarrow 24_p+81_p+60_s+24'_p.\tag{5.13.3}$$

Combining this with (5.12.3), and (5.11.3), we get

$$\begin{aligned}&(3\omega_2)\leftrightarrow 24_p,\\&(4\omega_1+\omega_2)\leftrightarrow 60_s,\\&(2\omega_1+2\omega_2)\leftrightarrow 81_p,\\&(6\omega_1)\leftrightarrow 24'_p.\end{aligned}\tag{5.13.4}$$

5.14. E7. In this case, genuine representations restrict to combinations of δ_8 and δ_8^*, so the proof will be more involved. We first note that

$$(\omega_4) \longleftrightarrow 7'_a, \tag{5.14.1}$$

because it is the representation of K on $\mathfrak{s}$ in the Cartan decomposition.

We record the following induced representations:

$$\begin{aligned} Ind_{A7}^{E7}[triv] &= 1_a + 35_b + 21'_b + 15'_a, \\ Ind_{E6}^{E7}[triv] &= 1_a + 27_a + 7'_a + 21'_b, \\ Ind_{D6}^{E7}[triv] &= 1_a + 27_a + 35_b. \end{aligned} \tag{5.14.2}$$

For the first one, $W_\delta \neq W_\delta^0$. It corresponds to the sum of

$$\begin{aligned} &(2\omega_1) \otimes (2\omega_1) = (4\omega_1) + (2\omega_1 + \omega_2) + (2\omega_2), \\ &\text{and} \\ &(2\omega_1) \otimes (2\omega_7) = (2\omega_1 + 2\omega_7) + (\omega_1 + \omega_7) + (0). \end{aligned} \tag{5.14.3}$$

The first line is $21'_b + 15_a$, the second one $1_a + 35_b$. Since

$$\begin{aligned} (\omega_1) \otimes (\omega_7) &= (\omega_1 + \omega_7) + (0), \\ (\omega_1) \otimes (\omega_1) &= (2\omega_1) + (\omega_2), \\ (\omega_7) \otimes (\omega_7) &= (2\omega_7) + (\omega_6), \end{aligned} \tag{5.14.4}$$

we conclude that none of the representations on the RHS of these representations (except (0)) are M-spherical. We conclude that

$$\begin{aligned} &(2\omega_1 + 2\omega_7) \longleftrightarrow 35_b, \\ &(4\omega_1) + (2\omega_1 + \omega_2) + (2\omega_2) \longleftrightarrow 21'_b + 15_a. \end{aligned} \tag{5.14.5}$$

For the second equation, $W_\delta \neq W_\delta^0$ as well. It corresponds to the sum

$$\begin{aligned} &(\omega_2) \otimes (\omega_2) = (2\omega_2) + (\omega_1 + \omega_3) + (\omega_4), \\ &\text{and} \\ &(\omega_2) \otimes (\omega_6) = (\omega_2 + \omega_6) + (\omega_1 + \omega_7) + (0). \end{aligned} \tag{5.14.6}$$

The first line is $7'_a + 21'_b$, the second one $1_a + 27_a$. The same argument as before implies

$$\begin{aligned} &(\omega_2 + \omega_6) \longleftrightarrow 27_a, \\ &(2\omega_2) + (\omega_1 + \omega_3) \longleftrightarrow 21'_b. \end{aligned} \tag{5.14.7}$$

But

$$(\omega_1) \otimes (\omega_3) = (\omega_1 + \omega_3) + (\omega_4), \tag{5.14.8}$$

and the dimension of ω_3 is $8 \cdot 7$, So $\omega_1 + \omega_3$ cannot have a 21-dimensional space of M-fixed vectors. It follows that

$$\begin{aligned} &(2\omega_2) \longleftrightarrow 21'_b, \\ &(\omega_3) \longleftrightarrow 21'_b \qquad \text{on } \delta_7^*. \end{aligned} \tag{5.14.9}$$

The dimension of $(\omega_1+\omega_2)$ is $8\cdot 21$, and

$$(\omega_1+\omega_2)\otimes(\omega_1)=(2\omega_1+\omega_2)+(\omega_1+\omega_3)+(2\omega_2). \tag{5.14.10}$$

