Unipotent Representations and the Dual Pairs Correspondence

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Introduction

I first met Roger Howe at a conference in Luminy in 1978. At the time I knew some results about the Segal-Shale-Weil representation from work of Rallis-Schiffmann, and the correspondence between finite dimensional representations of compact groups and highest weight modules from the work of Kashiwara-Vergne (independently done by Howe) and a seminar at MIT. I thought I had understood that the case when both groups in the dual pair were not compact was very problematic, very little of the previous work could be extended. To my amazement, Roger's talk was about that, and full of results. I felt rather awed and mystified. Roger had a profound influence on a lot of younger mathematicians (including me). Indirectly for me, Jeffrey Adams and Allen Moy. With Adams, some ten years later, I understood enough about what Roger was talking about to write a paper [AB1] where we described the correspondence for complex groups in detail; later we extended these results to some real classical groups [AB2]. Since then, it became somwhat of an ongoing project for me to try to understand the correspondence on the level of parameters of irreducible representations.

As a result I asked one of my students Shu-Yen Pan to investigate the correspondence in the case of p-adic groups. Another student, Daniel Wong, investigated an extension of the Theta correpondence.

Along different lines, at the same time that I started my collaboration with Adams, I met and started to collaborate with one of Roger's coworker Allen Moy. Another ten years later we gave a new proof of the Howe conjecture for p-adic groups.

In this talk I want to describe some consequences of the $\Theta-$correspondence as it relates to unipotent representations.
Some of this is old, well known to others, some still work in progress.
For the purpose of this talk I take the rather pragmatic viewpoint:

Definition

An irreducible $(\mathfrak{g}, K)-$module (Π, V) for a real reductive group G is called **unipotent** if

(1) $\mathrm{Ann}\,\Pi \subset U(\mathfrak{g})$ is a maximal primitive ideal,

(2) (π, V) is unitary.

In this talk I will mostly deal with complex groups viewed as real groups.
A lot of the material is available for real groups, still in progress.

The unipotent representations are classified in the case of complex classical groups; their Langlands parameters are explicitly given in [B1], and the metaplectic correspondence is well understood by [AB1].

Unipotent Representations

First recall the Langlands parametrization of irreducible modules. We use the standard realizations of the classical groups, roots, positive roots and simple roots. Let

- θ Cartan involution, K the fixed points of θ, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$,
- $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra,
- $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ a CSA, $\mathfrak{t} \subset \mathfrak{k}$, $\theta\mid_{\mathfrak{a}} = -Id$,
- $X(\mu, \nu) = \mathrm{Ind}_B^G(\mathbb{C}_\mu \otimes \mathbb{C}_\nu)$ standard module,
- $L(\mu, \nu)$, the unique subquotient containing $V_\mu \in \widehat{K}$,
- $\lambda_L = (\mu + \nu)/2$ and $\lambda_R = (-\mu + \nu)/2$.

The parameters of unipotent representations have real ν.

Theorem

1. *$L(\lambda_L, \lambda_R) \cong L(\lambda'_L, \lambda'_R)$ if and only if there is a $w \in W$ such that $w \cdot (\lambda_L, \lambda_R) = (\lambda'_L, \lambda'_R)$.*
2. *$L(\lambda_L, \lambda_R)$ is hermitian if and only if there is $w \in W$ such that $w \cdot (\mu, \nu) = (\mu, -\nu)$.*

We rely on [BV2] and [B1]. For each $\mathcal{O} \subset \mathfrak{g}$ we will give an infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$, and a set of parameters.

Main Properties of $\lambda_{\mathcal{O}}$:

- $\mathrm{Ann}\,\Pi \subset U(\mathfrak{g})$ is the maximal primitive ideal $I_{\lambda_{\mathcal{O}}}$ given the infinitesimal character,
- Π unitary.
- $|\{\Pi \;:\; \mathrm{Ann}\,\Pi = I_{\lambda_{\mathcal{O}}}\}| = |\widehat{A(\mathcal{O})}|$,
  where $A(\mathcal{O})$ is the component group of the centralizer of an $e \in \mathcal{O}$.

