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SPHERICAL UNITARY DUAL FOR COMPLEX CLASSICAL GROUPS

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1. Introduction

The full unitary dual for the complex classical groups viewed as real Lie groups is computed in [B1]. This was close to 20 years ago. Since then there have been many advances, but the general problem of classifying the unitary dual is far from solved.

In [B1] the complementary series are given in terms of an algorithm. In this talk I will give a closed form of the answer inspired by the spherical unitary dual for split groups. The groups are $G = Sp(2n, \mathbb{C})$, and $G = SO(n, \mathbb{C})$.

2. General Background

2.1. **Complex groups.** Assume that G is a complex group viewed as a real group. Let θ be Cartan involution, and $H = TA$ be a $\theta-$stable Cartan subgroup with Lie algebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be a $\theta-$stable Cartan subalgebra. Let $B = HN$ be a Borel subgroup.

Admissible irreducible representations of G are parametrized by $\Delta(W)$ conjugacy classes of pairs $(\lambda_L, \lambda_R) \in \mathfrak{h} \times \mathfrak{h}$. More precisely the following theorem holds.

Theorem. *Let (λ_L, λ_R) be such that $\mu := \lambda_L - \lambda_R$ is integral. Write $\nu := \lambda_L + \lambda_R$. We can view μ as a weight of T and ν as a weight of $\mathfrak{a}$. Let*

$$X(\lambda_L, \lambda_R) := \mathrm{Ind}_B^G[\mathbb{C}_\mu \otimes \mathbb{C}_\nu \otimes \mathbb{1}]_{K-finite}.$$

Then the $K-$type with extremal weight μ occurs with multiplicity 1, so let $L(\lambda_L, \lambda_R)$ be the unique irreducible subquotient containing this $K-$type. E very irreducible admissible $(\mathfrak{g}_c, K)$ module is of the form $L(\lambda_L, \lambda_R)$. Two such modules are equivalent if and only if the parameters are conjugate by $\Delta(W) \subset W_c \cong W \times W$.

The module $L(\lambda_L, \lambda_R)$ is hermitian if and only if there is $w \in W$ such that $w\mu = \mu, \quad w\nu = -\overline{\nu}$.

Denote by $\mathfrak{g}_c$ the complexification of the Lie algebra of G. In order to determine the unitary dual of a group, one proceeds as follows.

(1) Reduce the problem to an algebraic one about $(\mathfrak{g}_c, K)$ modules.
(2) Determine the irreducible $(\mathfrak{g}_c, K)$ modules.
(3) Determine the hermitian irreducible $(\mathfrak{g}_c, K)$ modules.

(4) Determine the hermitian irreducible $(\mathfrak{g}_c, K)$ modules which are unitarizable.

Item (1) is dealt with by using results of Harish-Chandra. Items (2) and (3) are addressed by the theorem above. For (4) we make the following observation. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} \subset \mathfrak{g}$ be a parabolic subgroup. If π is a unitary $(\mathfrak{m}, K \cap M)$ module, then so is every composition factor of

$$\Pi := \mathrm{Ind}_M^G[\pi].$$

Here Ind is normalized Harish-Chandra induction. When it is irreducible, we say that Π is unitarily induced from π. One can tensor π with a character χ_t depending continuously on a parameter $t \in \mathbb{R}$. Under appropriate assumptions, one can conclude that $\Pi_t := \mathrm{Ind}_M^G[\pi \otimes \chi_t]$ is irreducible and unitary for some interval $[0, a)$. There is a version involving a multidimensional parameter. Such representations are called *complementary series.*

Stein Complementary series. Let $\check{\mathfrak{m}} \subset \check{\mathfrak{g}}$ be $gl(n) \times gl(n) \subset gl(2a)$. Let $\chi_\nu(a) := |\det(a)|^\nu$. Then

$$\Pi_\nu := Ind_{\mathfrak{m}}^{\mathfrak{g}}[\chi_\nu \otimes \overline{\chi_{-\nu}}]$$

has a complementary series for $0 \le \nu < 1/2$ in the real case, $0 \le \nu < 1$ in the complex case. This is called *Stein Complementary Series.*

Thus one way to organize the answer in (4) is to find a special set of **basic** representations $\mathcal{B}$ for each reductive $\mathfrak{g}$. The main property, in addition to unitarity, would be that a basic representation cannot be obtained by unitary induction or complementary series from a unitary representation on a proper Levi subgroup. The main result in both the split and complex case is the following.

Theorem. *A spherical representation $L(\chi)$ is unitary if and only if it is a complementary series from a unitarily induced module*

$$\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}[L(\chi_1) \otimes \cdots \otimes L(\chi_k) \otimes L(\chi_0)]$$

where $\mathfrak{m} = gl(a_1) \times \cdots \times gl(a_k) \times \mathfrak{g}(n_0)$, and $L(\chi_0)$ is basic and $L(\chi_i)$ are Stein complementary series.

To make the theorem explicit, we will define basic representations case by case, and describe the complementary series.

2.2. **Unipotent Representations.**

Definition (1)**.** *An irreducible $(\mathfrak{g}_c, K)$ module π is called unipotent if*

(1) *The annihilator of π in $U(\mathfrak{g}_c)$ is maximal*
(2) *π is unitary.*

For a given an infinitesimal character χ, there are standard ways to determine the representations satisfying (1). Condition (2) is more difficult, usually one replaces it by a condition on the infinitesimal character. Let $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ be a nilpotent orbit in the Lie algebra dual to $\mathfrak{g}$. Then fix a triple

$\{\check{e}, \check{h}, \check{f}\}$ such that $\check{e} \in \check{\mathcal{O}}$. The semisimple middle element $\check{h}$ defines an infinitesimal character $\chi_{\check{\mathcal{O}}} = (\check{h}/2, \check{h}/2)$.

Definition (2). *An irreducible $(\mathfrak{g}_c, K)$ module π is called special unipotent if*

(1) *The annihilator of π in $U(\mathfrak{g}_c)$ is maximal.*
(2) *The infinitesimal character is $\chi_{\check{\mathcal{O}}}$.*

A significant component of the problem of determining the unitary dual is to show that special unipotent representations are unitary. But special unipotent representations do not contain a basic subset. The list of representations below can be taken as a definition of *unipotent* representations. It is known which have the property of not being unitarily induced or complementary series. In the exceptional cases there is a similar list, but it is only conjectured that the representations are unitary and contain a basic set.

The basic representations are unipotent representations attached to particular nilpotent orbits.

3. Spherical dual for split groups

Let G be a split symplectic or orthogonal group over a local field $\mathbb{F}$ which is either $\mathbb{R}$ or a $p-$adic field. Fix a maximal compact subgroup K. In the real case, there is only one conjugacy class. In the $p-$adic case, let $K = G(\mathcal{R})$ where $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$, with $\mathcal{R}$ the ring of integers and $\mathcal{P}$ the maximal prime ideal. Fix also a rational Borel subgroup $B = AN$. Then $G = KB$. A representation (π, V) (admissible) is called spherical if $V^K \neq (0)$.