It follows that neither $(2\omega_1+\omega_2)$ nor $(\omega_1+\omega_3)$ can be M-spherical. So

$$(4\omega_1)\longleftrightarrow 15'_a. \tag{5.14.11}$$

For the third equation, $W_\delta=W^0_\delta$. It corresponds to

$$\begin{aligned}(\omega_1+\omega_7)\otimes(\omega_1+\omega_7)&=(2\omega_1+2\omega_7)+(2\omega_1+\omega_6)+\\&+2(\omega_1+\omega_7)+(\omega_2+\omega_6)+(\omega_2+\omega_7)+(0).\end{aligned} \tag{5.14.12}$$

We conclude that neither of

$$(2\omega_1+\omega_6),\quad 2(\omega_1+\omega_7),\quad (\omega_2+\omega_7) \tag{5.14.13}$$

are M-spherical.

5.15. The following tensor products,

$$\begin{aligned}(\omega_1+\omega_6)\otimes(\omega_1)&=(2\omega_1+\omega_6)+(\omega_1+\omega_7)+(\omega_2+\omega_6),\\(\omega_1+2\omega_7)\otimes(\omega_1)&=(2\omega_1+2\omega_7)+(\omega_1+\omega_7)+(\omega_2+2\omega_7),\end{aligned} \tag{5.15.1}$$

and the results in section 5.14 allow us to conclude that

$$\begin{aligned}&(\omega_1)\longleftrightarrow 1_a,\\&(\omega_3)\longleftrightarrow 7'_a\\&(3\omega_1)\longleftrightarrow 15'_a\\&(\omega_1+\omega_2)\longleftrightarrow 21'_b,\\&(\omega_1+\omega_6)\longleftrightarrow 27_a,\\&(\omega_1+2\omega_7)\longleftrightarrow 35_b,\end{aligned} \tag{5.15.2}$$

on the δ_7-isotypic component. From

$$\begin{aligned}&(\omega_1+\omega_6)\otimes(\omega_1)=(2\omega_1+\omega_6)+(\omega_1+\omega_7)+(\omega_2+\omega_6)\\&(3\omega_1)\otimes(\omega_7)=(3\omega_1+\omega_7)+(2\omega_1)\end{aligned} \tag{5.15.3}$$

we conclude that $(3\omega_1+\omega_7)$ and $(2\omega_1+\omega_6)$ are not M-spherical.

5.16. Now consider

$$7'_a\otimes 7'_a=1_a+21_a+27_a, \tag{5.16.1}$$

and

$$\begin{aligned}&(\omega_3)\otimes(\omega_3)=(2\omega_3)+(\omega_1+\omega_5)+(\omega_2+\omega_4)+(\omega_6),\\&(\omega_3)\otimes(\omega_5)=(\omega_3+\omega_5)+(\omega_1+\omega_7)+(\omega_2+\omega_6)+(0).\end{aligned} \tag{5.16.2}$$

It follows that

$$(\omega_3+\omega_5)\longleftrightarrow 21_a, \tag{5.16.3}$$

and aside from the already listed cases, none of the representations occuring in (5.16.2) are M-spherical.

Next observe that

$$(\omega_5)\otimes(\omega_1+\omega_2)=(\omega_1+\omega_2+\omega_5)+(2\omega_1+\omega_6)+(\omega_1+\omega_7)+(\omega_2+\omega_6) \tag{5.16.4}$$

corresponds to

$$7'_a \otimes 21'_b = 27_a + 120_a. \tag{5.16.5}$$

Because $(2\omega_1 + \omega_6)$ does not have M-fixed vectors, we conclude that

$$(\omega_1 + \omega_2 + \omega_5) \longleftrightarrow 120_a. \tag{5.16.6}$$

The same argument using

$$(\omega_3) \otimes (\omega_1 + \omega_2) = (\omega_1 + \omega_2 + \omega_3) + (2\omega_1 + \omega_4) + (\omega_1 + \omega_5) + (\omega_2 + \omega_4) \tag{5.16.7}$$

shows that none of the representations in the RHS of (5.16.7) can be M-spherical.