The notation is as in [B1]. For special orbits whose dual is even, the infinitesimal character is one half the semisimple element of the Lie triple corresponding to the dual orbit. For the other orbits we need a case-by-case analysis. The parameter will always have integer and half-integer coordinates, the coresponding set of integral coroots is maximal.

Special orbits in the sense of Lusztig and in particular stably trivial orbits defined below will play a special role.

Definition

A special orbit $\mathcal{O}$ is called stably trivial if Lusztig's quotient $\overline{A}(\mathcal{O}) = A(\mathcal{O})$.

Example

$\mathcal{O} = (2222) \subset sp(8)$ is stably trivial, $A(\mathcal{O}) = \overline{A(\mathcal{O})} \cong \mathbb{Z}_2$, $\lambda_{\mathcal{O}} = (2, 1, 1, 0)$.
$\mathcal{O} = (222) \subset sp(6)$ is not, $A(\mathcal{O}) \cong \mathbb{Z}_2$, $\overline{A(\mathcal{O})} \cong 1$, $\lambda_{\mathcal{O}} = (3/2, 1/2, 1/2)$.
(222) is special, $h^{\vee}/2 = (1, 1, 0)$.

The partitions denote rows.

Type A

Nilpotent orbits are determined by their Jordan canonical form. An orbit is given by a partition, *i.e.* a sequence of numbers in decreasing order $(n_1, \dots, n_k)$ that add up to n. Let $(m_1, \dots, m_l)$ be the dual partition. Then the infinitesimal character is

$$(\frac{m_1 - 1}{2}, \dots, -\frac{m_1 - 1}{2}, \dots, \frac{m_l - 1}{2}, \dots, -\frac{m_l - 1}{2})$$

The orbit is induced from the trivial orbit on the Levi component $GL(m_1) \times \cdots \times GL(m_l)$. The corresponding unipotent representation is spherical and induced irreducible from the trivial representation on the same Levi component. All orbits are *special* and *stably trivial*.

Type B

We treat the case $SO(2m+1)$. A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \ldots, n_k)$ of $2m+1$ such that every even entry occurs an even number of times. Let $(m'_0, \ldots, m'_{2p'})$ be the dual partition (add an $m'_{2p'} = 0$ if necessary, in order to have an odd number of terms). If there are any $m'_{2j} = m'_{2j+1}$ then pair them together and remove them from the partition. Then relabel and pair up the remaining columns $(m_0)(m_1, m_2)\ldots(m_{2p-1}m_{2p})$. The members of each pair have the same parity and m_0 is odd. $\lambda_{\mathcal{O}}$ is given by the coordinates

$$\begin{aligned}
(m_0) &\longleftrightarrow (\frac{m_0-2}{2}, \ldots, \frac{1}{2}), \\
(m'_{2j} = m'_{2j+1}) &\longleftrightarrow (\frac{m_{2j}-1}{2}, \ldots, -\frac{m_{2j}-1}{2}) \\
(m_{2i-1}m_{2i}) &\longleftrightarrow (\frac{m_{2i-1}}{2}, \ldots, -\frac{m_{2i}-2}{2})
\end{aligned} \tag{1}$$

Type B, continued

In case $m'_{2j} = m'_{2j+1}$, $\mathcal{O}$ is induced from a $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m} = so(*) \times gl(m'_{2j})$ where $\mathfrak{m}$ is the Levi component of a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$. $\mathcal{O}_{\mathfrak{m}}$ is the trivial nilpotent on the $gl-$factor. The component groups satisfy $A_G(\mathcal{O}) \cong A_M(\mathcal{O}_{\mathfrak{m}})$. Each unipotent representation is unitarily induced from a unipotent representation attached to $\mathcal{O}_{\mathfrak{m}}$.

Similarly if some $m_{2i-1} = m_{2i}$, then $\mathcal{O}$ is induced from a $\mathcal{O}_{\mathfrak{m}} \subset so(*) \times gl(\frac{m_{2i-1}+m_{2i}}{2})$ with (0) on the $gl-$factor. $A_G(\mathcal{O}) \ncong A_M(\mathcal{O}_{\mathfrak{m}})$, but each unipotent representation is (not necessarily unitarily) induced irreducible from a $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m} \cong so(\ ) \times gl(\ )$.