The classification of irreducible admissible spherical modules is well known. For every irreducible spherical π, there is a character $\chi \in \widehat{A}$ such that $\chi|_{A\cap K} = triv$, and π is the unique spherical subquotient of $Ind_B^G[\chi \otimes \mathbb{1}]$. We will call a character χ whose restriction to $A \cap K$ is trivial, *unramified.* Write $X(\chi)$ for the induced module (principal series) and $L(\chi)$ for the irreducible spherical subquotient. Two such modules $L(\chi)$ and $L(\chi')$ are equivalent if and only if there is an element in the Weyl group W such that $w\chi = \chi'$. An $L(\chi)$ admits a nondegenerate hermitian form if and only if there is $w \in W$ such that $w\chi = -\overline{\chi}$.

The character χ is called *real* if it takes only positive real values. For real groups, χ is real if and only if $L(\chi)$ has real infinitesimal character ([K], chapter 16). As is proved there, any unitary representation of a real reductive group with nonreal infinitesimal character is unitarily induced from a unitary representation with real infinitesimal character on a proper Levi component. So for real groups it makes sense to consider only real infinitesimal character. In the $p-$adic case, χ is called real if the infinitesimal character is real in the sense of [BM2]. The results in [BM1] show that the problem of determining the unitary irreducible representations with Iwahori fixed vectors is equivalent to the same problem for the Iwahori-Hecke algebra. In [BM2], it is shown that the problem of classifying the unitary dual for the Hecke algebra reduces to determining the unitary dual with real infinitesimal character of some smaller Hecke algebra (not necessarily one for a proper Levi subgroup). So for $p-$adic groups as well it is sufficient to consider only real χ.

So we start by parametrizing real unramified characters of A. Since G is split, $A \cong (\mathbb{F}^\times)^n$ where n is the rank. Define

$$\mathfrak{a}^* = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{3.0.1}$$

where $X^*(A)$ is the lattice of characters of the algebraic torus A. Each element $\nu \in \mathfrak{a}^*$ defines an unramified character χ_ν of A, characterized by the formula

$$\chi_\nu(\tau(f)) = |f|^{\langle \tau, \nu \rangle}, \qquad f \in \mathbb{F}^\times, \tag{3.0.2}$$

where τ is an element of the lattice of one parameter subgroups $X_*(A)$. Since the torus is split, each element of $X_- * (A)$ can be regarded as a homomorphism of $\mathbb{F}^\times$ into A. The pairing in the exponent in (3.0.2) corresponds to the natural identification of $\mathfrak{a}^*$ with $\mathrm{Hom}[X_*(A), \mathbb{R}]$. The map $\nu \longrightarrow \chi_\nu$ from $\mathfrak{a}^*$ to real unramified characters of A is an isomorphism. We will often identify the two sets writing simply $\chi \in \mathfrak{a}^*$.

Let $\check{G}$ be the (complex) dual group, and let $\check{A}$ be the torus dual to A. Then $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to $\check{\mathfrak{a}}$, the Lie algebra of $\check{A}$. So we can regard χ as an element of $\check{\mathfrak{a}}$. We attach to each χ a nilpotent orbit $\check{\mathcal{O}}(\chi)$ as follows. By the Jacobson-Morozov theorem, there is a 1-1 correspondence between nilpotent orbits $\check{\mathcal{O}}$ and $\check{G}$-conjugacy classes of Lie triples $\{\check{e}, \check{h}, \check{f}\}$; the correspondence satisfies $\check{e} \in \check{\mathcal{O}}$. Choose the Lie triple such that $\check{h} \in \check{\mathfrak{a}}$. Then there are many $\check{\mathcal{O}}$ such that χ can be written as $w\chi = \check{h}/2 + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$, the centralizer in $\check{\mathfrak{g}}$ of the triple. For example this is always possible with $\check{\mathcal{O}} = (0)$. The results in [BM1] guarantee that for any χ there is a unique $\check{\mathcal{O}}(\chi)$ satisfying

(1) there exists $w \in W$ such that $w\chi = \frac{1}{2}\check{h} + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$,
(2) if χ satisfies property (1) for any other $\check{\mathcal{O}}'$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}(\chi)$.

Here is another characterization of the orbit $\check{\mathcal{O}}(\chi)$. Let

$$\check{\mathfrak{g}}_1 := \{\ x \in \check{\mathfrak{g}}\ :\ [\chi, x] = x\ \}, \qquad \check{\mathfrak{g}}_0 := \{x \in \check{\mathfrak{g}}\ :\ [\chi, x] = 0\ \}.$$

Then $\check{G}_0$, the Lie group corresponding to the Lie algebra $\check{\mathfrak{g}}_0$ has an open dense orbit in $\check{\mathfrak{g}}_1$. Its $\check{G}$ saturation in $\check{\mathfrak{g}}$ is $\check{\mathcal{O}}(\chi)$.

3.1. **Nilpotent orbits.** In this section we attach a set of parameters to each nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$. Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple so that $\check{e} \in \check{\mathcal{O}}$, and let $\mathfrak{z}(\check{\mathcal{O}})$ be its centralizer. In order for χ to be a parameter attached to $\check{\mathcal{O}}$ we require that

$$\chi = \check{h}/2 + \nu, \qquad \nu \in \mathfrak{z}(\check{\mathcal{O}}), \quad \text{semisimple}, \tag{3.1.1}$$

but also that if

$$\chi = \check{h}'/2 + \nu', \qquad \nu' \in \mathfrak{z}(\check{\mathcal{O}}'), \quad \text{semisimple} \tag{3.1.2}$$

for another nilpotent orbit $\check{\mathcal{O}}' \subset \check{\mathfrak{g}}$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$. In [BM1], it is shown that the orbit of χ, uniquely determines $\check{\mathcal{O}}$ and the conjugacy class of $\nu \in \mathfrak{z}(\check{\mathcal{O}})$. We describe the pairs $(\check{\mathcal{O}}, \nu)$ explicitly in the classical cases.

Nilpotent orbits are parametrized by partitions

$$(\underbrace{1,\dots,1}_{r_1}, \underbrace{2,\dots,2}_{r_2}, \dots, \underbrace{j,\dots,j}_{r_j}, \dots), \tag{3.1.3}$$

satisfying the following constraints.

A_{n-1}: $gl(n)$, partitions of n.

B_n: $so(2n+1)$, partitions of $2n+1$ such that every even part occurs an even number of times.

C_n: $sp(2n)$, partitions of $2n$ such that every odd part occurs an even number of times.

D_n: $so(2n)$, partitions of $2n$ such that every even part occurs an even number of times. In the case when every part of the partition is even, there are two conjugacy classes of nilpotent orbits with the same Jordan blocks, labelled (I) and (II). The two orbits are conjugate under the action of $O(2n)$.