5.17. Note that

$$\begin{aligned}(\omega_3) \otimes (\omega_1 + \omega_6) &= (\omega_1 + \omega_3 + \omega_6) + (2\omega_1) + \\ &\quad + (\omega_1 + \omega_2 + \omega_7) + (\omega_2) + (\omega_3 + \omega_7) + (\omega_4 + \omega_6) \\ (\omega_5) \otimes (\omega_1 + \omega_6) &= (\omega_1 + \omega_5 + \omega_6) + (\omega_1 + \omega_4 + \omega_7) + \\ &\quad + (\omega_1 + \omega_3) + (\omega_4) + (\omega_5 + \omega_7) + (2\omega_6).\end{aligned} \tag{5.17.1}$$

The representations in the first equation do not have M-fixed vectors, while the ones in the second equation correspond to

$$7'_a \otimes 27_a = 7'_a + 21'_b + 56'_a + 105'_a. \tag{5.17.2}$$

Since

$$(\omega_7) \otimes (\omega_1 + \omega_4) = (\omega_1 + \omega_4 + \omega_7) + (\omega_1 + \omega_3) + (\omega_4), \tag{5.17.3}$$

and the dimension of $(\omega_1 + \omega_4)$ is $8 \cdot 63$, we conclude that the dimension of the M-fixed vectors in $(\omega_1 + \omega_4 + \omega_7)$ is 56. So it follows that

$$\begin{aligned}(\omega_1 + \omega_5 + \omega_6) &\longleftrightarrow 105'_a, \\ (\omega_1 + \omega_4 + \omega_7) &\longleftrightarrow 56'_a.\end{aligned} \tag{5.17.4}$$

We also conclude that

$$(\omega_1 + \omega_4) \longleftrightarrow 7'_a + 56'_a. \tag{5.17.5}$$

5.18. Note that

$$\begin{aligned}(\omega_3) \otimes (\omega_1 + 2\omega_7) =& (\omega_1 + \omega_3 + 2\omega_7) + (\omega_1 + \omega_2 + \omega_7) + \\ &+ (\omega_3 + \omega_7) + (\omega_4 + 2\omega_7), \\ (\omega_3) \otimes (\omega_1 + 2\omega_7) =& (\omega_1 + \omega_3 + 2\omega_7) + (\omega_1 + \omega_2 + \omega_7) + \\ &+ (\omega_3 + \omega_7) + (\omega_4 + 2\omega_7).\end{aligned} \tag{5.18.1}$$

The first equation consists of representations without M-fixed vectors, the second one corresponds to

$$7'_a \otimes 35_b = 56'_a + 189'_b. \tag{5.18.2}$$

It follows that

$$(\omega_1 + \omega_5 + 2\omega_7) \leftrightarrow 189'_b. \tag{5.18.3}$$

5.19. Note that

$$\begin{aligned}(\omega_3)\otimes(3\omega_1)&=(3\omega_1+\omega_3)+(2\omega_1+\omega_4),\\(\omega_5)\otimes(3\omega_1)&=(3\omega_1+\omega_5)+(2\omega_1+\omega_6).\end{aligned}\tag{5.19.1}$$

The first equation does not contain any representations with M-fixed vectors, while the second one corresponds to

$$7'_a\otimes 15'_a=105_b.\tag{5.19.2}$$

So

$$(3\omega_1+\omega_5)\longleftrightarrow 105_b.\tag{5.19.3}$$

The proof of corollary 4.10 is complete.

5.20. The tensor products decompose

$$\begin{aligned}(\omega_1+\omega_2)\otimes(\omega_1+\omega_2)=&(2\omega_1+2\omega_2)+(3\omega_1+\omega_3)+(2\omega_1+\omega_4)+\\&+2(\omega_1+\omega_2+\omega_3)+(3\omega_2)+(\omega_2+\omega_4)+(2\omega_3),\\(\omega_1+\omega_2)\otimes(\omega_6+\omega_7)=&(\omega_1+\omega_2+\omega_6+\omega_7)+(2\omega_1+\omega_6)+\\&+(2\omega_1+2\omega_7)+2(\omega_1+\omega_7)+(\omega_2+\omega_6)+\\&+(\omega_2+2\omega_7)+(0).\end{aligned}\tag{5.20.1}$$