The *stably trivial* orbits are the ones such that every odd sized part appears an even number of times, except for the largest size. An orbit is called triangular if it has partition

$$\mathcal{O} \longleftrightarrow (2m+1, 2m-1, 2m-1, \dots, 3, 3, 1, 1).$$

Type B, continued

We give the explicit Langlands parameters of the unipotent representations in terms of their . There are $|A_G(\mathcal{O})|$ distinct representations. Let

$$(\underbrace{1, \dots, 1}_{r_1}, \dots, \underbrace{k, \dots k}_{r_k})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers r_{2i} are even. The reductive part of the centralizer (when $G = O(*)$) of the nilpotent element is a product of $O(r_{2i+1})$, and $Sp(r_{2j})$.

Type B, continued

The columns are paired as in (1). The pairs $(m'_{2j} = m'_{2j+1})$ contribute to the spherical part of the parameter,

$$(m'_{2j} = m'_{2j+1}) \longleftrightarrow \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} \frac{m'_{2j}-1}{2} & , & \cdots & , & -\frac{m'_{2j}-1}{2} \\ \frac{m'_{2j}-1}{2} & , & \cdots & , & -\frac{m'_{2j}-1}{2} \end{pmatrix}. \qquad (2)$$

The singleton (m_0) contributes to the spherical part,

$$(m_0) \longleftrightarrow \begin{pmatrix} \frac{m_0-2}{2} & , & \cdots & , & \frac{1}{2} \\ \frac{m_0-2}{2} & , & \cdots & , & \frac{1}{2} \end{pmatrix}. \qquad (3)$$

Let $(\eta_1, \ldots, \eta_p)$ with $\eta_i = \pm 1$, one for each (m_{2i-1}, m_{2i}). An $\eta_i = 1$ contributes to the spherical part of the parameter, with coordinates as in (1). An $\eta_i = -1$ contributes

$$\begin{pmatrix} \frac{m_{2i-1}}{2} & , & \cdots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}}{2} & , & \cdots & , & -\frac{m_{2i}-2}{2} \\ \frac{m_{2i-1}}{2} & , & \cdots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}-2}{2} & , & \cdots & , & -\frac{m_{2i}}{2} \end{pmatrix}. \qquad (4)$$

If $m_{2p} = 0$, $\eta_p = 1$ only.

Explanation

1. Odd sized rows contribute a $\mathbb{Z}_2$ to $A(\mathcal{O})$, even sized rows a 1.
2. When there are no $m'_{2j} = m'_{2j+1}$, every row size occurs. $\ldots(m_{2i-1} \geq m_{2i}) > (m_{2i+1} \geq m_{2i+2})\ldots$ determines that there are $m_{2i} - m_{2i+1}$ rows of size $2i+1$. The pair $(m_{2i-1} \geq m_{2i})$ contributes exactly 2 parameters corresponding to the $\mathbb{Z}_2$ in $A(\mathcal{O})$.
3. The pairs $(m'_{2j} = m'_{2j+1})$ lengthen the sizes of the rows without changing their parity. The component group does not change, they do not affect the number of parameters.

Type C

A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \ldots, n_k)$ of $2n$ such that every odd part occurs an even number of times. Let $(c'_0, \ldots, c'_{2p'})$ be the dual partition (add a $c'_{2p'} = 0$ if necessary in order to have an odd number of terms). If there are any $c'_{2j-1} = c'_{2j}$ pair them up and remove them from the partition. Then relabel and pair up the remaining columns $(c_0 c_1) \ldots (c_{2p-2} c_{2p-1})(c_{2p})$. The members of each pair have the same parity. The last one, c_{2p}, is always even. Then form a parameter

$$(c'_{2j-1} = c'_{2j}) \leftrightarrow (\frac{c_{2j}-1}{2}, \ldots, -\frac{c_{2j}-1}{2}), \tag{5}$$

$$(c_{2i} c_{2i+1}) \leftrightarrow (\frac{c_{2i}}{2}, \ldots, -\frac{c_{2i+1}-2}{2}), \tag{6}$$

$$c_{2p} \leftrightarrow (\frac{c_{2p}}{2}, \ldots, 1). \tag{7}$$

Type C, continued

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.
An orbit is called triangular if it corresponds to the partition $(2m, 2m, \dots, 4, 4, 2, 2)$.