The Bala-Carter classification is particularly well suited for describing the parameter spaces attached to the $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$. An orbit is called *distinguished* if it does not meet any proper Levi component. In type A, the only distinguished orbit is the principal nilpotent orbit, where the partition has only one part. In the other cases, the distinguished orbits are the ones where each part of the partition occurs at most once. In particular, these are *even nilpotent orbits*, *i.e.* $\operatorname{ad}\check{h}$ has even eigenvalues only. Let $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ be an arbitrary nilpotent orbit. We need to put it into as small as possible Levi component $\check{\mathfrak{m}}$. In type A, if the partition is $(a_1, \dots, a_k)$, the Levi component is $\check{\mathfrak{m}}_{BC} = gl(a_1) \times \dots \times gl(a_k)$. In the other classical types, the orbit $\check{\mathcal{O}}$ meets a proper Levi component if and only if one of the $r_j > 1$. So separate as many pairs (a, a) from the partition as possible, and rewrite it as

$$((a_1, a_1), \dots, (a_k, a_k); d_1, \dots, d_l), \tag{3.1.4}$$

with $d_i < d_{i+1}$. The Levi component $\check{\mathfrak{m}}_{BC}$ attached to this nilpotent by Bala-Carter is

$$\check{\mathfrak{m}}_{BC} = gl(a_1) \times \dots \times gl(a_k) \times \check{\mathfrak{g}}_0(n_0) \quad n_0 := n - \sum a_i, \tag{3.1.5}$$

The distinguished nilpotent orbit is the one with partition (d_i) on $\check{\mathfrak{g}}(n_0)$, principal nilpotent on each $gl(a_j)$. The χ of the form $\check{h}/2 + \nu$ are the ones with ν an element of the center of $\check{\mathfrak{m}}_{BC}$. The explicit form is

$$(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i, \dots; \check{h}_0/2), \tag{3.1.6}$$

where $\check{h}_0$ is the middle element of a triple corresponding to (d_i). We will write out (d_i) and $\check{h}_0/2$ in sections 3.2-3.5.

We will consider more general cases where we write the partition of $\check{\mathcal{O}}$ in the form (3.1.4) so that the d_i are not necessarily distinct, but (d_i) forms an even nilpotent orbit in $\check{\mathfrak{g}}_0(n_0)$.

The parameter χ determines an irreducible spherical module $L(\chi)$ for G as well as an $L_M(\chi)$ for $M = M_{BC}$ of the form

$$L_1(\chi_1) \otimes \cdots \otimes L_k(\chi_k) \otimes L_0(\chi_0), \tag{3.1.7}$$

where the $L_i(\chi_i)$ $i = 1, \dots, k$ are one dimensional.

3.2. **G of Type A.** We write the $\check{h}/2$ for a nilpotent $\check{\mathcal{O}}$ corresponding to $(a_1, \dots, a_k)$ with $a_i \le a_{i+1}$ as

$$(\dots; -\frac{a_i - 1}{2}, \dots, \frac{a_i - 1}{2}; \dots).$$

The parameters of the form $\chi = \check{h}/2 + \nu$ are then

$$(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots). \tag{3.2.1}$$

Conversely, given a parameter as a concatenation of strings

$$\chi = (\dots; A_i, \dots, B_i; \dots), \tag{3.2.2}$$

it is of the form $\check{h}/2+\nu$ where $\check{h}$ is the neutral element for the nilpotent orbit with partition $(A_i + B_i + 1)$ (the parts need not be in any particular order) and $\nu_i = \frac{A_i - B_i}{2}$. We recall the following well known result about closures of nilpotent orbits.

Lemma. *Assume $\check{\mathcal{O}}$ and $\check{\mathcal{O}}'$ correspond to the (increasing) partitions $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ respectively, where some of the a_i or b_j may be zero in order to have the same number k. The following are equivalent*

1. $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$.
2. $\sum_{i \ge s} a_i \ge \sum_{i \ge s} b_i$ *for all* $k \ge s \ge 1$.

Proposition. *A parameter χ as in (3.2.1) is attached to $\check{\mathcal{O}}$ in the sense of satisfying (3.1.1) and (3.1.2) if and only if it is nested.*

Proof. Assume the strings are not nested. There must be two strings

$$(A, \dots, B), \qquad (C, \dots, D) \tag{3.2.3}$$

such that $A - C \in \mathbb{Z}$, and $A < C \le B < D$, or $C = B + 1$. Then by conjugating χ by the Weyl group to a χ', we can rearrange the coordinates of the two strings in (3.2.3) so that the strings

$$(A, \dots, D), \quad (C, \dots B), \qquad \text{or} \qquad (A, \dots, D). \tag{3.2.4}$$

appear. Then by the lemma, $\chi' = \check{h}'/2 + \nu'$ for a strictly larger nilpotent $\check{\mathcal{O}}'$.

Conversely, assume $\chi = \check{h}/2 + \nu$, so it is written as strings, and they are nested. The nilpotent orbit for which the neutral element is $\check{h}/2$ has partition given by the lengths of the strings, say $(a_1, \dots a_k)$ in increasing order. If χ is nested, then a_k is the length of the longest string of entries we can extract from the coordinates of χ, a_{k-1} the longest string we can extract from the remaining coordinates and so on. Then (2) of lemma 3.2 precludes the possibility that some conjugate χ' equals $\check{h}'/2 + \nu'$ for a strictly larger nilpotent orbit. □

3.3. **G of Type B.** Rearrange the parts of the partition of $\check{\mathcal{O}} \subset sp(2n, \mathbb{C})$, in the form (3.1.4),

$$((a_1, a_1), \dots, (a_k, a_k); 2x_0, \dots, 2x_{2m}) \tag{3.3.1}$$

The d_i have been relabeled as $2x_i$ and a $2x_0 = 0$ is added if necessary, to insure that there is an odd number. The x_i are integers, because all the odd parts of the partition of $\check{\mathcal{O}}$ occur an even number of times, and were therefore extracted as (a_i, a_i). The χ of the form $\check{h}/2 + \nu$ are

$$\begin{aligned}(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots;\\ \underbrace{1/2, \dots, 1/2}_{n_{1/2}}, \dots, \underbrace{x_{2m} - 1/2, \dots, x_{2m} - 1/2}_{n_{x_{2m}-1/2}}).\end{aligned} \tag{3.3.2}$$

where

$$n_{l-1/2} = \#\{x_i \geq l\}. \tag{3.3.3}$$

Lemma 3.2 holds for this type verbatim. So the following proposition holds.