The first equation does not have any representations with M-fixed vectors, while the second one corresponds to

$$21'_b\otimes 21'_b=1_a+27_a+35_b+168_a+210_a.\tag{5.20.2}$$

Thus

$$(\omega_1+\omega_2+\omega_6+\omega_7)\longleftrightarrow 168_a+210_a.\tag{5.20.3}$$

5.21. We decompose the tensor products

$$\begin{aligned}&(\omega_1+\omega_2)\otimes(\omega_1+\omega_6)=(3\omega_1+\omega_7)+(2\omega_1+\omega_2+\omega_6)+(2\omega_1)+\\&2(\omega_1+\omega_2+\omega_7)+(\omega_1+\omega_3+\omega_6)+(2\omega_2+\omega_6)+(\omega_2)+(\omega_3+\omega_7),\\&(\omega_1+\omega_2)\otimes(\omega_2+\omega_7)=(\omega_1+2\omega_2+\omega_7)+(2\omega_1+\omega_3+\omega_7)+(2\omega_1+\omega_2)+\\&2(\omega_1+\omega_3)+(\omega_1+\omega_4+\omega_7)+(\omega_1+\omega_4+\omega_7)+(2\omega_2)+(\omega_2+\omega_3+\omega_7).\end{aligned}\tag{5.21.1}$$

Since also

$$21'_b\otimes 27_a=7'_a+21'_b+56'_a+105'_a+189'_b+189'_c,\tag{5.21.2}$$

and the first equation does not contain any representations with M-fixed vectors, we conclude

$$\begin{aligned}(2\omega_1+\omega_3+\omega_7)&\longleftrightarrow 189'_{b,c},\\(\omega_1+2\omega_2+\omega_7)&\longleftrightarrow 189'_{b,c}.\end{aligned}\tag{5.21.3}$$

The ambiguity is resolved in section 5.25.

5.22. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(3\omega_1)=(6\omega_1)+(4\omega_1+\omega_2)+(2\omega_1+2\omega_2)+(3\omega_2),\\&(3\omega_1)\otimes(3\omega_7)=(3\omega_1+3\omega_7)+(2\omega_1+2\omega_7)+(\omega_1+\omega_7)+(0).\end{aligned}\tag{5.22.1}$$

The first equation does not contain any representations with M-fixed vectors, the second one corresponds to

$$15'_a\otimes 15'_a=1_a+35_b+84_a+105_c.\tag{5.22.2}$$

It follows that

$$(3\omega_1+3\omega_2)\longleftrightarrow 84_a+105_c.\tag{5.22.3}$$

5.23. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+\omega_6)=(4\omega_1+\omega_6)+(3\omega_1+\omega_7)+\\&(2\omega_1+\omega_2+\omega_6)+(\omega_1+\omega_2+\omega_7),\\&(3\omega_1)\otimes(\omega_2+\omega_7)=(3\omega_1+\omega_2+\omega_7)+(2\omega_1+\omega_2)+\\&(2\omega_1+\omega_3+\omega_7)+(\omega_1+\omega_3).\end{aligned}\tag{5.23.1}$$

The first equation does not contain any M-spherical representations, while the second one corresponds to

$$15'_a\otimes 27_a=189'_b+216'_a.\tag{5.23.2}$$

We conclude that

$$(3\omega_1+\omega_2+\omega_7)\longleftrightarrow 216'_a.\tag{5.23.3}$$

5.24. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+\omega_2)=(4\omega_1+\omega_2)+(3\omega_1+\omega_3)+(2\omega_1+2\omega_2)+\\&(\omega_1+\omega_2+\omega_3),\\&(3\omega_1)\otimes(\omega_6+\omega_7)=(3\omega_1+\omega_6+\omega_7)+(2\omega_1+\omega_6)+\\&(2\omega_1+2\omega_7)+(\omega_1+\omega_7).\end{aligned}\tag{5.24.1}$$

The first equation does not contain any M-spherical representations, while the second on corresponds to

$$15'_a\otimes 21'_b=35_b+280_b\tag{5.24.2}$$

We conclude that

$$(3\omega_1+\omega_6+\omega_7)\longleftrightarrow 280_b.\tag{5.24.3}$$

5.25. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+2\omega_7)=(4\omega_1+2\omega_2)+(3\omega_1+\omega_7)+\\&(2\omega_1+\omega_2+2\omega_7)+(2\omega_1)+(\omega_1+\omega_2+\omega_7)+(\omega_2),\\&(3\omega_1)\otimes(2\omega_1+\omega_7)=(5\omega_1+\omega_7)+(4\omega_1)+\\&(3\omega_1+\omega_2+\omega_7)+(2\omega_1+\omega_2)+(\omega_1+2\omega_2+\omega_7)+(2\omega_2).\end{aligned}\tag{5.25.1}$$

The first equation does not contain any M-spherical representations, while the second on corresponds to

$$15'_a \otimes 35_b = 15'_a + 21'_b + 84'_a + 189'_c + 216'_a. \tag{5.25.2}$$

Combining this with section 5.21, we conclude that

$$\begin{aligned}(5\omega_1+\omega_7) &\longleftrightarrow 84'_a,\\ (\omega_1+2\omega_2+\omega_7) &\longleftrightarrow 189'_c,\\ (2\omega_1+\omega_3+\omega_7) &\longleftrightarrow 189'_b\end{aligned} \tag{5.25.3}$$

5.26. E8. Let (μ, V) be the fine K-type (ω_2). Then

$$\begin{aligned}&\mathrm{Hom}_M[V,V] \cong Ind_{W(E7A1)}^{W(E8)}[triv] = 1_x + 35_x + 84_x,\\ &(\omega_2)\otimes(\omega_2) = (2\omega_2)+(\omega_1+\omega_3)+(2\omega_1)+(\omega_2)+(\omega_4)+(0).\end{aligned} \tag{5.26.1}$$

This is because the stabilizer of δ_{120} in $W(E8)$ is $W(E7A1)$.

Similarly, if (μ, V) is the fine K-type $(2\omega_1)$,

$$\begin{aligned}&\mathrm{Hom}_M[V,V] \cong Ind_{W(D8)}^{W(E8)}[triv] = 1_x + 84_x + 50_x,\\ &(2\omega_1)\otimes(2\omega_1) = (4\omega_1)+(2\omega_1+\omega_2)+(2\omega_1)+(2\omega_2)+(\omega_2)+(0).\end{aligned} \tag{5.26.2}$$

Then (ω_3) has dimension $35\cdot 16$, so δ_{16} occurs 35 times. The tensor product is

$$(\omega_1)\otimes(\omega_3) = (\omega_1+\omega_3)+(\omega_2)+(\omega_4), \tag{5.26.3}$$

and thus the multiplicity of δ_1 in $(\omega_1+\omega_3)+(\omega_4)$ is 35, and (5.26.1) implies that the representation on the M fixed vectors is 35_x. On the other hand, $\dim\omega_4 = 1820$, so the multiplicity of δ_1 in (ω_4) is nonzero. This is because 1820 is not divisible by 3, but the contribution of the other characters occuring must be divisible by 3. Using (5.26.1) again, we conclude that

$$(\omega_4) \longleftrightarrow 35_x, \quad (2\omega_2) \longleftrightarrow 84_x. \tag{5.26.4}$$

We also conclude that the multiplicity of δ_1 in $(\omega_1+\omega_3)$ is zero, and

$$(\omega_3) \longleftrightarrow 35_x \tag{5.26.5}$$

(on the δ_{16} isotypic component).