Proposition. *A parameter $\chi = \check{h}/2 + \nu$ cannot be conjugated to one of the form $\check{h}'/2 + \nu'$ for any larger nilpotent $\check{\mathcal{O}}'$ if and only if*

1. *the set of strings satisfying $\frac{a_i-1}{2} + \nu_i - \frac{a_j-1}{2} - \nu_j \in \mathbb{Z}$ are nested.*
2. *the strings satisfying $\frac{a_i-1}{2} + \nu_i \in 1/2 + \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1/2 < -\frac{a_i-1}{2} + \nu_i$ or there is j such that*

$$x_j + 1/2 < -\frac{a_i - 1}{2} + \nu_i \leq \frac{a_i - 1}{2} + \nu_i < x_{j+1} + 1/2. \tag{3.3.4}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is obtained from $\check{\mathfrak{m}}_{BC}$ as follows. Consider the strings for which a_i is even, and $\nu_i = 0$. If a_i is not equal to any $2x_j$, then remove one pair (a_i, a_i), and add two $2x_j = a_i$ to the last part of (3.3.1). For example, if the nilpotent orbit $\check{\mathcal{O}}$ is

$$(2, 2, 2, 3, 3, 4, 4), \tag{3.3.5}$$

then the parameters of the form $\check{h}/2 + \nu$ are

$$\begin{aligned}(&-1/2 + \nu_1, 1/2 + \nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2;\\ &-3/2 + \nu_3, -1/2 + \nu_3, 1/2 + \nu_3, 3/2 + \nu_3; 1/2)\end{aligned} \tag{3.3.6}$$

The Levi component is $\check{\mathfrak{m}}_{BC} = gl(2) \times gl(3) \times gl(4) \times \check{\mathfrak{g}}(1)$. If $\nu_3 \neq 0$, then $\check{\mathfrak{m}}_{BC} = \check{\mathfrak{m}}_{KL}$. But if $\nu_3 = 0$, then $\check{\mathfrak{m}}_{KL} = gl(2) \times gl(3) \times \check{\mathfrak{g}}(5)$. The parameter is rewritten

$$\begin{aligned} &\check{\mathcal{O}} \longleftrightarrow ((2,2)(3,3);2,4,4) &(3.3.7)\\ &\chi \longleftrightarrow (-1/2+\nu_1, 1/2+\nu_1; -1+\nu_2, \nu_2, 1+\nu_2; 1/2, 1/2, 1/2, 3/2, 3/2). \end{aligned}$$

The explanation is as follows. For a partition (3.1.3),

$$\mathfrak{z}(\check{\mathcal{O}}) = sp(r_1) \times so(r_2) \times sp(r_3) \times \dots \tag{3.3.8}$$

and the centralizer in $\check{G}$ is a product of $Sp(r_{2j+1})$ and $O(r_{2j})$, *i.e.* Sp for the odd parts, O for the even parts. Thus the component group $A(\check{h}, \check{e})$, which by [BV2] also equals $A(\check{e})$, is a product of $\mathbb{Z}_2$, one for each $r_{2j} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$. In general $A_{M_{BC}}(\chi, \check{e}) = A_{M_{BC}}(\check{e})$ embeds canonically into $A(\chi, \check{e})$, but the two are not necesarily equal. In this case they are unless one of the $\nu_i = 0$ for an even a_i with the additional property that there is no $2x_j = a_i$.

We can rewrite each of the remaining strings

$$(-\frac{a_i-1}{2}+\nu_i, \dots, \frac{a_i-1}{2}+\nu_i) \tag{3.3.9}$$

as

$$\begin{aligned} \chi_i :=& (f_i+\tau_i, f_i+1+\tau_i, \dots, F_i+\tau_i), &(3.3.10)\\ &\text{satisfying}\\ &f_i \in \mathbb{Z}+1/2, \quad 0 \le \tau_i \le 1/2, \quad F_i = f_i + a_i, . &(3.3.11)\\ &|f_i+\tau_i| \ge |F_i+\tau_i| \text{ if } \tau_i = 1/2 \end{aligned}$$

This is done as follows. We can immediately get an expression like (3.3.10) with $0 \le \tau_i < 1$, by defining f_i to be the largest element in $\mathbb{Z}+1/2$ less than or equal to $-\frac{a_i-1}{2}+\nu_i$. If $\tau_i \le 1/2$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i-1$, and τ_i by $1-\tau_i$. The presentation of the strings subject to (3.3.11) is unique except when $\tau_i = 1/2$. In this case the argument just given provides the presentation $(f_i+1/2, \dots, F_i+1/2)$, but also provides the presentation

$$(-F_i-1+1/2, \dots, -f_i-1+1/2). \tag{3.3.12}$$

We choose between (3.3.10) and (3.3.12) the one whose leftmost term is larger in absolute value. That is, we require $|f_i+\tau_i| \ge |F_i+\tau_i|$ whenever $\tau_i = 1/2$.

3.4. **G of Type C.** Rearrange the parts of the partition of $\check{\mathcal{O}} \subset so(2n+1, \mathbb{C})$, in the form (3.1.4),

$$((a_1,a_1), \dots, (a_k,a_k); 2x_0+1, \dots, 2x_{2m}+1); \tag{3.4.1}$$

The d_i have been relabeled as $2x_i + 1$. In this case it is automatic that there is an odd number of nonzero x_i. The x_i are integers, because all the even parts of the partition of $\check{\mathcal{O}}$ occur an even number of times, and were threrefore extracted as (a_i, a_i). The χ of the form $\check{h}/2 + \nu$ are

$$(\dots; -\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i; \dots; \underbrace{0, \dots, 0}_{n_0}, \dots, \underbrace{x_{2m}, \dots, x_{2m}}_{n_{x_{2m}}}). \quad (3.4.2)$$

where

$$n_l = \begin{cases} m & \text{if } l = 0, \\ \#\{x_i \geq l\} & \text{if } l \neq 0. \end{cases} \quad (3.4.3)$$

Lemma 3.2 holds for this type verbatim. So the following proposition holds.

Proposition. *A parameter $\chi = \check{h}/2 + \nu$ cannot be conjugated to one of the form $\check{h}'/2 + \nu'$ for any larger nilpotent $\check{\mathcal{O}}'$ if and only if*

(1) *the set of strings satisfying $\frac{a_i-1}{2} + \nu_i - \frac{a_j-1}{2} - \nu_j \in \mathbb{Z}$ are nested.*
(2) *the strings satisfying $\frac{a_i-1}{2} + \nu_i \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1 < -\frac{a_i-1}{2} + \nu_i$ or there is j such that*

$$x_j + 1 < -\frac{a_i-1}{2} + \nu_i \leq \frac{a_i-1}{2} + \nu_i < x_{j+1} + 1. \quad (3.4.4)$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is obtained from $\check{\mathfrak{m}}_{BC}$ as follows. Consider the strings for which a_i is odd and $\nu_i = 0$. If a_i is not equal to any $2x_j + 1$, then remove one pair (a_i, a_i), and add two $2x_j + 1 = a_i$ to the last part of (3.4.1). For example, if the nilpotent orbit is

$$(1, 1, 1, 3, 3, 4, 4) = ((1, 1), (3, 3), (4, 4); 1), \quad (3.4.5)$$

then the parameters of the form $\check{h}/2 + \nu$ are

$$\begin{aligned} &(\nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2; \\ &-3/2 + \nu_3, -1/2 + \nu_3, 1/2 + \nu_3, 3/2 + \nu_3) \end{aligned} \quad (3.4.6)$$