Consider $(\omega_1+\omega_2)$ which has dimension $84\cdot 16$, so δ_{16} occurs 84 times. Then

$$(\omega_1+\omega_2)\otimes(\omega_1) = (2\omega_1+\omega_2)+(2\omega_1)+(\omega_1+\omega_3)+(2\omega_2)+(\omega_2). \tag{5.26.6}$$

Thus only $(2\omega_2)$ contains δ_1. Combined with (5.26.2) we get

$$(4\omega_1) \longleftrightarrow 50_x. \tag{5.26.7}$$

5.27. The representation ω_7 equals $8\delta_{16}$. Also, $\omega_8 = 8\delta_1 + \delta_{120}$, and the W-representation on $(\omega_8)^M$ is 8_x, because this is the representation on $\mathfrak{s}$. Then

$$(\omega_1) \otimes (\omega_7) = (\omega_1 + \omega_7) + (\omega_8). \tag{5.27.1}$$

So δ_1 does not occur in $(\omega_1 + \omega_7)$, and it occurs 8 times in (ω_8). This implies that $\mathrm{Hom}_M[\omega_1, \omega_7]$ is 8_x. We combine this with

$$\begin{aligned} &35_x \otimes 8_x = 160_z + 112_x + 8_x,\\ &(\omega_3) \otimes (\omega_7) = (\omega_3 + \omega_7) + (\omega_2 + \omega_8) + (\omega_1 + \omega_7) + (\omega_8). \end{aligned} \tag{5.27.2}$$

We have to distribute the other two representations among $(\omega_3 + \omega_7)$ and $(\omega_2 + \omega_8)$. We have

$$(\omega_2) \otimes (\omega_8) = (\omega_2 + \omega_8) + (\omega_2) + (\omega_8), \tag{5.27.3}$$

and δ_1 occurs 120 times. Thus $(\omega_2 + \omega_8)$ contains δ_1 112 times, and combined with (5.27.2) we get

$$(\omega_2 + \omega_8) \longleftrightarrow 112_x, \qquad (\omega_3 + \omega_7) \longleftrightarrow 160_z. \tag{5.27.4}$$

5.28. We apply the previous reasoning to

$$\begin{aligned} 35_x \otimes 35_x =& 210_x + 84_x + 567_x + 300_x + 35_x + 28_x + 1_x,\\ (\omega_3) \otimes (\omega_3) =& (2\omega_3) + (\omega_1 + \omega_3) + (\omega_1 + \omega_5) + (2\omega_1) +\\ &+ (2\omega_2) + (\omega_2 + \omega_4) + (\omega_2) + (\omega_4) + (\omega_6) + (0). \end{aligned} \tag{5.28.1}$$

Then

$$(\omega_1) \otimes (\omega_5) = (\omega_1 + \omega_5) + (\omega_6) + (\omega_4), \tag{5.28.2}$$

and the dimension of ω_5 being $273 \cdot 16$, we get that the multiplicity of δ_1 in $(\omega_6) + (\omega_1 + \omega_5)$ is 238. The tensor product decomposes

$$(\omega_2) \otimes (\omega_4) = (\omega_2 + \omega_4) + (\omega_1 + \omega_3) + (\omega_1 + \omega_5) + (\omega_2) + (\omega_4) + (\omega_6), \tag{5.28.3}$$

Using the fact that $\omega_4 = 35\delta_1 + 7\delta_{120} + 7\delta_{135}$, we conclude that δ_1 must occur 567 times in $(\omega_2 + \omega_4)$. The formula for (ω_4) is a consequence of (5.6.7) which yields the system

$$\begin{aligned} &1820 = 35 + 120a_1 + 135b_1,\\ &7020 = 120a_2 + 135b_2,\\ &a_1 + a_2 = 34,\\ &b_1 + b_2 = 35. \end{aligned} \tag{5.28.4}$$

The numbers 1820, 7020 are the dimensions of ω_4 and $(\omega_1 + \omega_3)$, and the two last equations come from

$$\begin{aligned} &(\omega_1) \otimes (\omega_1) = (2\omega_1) + (\omega_2) + (0)\\ &\delta_{16} \otimes \delta_{16} = \delta_{135} + \delta_{120} + \delta_1. \end{aligned} \tag{5.28.5}$$

Furthermore,

$$(\omega_8) \otimes (\omega_8) = (2\omega_8) + (\omega_6) + (\omega_4) + (\omega_2) + (0) \tag{5.28.6}$$

and δ_1 occurs 184 times. Since the dimension of (ω_6) is $8008 = a + 120b + 135c$, it follows that $a > 0$, and a cannot be divisible by 5. We conclude that the only choice from (5.28.3) given that 84_x, 567_x are not available, is

$$\begin{aligned}&(\omega_1+\omega_5) \longleftrightarrow 210_x,\\&(\omega_6) \longleftrightarrow 28_x,\\&(\omega_2+\omega_4) \longleftrightarrow 567_x,\\&(2\omega_3) \longleftrightarrow 300x,\\&(\omega_5) \longleftrightarrow 28_x + 35_x + 210_x.\end{aligned} \tag{5.28.7}$$

5.29. The previous arguments also imply that

$$(3\omega_1) \longleftrightarrow 50_x. \tag{5.29.1}$$

Then

$$\begin{aligned}&50_x \otimes 8_x = 400_x,\\&(3\omega_1) \otimes (\omega_7) = (3\omega_1+\omega_7) + (2\omega_1+\omega_8).\end{aligned} \tag{5.29.2}$$

But

$$(\omega_1+\omega_8)\otimes(\omega_1) = (2\omega_1+\omega_8) + (\omega_1+\omega_7) + (\omega_2+\omega_8) + (\omega_8). \tag{5.29.3}$$

Since $\dim(\omega_1+\omega_8) = 120\cdot 16$, the previous results show that δ_1 cannot occur in $(2\omega_1+\omega_8)$, and we conclude

$$(3\omega_1+\omega_7) \longleftrightarrow 400_x. \tag{5.29.4}$$

5.30. Recall again that

$$(3\omega_1) \longleftrightarrow 50_x, \qquad (\omega_3) \longleftrightarrow 35_x. \tag{5.30.1}$$

The tensor product decomposes

$$50_x \otimes 35_x = 1050_x + 700_x. \tag{5.30.2}$$

On the other hand,

$$\begin{aligned}&(3\omega_1)\otimes(\omega_3) = (3\omega_1+\omega_3) + (2\omega_1+\omega_2) + (2\omega_1+\omega_4) + (\omega_1+\omega_3),\\&(\omega_1)\otimes(\omega_1+\omega_4) =\\&\qquad (2\omega_1+\omega_4) + (\omega_2+\omega_4) + (\omega_1+\omega_5) + (\omega_1+\omega_3) + (\omega_4).\end{aligned} \tag{5.30.3}$$

In the first equation $2(\omega_1+\omega_4) + (3\omega_1+\omega_3)$ contains all the M-spherical vectors, and their dimensions add up to 1750. Counting dimensions of M-fixed vectors in the second equation, we find that the dimension of the M-fixed vectors in $(2\omega_1+\omega_4)$ is 700. Thus $(2\omega_1+\omega_4)$ must contain 700_x and $(3\omega_1+\omega_3)$ must contain 1050_x.

This completes the proof of corollary 4.11.

5.31. Recall that $50_x \longleftrightarrow (3\omega_1)$. The tensor product decomposes

$$50_x \otimes 50_x = 1_x + 50_x + 84_x + 168_y + 525_x + 700_{xx} + 972_x. \tag{5.31.1}$$

It corresponds to

$$\begin{aligned}&(3\omega_1) \otimes (3\omega_1) = \\ &(6\omega_1) + (4\omega_1 + \omega_2) + (4\omega_1) + (2\omega_1 + 2\omega_2) + (2\omega_1 + \omega_2) + \\ &(2\omega_1) + (3\omega_2) + (2\omega_2) + (\omega_2) + (0).\end{aligned} \tag{5.31.2}$$

Since

$$\begin{aligned}&(\omega_1 + \omega_2) \otimes \omega_1 = (2\omega_1 + \omega_2) + (2\omega_2) + (\omega_1 + \omega_3) + (2\omega_1) + (\omega_2), \\ &(\omega_1) \otimes (\omega_3) = (\omega_4) + (\omega_2) + (\omega_1 + \omega_3),\end{aligned} \tag{5.31.3}$$

and taking the earlier results into account, we find that $(\omega_1+\omega_3)$ and $(2\omega_1+\omega_2)$ cannot have M-fixed vectors. This shows that the sum of Weyl group representations in (5.31.1) corresponds to the M-fixed vectors in (5.31.2). We omit the details of how they match the irreducible modules. Finally, recall that $(\omega_1 + \omega_2)$ corresponds to 84_x, while ω_3 corresponds to 35_x. Then