The Levi component is $\check{\mathfrak{m}}_{BC} = gl(1) \times gl(3) \times gl(4)$. If $\nu_2 \neq 0$, then $\check{\mathfrak{m}}_{BC} = \check{\mathfrak{m}}_{KL}$. But if $\nu_2 = 0$, then $\check{\mathfrak{m}}_{KL} = gl(1) \times gl(4) \times \check{\mathfrak{g}}(3)$. The parameter is rewritten

$$\begin{aligned} &\check{\mathcal{O}} \longleftrightarrow ((1, 1), (4, 4); 1, 3, 3) \quad (3.4.7) \\ &\chi \longleftrightarrow (\nu_1; -3/2 + \nu_3, -1/2 + \nu_3, 1/2 + \nu_3, 3/2 + \nu_3; 0, 1, 1). \end{aligned}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is unchanged if $\nu_1 = 0$.

The explanation is as follows. For a partition (3.1.3),

$$\mathfrak{z}(\check{\mathcal{O}}) = so(r_1) \times sp(r_2) \times so(r_3) \times \dots \quad (3.4.8)$$

and the centralizer in $\check{G}$ is a product of $O(r_{2j+1})$ and $Sp(r_{2j})$, *i.e.* O for the odd parts, Sp for the even parts. Thus the component group is a product of $\mathbb{Z}_2$, one for each $r_{2j+1} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$, and so $A_{M_{BC}}(\chi, \check{e}) = A(\chi, \check{e})$ unless one of the $\nu_i = 0$ for an odd a_i with the additional property that there is no $2x_j + 1 = a_i$.

We can rewrite each of the remaining strings

$$(-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i) \tag{3.4.9}$$

as

$$\begin{aligned}\chi_i :=&(f_i+\tau_i, f_i+1+\tau_i,\dots,F_i+\tau_i), && (3.4.10)\\ &\text{satisfying}\\ &f_i\in\mathbb{Z},\quad 0\le\tau_i\le 1/2,\quad F_i=f_i+a_i && (3.4.11)\\ &|f_i+\tau_i|\ge|F_i+\tau_i| \text{ if } \tau_i=1/2.\end{aligned}$$

This is done as follows. We can immediately get an expression like (3.4.10) with $0\le\tau_i<1$, by defining f_i to be the largest element in $\mathbb{Z}$ less than or equal to $-\frac{a_i-1}{2}+\nu_i$. If $\tau_i\le 1/2$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i-1$, and τ_i by $1-\tau_i$. The presentation of the strings subject to (3.4.11) is unique except when $\tau_i=1/2$. In this case the argument just given also provides the presentation

$$(-F_i-1+1/2,\dots,-f_i-1+1/2). \tag{3.4.12}$$

We choose between (3.4.10) and (3.4.12) the one whose leftmost term is larger in absolute value. That is, we require $|f_i+\tau_i|\ge|F_i+\tau_i|$ whenever $\tau_i=1/2$.

3.5. **G of Type D.** Rearrange the parts of the partition of $\check{\mathcal{O}}\subset so(2n,\mathbb{C})$, in the form (3.1.4),

$$((a_1,a_1),\dots,(a_k,a_k);2x_0+1,\dots,2x_{2m-1}+1) \tag{3.5.1}$$

The d_i have been relabeled as $2x_i+1$. In this case it is automatic that there is an even number of nonzero $2x_i+1$. The x_i are integers, because all the even parts of the partition of $\check{\mathcal{O}}$ occur an even number of times, and were therefore extracted as (a_i,a_i). The χ of the form $\check{h}/2+\nu$ are

$$(\dots;-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i;\dots;\underbrace{0,\dots,0}_{n_0},\dots,\underbrace{x_{2m-1},\dots,x_{2m-1}}_{n_{x_{2m-1}}}). \tag{3.5.2}$$

where

$$n_l=\begin{cases}m & \text{if } l=0,\\ \#\{x_i\ge l\} & \text{if } l\ne 0.\end{cases} \tag{3.5.3}$$

Lemma 3.2 holds for this type verbatim. So the following proposition holds.

Proposition. *A parameter $\chi=\check{h}/2+\nu$ cannot be conjugated to one of the form $\check{h}'/2+\nu'$ for any larger nilpotent $\check{\mathcal{O}}'$ if and only if*

(1) *the set of strings satisfying $\frac{a_i-1}{2}+\nu_i-\frac{a_j-1}{2}-\nu_j\in\mathbb{Z}$ are nested.*

(2) *the strings satisfying* $\frac{a_i-1}{2}+\nu_i\in\mathbb{Z}$ *satisfy the additional condition that either* $x_{2m-1}+1<-\frac{a_i-1}{2}+\nu_i$ *or there is* j *such that*

$$x_j+1<-\frac{a_i-1}{2}+\nu_i\le\frac{a_i-1}{2}+\nu_i<x_{j+1}+1. \tag{3.5.4}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is obtained from $\check{\mathfrak{m}}_{BC}$ as follows. Consider the strings for which a_i is odd and $\nu_i=0$. If a_i is not equal to any $2x_j+1$, then remove one pair (a_i,a_i), and add two $2x_j+1=a_i$ to the last part of (3.5.1). For example, if the nilpotent orbit is

$$(1,1,3,3,4,4), \tag{3.5.5}$$

then the parameters of the form $\check{h}/2+\nu$ are

$$\begin{aligned}&(\nu_1;-1+\nu_2,\nu_2,1+\nu_2;\\&-3/2+\nu_3,-1/2+\nu_3,1/2+\nu_3,3/2+\nu_3)\end{aligned} \tag{3.5.6}$$

The Levi component is $\check{\mathfrak{m}}_{BC}=gl(1)\times gl(3)\times gl(4)$. If $\nu_2\neq0$ and $\nu_1\neq0$, then $\check{\mathfrak{m}}_{BC}=\check{\mathfrak{m}}_{KL}$. If $\nu_2=0$ and $\nu_1\neq0$, then $\check{\mathfrak{m}}_{KL}=\check{\mathfrak{g}}(3)\times gl(1)\times gl(4)$. If $\nu_2\neq0$ and $\nu_1=0$, then $\check{\mathfrak{m}}_{KL}=gl(3)\times gl(4)\times\check{\mathfrak{g}}(1)$. If $\nu_1=\nu_2=0$, then $\check{\mathfrak{m}}_{KL}=gl(4)\times\check{\mathfrak{g}}(4)$. The parameter is rewritten

$$\begin{aligned}&\check{\mathcal{O}}\longleftrightarrow((1,1),(4,4);3,3)\\&\chi\longleftrightarrow(\nu_1;-3/2+\nu_3,-1/2+\nu_3;1/2+\nu_3,3/2+\nu_3;0,1,1).\end{aligned} \tag{3.5.7}$$