$$84_x \otimes 35_x = 1344_x + 700_x + 210_x + 567_x + 84_x + 35_x, \tag{5.31.4}$$

while

$$\begin{aligned}&(\omega_1 + \omega_2) \otimes (\omega_3) = \\ &(2\omega_1 + \omega_2) + (2\omega_1 + \omega_4) + (2\omega_1) + (\omega_1 + \omega_2 + \omega_3) + 2(\omega_1 + \omega_3) + \\ &+ (\omega_1 + \omega_5) + (2\omega_2) + (\omega_2 + \omega_4) + (2\omega_2) + (\omega_2) + (\omega_4).\end{aligned} \tag{5.31.5}$$

The fact that the representation of $W(E_8)$ on the M-fixed vectors of $(\omega_1 + \omega_2 + \omega_3)$ is 1344_x follows from these two equations.

References

[B1] D. Barbasch, *The unitary dual of complex classical groups*, Inv. Math. **96** (1989), 103–176.

[B2] ———, *Unipotent representations for real reductive groups*, Proceedings of ICM, Kyoto 1990, Springer-Verlag, The Mathematical Society of Japan, 1990, pp. 769–777.

[B3] ———, *The spherical unitary dual for split classical p-adic groups*, Geometry and representation theory of real and p-adic groups (J. Tirao, D. Vogan, and J. Wolf, eds.), Birkhauser-Boston, Boston-Basel-Berlin, 1996, pp. 1–2.

[B4] ———, *The associated variety of a unipotent representation* preprint in preparation

[B5] ———, *The spherical unitary dual of split real and p-adic groups* preprint in preparation, http://www.math.cornell.edu/~barbasch

[BC] D. Barbasch, D. Ciubotaru, *The spherical unitary dual of split p-adic groups of type E*, preprint, in preparation

[BM1] D. Barbasch and A. Moy *A unitarity criterion for p-adic groups*, Inv. Math. **98** (1989), 19–38.

[BM2] ———, *Reduction to real infinitesimal character in affine Hecke algebras*, Journal of the AMS **6 no. 3** (1993), 611-635.

[BM3] ———, *Unitary spherical spectrum for p-adic classical groups*, Acta Applicandae Math **5 no. 1** (1996), 3-37.

[C] D. Ciubotaru *The Iwahori spherical unitary dual of the split group of type F4* preprint, to appear in Represent. Theory

[K] A. Knapp *Representation theory of real semisimple groups: an overview based on examples* Princeton University Press, Princeton, New Jersey (1986)

[KnV] A. Knapp, D. Vogan *Cohomological induction and unitary representations* Princeton University Press, (1995).

[L] G. Lusztig *Characters of reductive groups over a finite field* Annals of Math. Studies, Princeton University Press, Princeton New Jersey (1984)

[P] A. Pantano, Ph.D. thesis, Princeton University, (2004)

[T] M. Tadić *Classification of unitary representations in irreducible representations of general linear groups*, Ann. Sci. École Norm. Sup. **(4) 19, no. 3**, (1986), 335-382.

[V1] D. Vogan *The unitary dual for real G2* Inv. Math., vol. **116**, (1994), 677-791

[V2] D. Vogan *The unitary dual of GL(n) over an archimedean field*, Inv. Math., vol. **83**, (1986), 449-505.

[V3] ——— *Irreducible characters of semisimple groups IV* Duke Math. J., vol. **49**, (1982), 943-1073

[V4] ——— *Representations of real reductive Lie groups* Progress in Mathematics, vol. **15**, (1981) , Birkhäuser, Boston-Basel-Stuttgart

[ZE] A. Zelevinsky *Induced representations of reductive $\mathfrak{p}$-adic groups II. On irreducible representations of* GL(n), Ann. Sci. cole Norm. Sup. **(4) 13 no. 2**, 165-210

(D. Barbasch) Dept. of Mathematics, Cornell University, Ithaca, NY 14850
E-mail address: barbasch@math.cornell.edu