The explanation is as follows. For a partition (3.1.3),

$$\mathfrak{z}(\check{\mathcal{O}})=so(r_1)\times sp(r_2)\times so(r_3)\times\dots \tag{3.5.8}$$

and the centralizer in $\check{G}$ is a product of $O(r_{2j+1})$ and $Sp(r_{2j})$, *i.e.* O for the odd parts, Sp for the even parts. Thus the component group is a product of $\mathbb{Z}_2$, one for each $r_{2j+1}\neq0$. Then $A(\chi,\check{e})=A(\nu,\check{h},\check{e})$, and so $A_{M_{BC}}(\chi,\check{e})=A(\chi,\check{e})$ unless one of the $\nu_i=0$ for an odd a_i with the additional property that there is no $2x_j+1=a_i$.

We can rewrite each of the remaining strings

$$(-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i) \tag{3.5.9}$$

as

$$\chi_i:=(f_i+\tau_i,f_i+1+\tau_i,\dots,F_i+\tau_i), \tag{3.5.10}$$

$$\begin{aligned}&\text{satisfying}\quad f_i\in\mathbb{Z},\quad 0\le\tau_i\le1/2,\quad F_i=f_i+a_i\\&|f_i+\tau_i|\ge|F_i+\tau_i|\text{ if }\tau_i=1/2.\end{aligned} \tag{3.5.11}$$

This is done as in types B and C, but see the remarks which have to do with the fact that $-Id$ is not in the Weyl group. We can immediately get an expression like (3.5.10) with $0\le\tau_i<1$, by defining f_i to be the largest element in $\mathbb{Z}$ less than or equal to $-\frac{a_i-1}{2}+\nu_i$. If $\tau_i\le1/2$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i-1$, and τ_i by $1-\tau_i$. The presentation of the strings subject to (3.5.11) is unique except when $\tau_i = 1/2$. In this case the argument just given also provides the presentation

$$(-F_i - 1 + 1/2, \dots, -f_i - 1 + 1/2). \tag{3.5.12}$$

We choose between (3.5.10) and (3.5.12) the one whose leftmost term is larger in absolute value. That is, we require $|f_i + \tau_i| \geq |F_i + \tau_i|$ whenever $\tau_i = 1/2$.

Remarks

(1) A (real) spherical parameter χ is hermitian if and only if there is $w \in W(D_n)$ such that $w\chi = -\chi$. This is the case if the parameter has a coordinate equal to zero, or if none of the coordinates are 0, but then n must be even.

(2) Assume the nilpotent orbit $\check{\mathcal{O}}$ is very even, *i.e.* all the parts of the partition are even (and therefore occur an even number of times). The nilpotent orbits labelled (I) and (II) are characterized by the fact that $\check{\mathfrak{m}}_{BC}$ is of the form

$$(I) \longleftrightarrow gl(a_1) \times \cdots \times gl(a_{k-1}) \times gl(a_k),$$
$$(II) \longleftrightarrow gl(a_1) \times \cdots \times gl(a_{k-1}) \times gl(a_k)'.$$

The last gl factors differ by which extremal root of the fork at the end of the diagram for D_n is in the Levi component. The string for k is

$$(I) \longleftrightarrow (-\frac{a_k - 1}{2} + \nu_k, \dots, \frac{a_k - 1}{2} + \nu_k),$$
$$(II) \longleftrightarrow (-\frac{a_k - 1}{2} + \nu_k, \dots \frac{a_k - 3}{2} + \nu_k, -\frac{a_k - 1}{2} - \nu_k).$$

We can put the parameter in the form (3.5.10) and (3.5.11), because all strings are even length. In any case (I) and (II) are conjugate by the outer automorphism, and for unitarity it is enough to consider the case of (I).

The assignment of a nilpotent orbit (I) or (II) to a parameter is unambiguous. If a χ has a coordinate equal to 0, it might be written as $h_I/2 + \nu_I$ or $h_{II}/2 + \nu_{II}$. But then it can also be written as $h'/2 + \nu'$ for a larger nilpotent orbit. For example, in type D_2, the two cases are $(2,2)_I$ and $(2,2)_{II}$, and we can write

$$(I) \longleftrightarrow (1/2, -1/2) + (\nu, \nu),$$
$$(II) \longleftrightarrow (1/2, 1/2) + (\nu, -\nu).$$

The two forms are not conjugate unless the parameter contains a 0. But then it has to be $(1, 0)$ and this corresponds to $(1, 3)$, the larger principal nilpotent orbit.

(3) Because we can only change an even number of signs using the Weyl group, we might not be able to change all the signs of a string. We can always do this if the parameter contains a coordinate equal to 0, or if the length of the string is even. If there is an odd length

string, and none of the coordinates of χ are 0, changing all of the signs of the string cannot be achieved unless some other coordinate changes sign as well. However if $\chi = \check{h}/2 + \nu$ cannot be made to satisfy (3.5.10) and (3.5.11), then χ', the parameter obtained from χ by applying the outer automorphism, can. Since $L(\chi)$ and $L(\chi')$ are either both unitary of both nonunitary, it is enough to consider just the cases that can be made to satisfy (3.5.10) and (3.5.11). For example, the parameters

$$(-1/3, 2/3, 5/3; -7/4, -3/4, 1/4),$$
$$(-5/3, -2/3, 1/3; -7/4, -3/4, 1/4)$$

in type D_6 are of this kind. Both parameters are in a form satisfying (3.5.10) but only the second one satisfies (3.5.11). The first one cannot be conjugated by $W(D_6)$ to one satisfying (3.5.11).

4. The Main Result

4.1. Recall that $\check{G}$ is the (complex) dual group, and $\check{A} \subset \check{G}$ the maximal torus dual to A. Assuming as we may that the parameter is real, a spherical irreducible representation corresponds to an orbit of a hyperbolic element $\chi \in \check{\mathfrak{a}}$, the Lie algebra of $\check{A}$. In section 2 we attached a nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$ with partition $(\underbrace{a_1, \dots a_1}_{r_1}, \dots, \underbrace{a_k, \dots, a_k}_{r_k})$ to such a parameter. Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple attached to $\check{\mathcal{O}}$. Let $\chi := \check{h}/2 + \nu$ satisfy (3.1.1)-(3.1.2).

Definition. *A representation $L(\chi)$ is said to be in the complementary series for $\check{\mathcal{O}}$, if the parameter χ is attached to $\check{\mathcal{O}}$ in the sense of satisfying (3.1.1) and (3.1.2), and is unitary.*

We will describe the complementary series explicitly in coordinates.

The centralizer $Z_{\check{G}}(\check{e}, \check{h}, \check{f})$ has Lie algebra $\mathfrak{z}(\check{\mathcal{O}})$ which is a product of $sp(r_l, \mathbb{C})$ or $so(r_l, \mathbb{C})$, $1 \le l \le k$, according to the rule

Ǧ of type B, D: $sp(r_l)$ for a_l even, $so(r_l)$ for a_l odd,

Ǧ of type C: $sp(r_l)$ for a_l odd, $so(r_l)$ for a_l even.

The parameter ν determines a spherical irreducible module $L_{\check{\mathcal{O}}}(\nu)$ for the split group whose dual is $Z_{\check{G}}(\check{e}, \check{h}, \check{f})^0$. It is attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$.

Theorem. *The complementary series attached to $\check{\mathcal{O}}$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$. For the trivial orbit* (0) *in each of the classical cases, the complementary series are*

G of type B:

$$0 \le \nu_1 \le \cdots \le \nu_k < 1/2.$$

G of type C, D:

$$0 \le \nu_1 \le \cdots \le \nu_k \le 1/2 < \nu_{k+1} < \cdots < \nu_{k+l} < 1$$

so that $\nu_i + \nu_j \le 1$. There are

(1) *an even number of* ν_i *such that* $1-\nu_{k+1}<\nu_i\le 1/2$,
(2) *for every* $1\le j\le l$, *there is an odd number of* ν_i *such that* $1-\nu_{k+j+1}<\nu_i<1-\nu_{k+j}$.
(3) *In type* D *of odd rank,* $\nu_1=0$ *or else the parameter is not hermitian.*

Remarks.

(1) The complementary series for $\check{\mathcal{O}}=(0)$ consists of representations which are both spherical and generic in the sense that they have Whittaker models.
(2) The condition that $\nu_i+\nu_j\neq 1$ implies that in types C,D there is at most one $\nu_k=1/2$.
(3) In the case of $\check{\mathcal{O}}\neq(0)$, $\chi=\check{h}/2+\nu$, and each of the coordinates ν_i for the parameter on $\mathfrak{z}(\check{\mathcal{O}})$ comes from a string, *i.e.* each ν_i comes from $(-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i)$. The parameter does not satisfy (3.5.11). For (3.5.11) to hold, it suffices to change ν_{k+j} for types C, D to $1-\nu_{k+j}$. More precisely, for $1/2<\nu_{k+j}<1$ the connection with the strings in the form (3.5.10) and (3.5.11) is as follows. Write $(-\frac{a_{k+j}-1}{2}+\nu_{k+j},\dots,\frac{a_{k+j}-1}{2}+\nu_{k+j})$ as $(-\frac{a_{k+j}-3}{2}+(\nu_{k+j}-1),\dots,\frac{a_{k+j}+1}{2}+(\nu_{k+j}-1))$ and then conjugate each entry to its negative to form $(-\frac{a_{k+j}-3}{2}+\nu'_{k+j},\dots,\frac{a_{k+j}+1}{2}+\nu'_{k+j})$, with $0<\nu'_{k+j}=1-\nu_{k+j}<1/2$.

5. Spherical modules for complex groups

5.1. G of Type B.

I: The partition for $\check{\mathcal{O}}$ is labeled

$$(m_0\le m_1\le\dots\le m_{2p'}) \tag{5.1.1}$$

by adding a 0 to have an odd number. Then a given size *starts at an even or odd label.*

II: From each size extract pairs

$$(a_1,a_1)\dots(a_k,a_k) \tag{5.1.2}$$

as many as possible *leaving one pair for each even size starting at an* **odd** *label.* Thus to each $\check{\mathcal{O}}$ we have associated

$$((a_1a_1)\dots(a_ka_k);2x_0,\dots,2x_{2p}). \tag{5.1.3}$$

Pair up the x_i

$$(x_0x_1)\dots(x_{2p-2}x_{2p-1})(x_{2p}) \tag{5.1.4}$$

Note that $x_{2i}<x_{2i+1}$. Form a vector

$$\underline{\epsilon}=(\epsilon_1,\dots,\epsilon_p),\qquad \epsilon_i=\pm 1. \tag{5.1.5}$$

III: Define a Levi component $\check{\mathfrak{m}}_u$,

$$\check{\mathfrak{m}}_u := gl(a_1) \times \cdots \times gl(a_k) \times \check{\mathfrak{g}}_0(n_0), \qquad n_0 = n - \sum a_i. \tag{5.1.6}$$

The nilpotent orbit with partition $(2x_i)$ in $\check{\mathfrak{g}}_0$ is denoted $\check{\mathcal{O}}_0$.

A typical parameter for a spherical representation will be $(\chi_1, \dots, \chi_k, \chi_{\underline{\epsilon}})$ with coordinates as follows:

- $\chi_i \longleftrightarrow (-\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i)$,
- $(x_{2p}) \longleftrightarrow (1/2, \dots, x_{2p} - 1/2)$,
- $\epsilon_i = 1 \longleftrightarrow (-x_{2i+1} + 1/2, \dots, x_{2i} - 1/2)$
- $\epsilon = -1 \longleftrightarrow (-x_{2i+1}, \dots, x_{2i} - 1)$.

IV: For each size n_j group the corresponding ν_i with $a_i = n_j$ into a parameter

$$\underline{\nu}^j = (\nu_1^j, \dots, \nu_{k_j}^j), \qquad 0 \le \dots \nu_i^j \le \nu_{i+1}^j \dots. \tag{5.1.7}$$

The unitarity conditions for an $L(\chi)$ are in terms of the $\underline{\nu}^j$ as complementary series for types B,C,D plus something extra:

- a_l even starting at an even label: type B.
- a_l even starting at an odd label: $\begin{cases} \epsilon_l = 1, & \text{type B} \\ \epsilon_l = -1 & \text{type C} \end{cases}$
- a_i odd, $2x_{2i} < a_l < 2x_{2i+1}$, $\begin{cases} \epsilon_i = 1 & \text{type C} \\ \epsilon_i = -1 & \text{type D} \end{cases}$
- a_l odd, not as above, type B.

V: For two adjacent sizes, any sum of coordinates $\nu_*^j + \nu_*^{j+1} < 3/2$. This is a restricition only if both sizes have complementary series of type B, D.

Definition. *A parameter $L(\chi_{\underline{\epsilon}})$ corresponding to a $\check{\mathcal{O}}$ with partition $(2x_0, \dots, 2x_{2p})$ satisfying $x_{2i} < x_{2i+1}$ is called basic.*

By [B1], basic parameters are unitary. This completes the statement of theorem 2.1 in this case.

5.2. G of Type C.

I: The partition for $\check{\mathcal{O}}$ is labeled

$$(m_0 \le m_1 \le \cdots \le m_{2p'}) \tag{5.2.1}$$

by adding a 0 to have an odd number. Then a given size *starts at an even or odd label.*

II: From each size extract pairs

$$(a_1, a_1) \dots (a_k, a_k) \tag{5.2.2}$$

as many as possible *leaving one pair for each odd size starting at an* **even** *label.* Thus to each $\check{\mathcal{O}}$ we have associated

$$((a_1 a_1) \dots (a_k a_k); 2x_0 + 1, \dots, 2x_{2p} + 1). \tag{5.2.3}$$

Pair up the x_i

$$(x_0)(x_1x_2)\dots(x_{2p-1}x_{2p}) \tag{5.2.4}$$

Note that $x_{2i-1} < x_{2i}$. Form a vector

$$\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_p), \qquad \epsilon_i = \pm 1. \tag{5.2.5}$$

III: Define a Levi component $\check{\mathfrak{m}}_u$,

$$\check{\mathfrak{m}}_u := gl(a_1) \times \dots \times gl(a_k) \times \check{\mathfrak{g}}_0(n_0), \qquad n_0 = n - \sum a_i. \tag{5.2.6}$$

The nilpotent orbit with partition $(2x_i + 1)$ in $\check{\mathfrak{g}}_0$ is denoted $\check{\mathcal{O}}_0$.

A typical parameter for a spherical representation will be $(\chi_1, \dots, \chi_k, \chi_{\underline{\epsilon}})$ with coordinates as follows:

- $\chi_i \longleftrightarrow (-\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i)$,
- $(x_0) \longleftrightarrow (1, \dots, x_0)$,
- $\epsilon_i = 1 \longleftrightarrow (-x_{2i}, \dots, x_{2i-1})$
- $\epsilon = -1 \longleftrightarrow (-x_{2i} - 1/2, \dots, x_{2i-1} - 1/2)$.

IV: For each size n_j group the corresponding ν_i with $a_i = n_j$ into a parameter

$$\underline{\nu}^j = (\nu^j_1, \dots, \nu^j_{k_j}), \qquad 0 \le \dots \nu^j_i \le \nu^j_{i+1} \dots . \tag{5.2.7}$$

The unitarity conditions for an $L(\chi)$ are in terms of the $\underline{\nu}^j$ as complementary series for types B,C,D plus something extra:

- a_l odd starting at an odd label: type B.
- a_l odd starting at an even label: $\begin{cases} \epsilon_l = 1, & \text{type B} \\ \epsilon_l = -1 & \text{type C} \end{cases}$
- a_l even, $2x_{2i-1} + 1 < a_l < 2x_{2i} + 1$, $\begin{cases} \epsilon_i = 1 & \text{type C} \\ \epsilon_i = -1 & \text{type D} \end{cases}$
- a_l even, not as above, type B.

V: For two adjacent sizes, any sum of coordinates $\nu^j_* + \nu^{j+1}_* < 3/2$. This is a restricition only if both sizes have complementary series of type B, D.

Definition. *A parameter $L(\chi_{\underline{\epsilon}})$ corresponding to a $\check{\mathcal{O}}$ with partition $(2x_0 + 1, \dots, 2x_{2p} + 1)$ satisfying $x_{2i-1} < x_{2i}$ is called basic.*

By [B1], basic parameters are unitary. This completes the statement of theorem 2.1 in this case.

5.3. G of Type D.

I: The partition for $\check{\mathcal{O}}$ is labeled

$$(m_0 \le m_1 \le \dots \le m_{2p'-1}) \tag{5.3.1}$$

by adding a 0 to have an even number. Then a given size *starts at an even or odd label.*

II: From each size extract pairs

$$(a_1, a_1) \dots (a_k, a_k) \tag{5.3.2}$$

as many as possible *leaving one pair for each odd size starting at an* **even** *label.* Thus to each $\check{\mathcal{O}}$ we have associated

$$((a_1 a_1) \dots (a_k a_k); 2x_0 + 1, \dots, 2x_{2p-1} + 1). \tag{5.3.3}$$

Pair up the x_i

$$(x_0 x_{2p-1})(x_1 x_2) \dots (x_{2p-3} x_{2p-2}) \tag{5.3.4}$$

Note that $x_{2i-1} < x_{2i}$. Form a vector

$$\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_{p-1}), \qquad \epsilon_i = \pm 1. \tag{5.3.5}$$

III: Define a Levi component $\check{\mathfrak{m}}_u$,

$$\check{\mathfrak{m}}_u := gl(a_1) \times \dots \times gl(a_k) \times \check{\mathfrak{g}}_0(n_0), \qquad n_0 = n - \sum a_i. \tag{5.3.6}$$

The nilpotent orbit with partition $(2x_i + 1)$ in $\check{\mathfrak{g}}_0$ is denoted $\check{\mathcal{O}}_0$.

A typical parameter for a spherical representation will be $(\chi_1, \dots, \chi_k, \chi_{\underline{\epsilon}})$ with coordinates as follows:

- $\chi_i \longleftrightarrow (-\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i)$,
- $(x_0 x_{2p-1}) \longleftrightarrow (-x_{2p-1}, \dots, x_0)$,
- $\epsilon_i = 1 \longleftrightarrow (-x_{2i}, \dots, x_{2i-1})$
- $\epsilon = -1 \longleftrightarrow (-x_{2i} - 1/2, \dots, x_{2i-1} - 1/2)$.

IV: For each size n_j group the corresponding ν_i with $a_i = n_j$ into a parameter

$$\underline{\nu}^j = (\nu_1^j, \dots, \nu_{k_j}^j), \qquad 0 \le \dots \nu_i^j \le \nu_{i+1}^j \dots. \tag{5.3.7}$$

The unitarity conditions for an $L(\chi)$ are in terms of the $\underline{\nu}^j$ as complementary series for types B,C,D plus something extra:

- a_l odd starting at an odd label: type B.
- a_l odd starting at an even label: $\begin{cases} \epsilon_l = 1, & \text{type B} \\ \epsilon_l = -1 & \text{type C} \end{cases}$
- a_l even, $2x_{2i-1} + 1 < a_l < 2x_{2i} + 1$, $\begin{cases} \epsilon_i = 1 & \text{type C} \\ \epsilon_i = -1 & \text{type D} \end{cases}$
- a_l even, not as above, type B.

V: For two adjacent sizes, any sum of coordinates $\nu_*^j + \nu_*^{j+1} < 3/2$. This is a restricition only if both sizes have complementary series of type B, D.

Definition. *A parameter $L(\chi_{\underline{\epsilon}})$ corresponding to a $\check{\mathcal{O}}$ with partition* $(2x_0 + 1, \dots, 2x_{2p} + 1)$ *satisfying $x_{2i-1} < x_{2i}$ is called basic.*

By [B1], basic parameters are unitary. This completes the statement of theorem 2.1 in this case.

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