We give a parametrization of the unipotent representations in terms of their Langlands parameters. There are $\mid A_G(\mathcal{O}) \mid$ representations.
Let

$$(\underbrace{1, \dots, 1}_{r_1}, \dots, \underbrace{k, \dots k}_{r_k})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers r_{2i+1} are even.

Type C, continued

The reductive part of the centralizer of the nilpotent element is a product of $Sp(r_{2i+1})$, and $O(r_{2j})$.
The elements $(c'_{2j-1} = c'_{2j})$ and c_{2p} contribute to the spherical part of the parameter as in (2) and (3). Let $(\epsilon_1, \dots, \epsilon_p)$ be such that $\epsilon_i = \pm 1$, one for each (c_{2i}, c_{2i+1}). An $\epsilon_i = 1$ contributes to the spherical paramater according to the infinitesimal character. An $\epsilon_i = -1$ contributes

$$\left(\begin{array}{cccccccc} \frac{c_{2i}}{2} & , & \dots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}}{2} & \dots & , & -\frac{c_{2i+1}-2}{2} \\ \frac{c_{2i}}{2} & , & \dots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}-2}{2} & \dots & , & -\frac{c_{2i+1}}{2} \end{array} \right). \tag{8}$$

The explanation is similar to type B.

Type D

We treat the case $G = SO(2m)$. A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \ldots, n_k)$ of $2m$ such that every even part occurs an even number of times. Let $(m'_0, \ldots, m'_{2p'-1})$ be the dual partition (add a $m'_{2p'-1} = 0$ if necessary). If there are any $m'_{2j} = m'_{2j+1}$ pair them up and remove from the partition. Then pair up the remaining columns $(m_0, m_{2p-1})(m_1, m_2)\ldots(m_{2p-3}, m_{2p-2})$. The members of each pair have the same parity and m_0, m_{2p-1} are both even. The infinitesimal character is

$$\begin{aligned}
(m'_{2j} = m'_{2j+1}) &\longleftrightarrow (\frac{m'_{2j}-1}{2}\ldots, -\frac{m'_{2j}-1}{2}) \\
(m_0 m_{2p-1}) &\longleftrightarrow (\frac{m_0-2}{2}, \ldots, -\frac{m_{2p-1}}{2}), \\
(m_{2i-1} m_{2i}) &\longleftrightarrow (\frac{m_{2i-1}}{2}\ldots, -\frac{m_{2i}-2}{2})
\end{aligned} \qquad (9)$$

Type D, continued

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. An exception occurs for $G = SO(2m)$ when the partition is formed of pairs $(m'_{2j} = m'_{2j+1})$ only. In this case there are two nilpotent orbits corresponding to the partition. There are also two nonconjugate Levi components of the form $gl(m'_0) \times gl(m'_2) \times \dots gl(m'_{2p'-2})$ of parabolic subalgebras. There are two unipotent representations each induced irreducible from the trivial representation on the corresponding Levi component.

The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.

A nilpotent orbit is triangular if it corresponds to the partition $(2m-1, 2m-1, \dots, 3, 3, 1, 1)$.

Type D, continued

The parametrization of the unipotent representations follows types B,C, with the pairs $(m'_{2j} = m'_{2j+1})$ and (m_0, m_{2p-1}) contributing to the spherical part of the parameter only. Similarly for (m_{2i-1}, m_{2i}) with $\epsilon_i = 1$ spherical only, while $\epsilon_i = -1$ contributes analogous to (4) and (8).

The explanation parallels that for types B,C.

Metaplectic Correspondence

The next results are motivated by [KP1].
Restrict attention to the cases

(B) $(m_0)(m_1, m_2)\dots(m_{2p-1}, m_{2p})$ with $m_{2k} > m_{2k+1}$,
(C) $(c_0, c_1)\dots(c_{2p-2}, c_{2p-1})(c_{2p})$ with $c_{2j-1} > c_{2j}$,
(D) $(m_0, m_{2p+1})(m_1, m_2)\dots(m_{2p-1}, m_{2p})$ with $m_{2j} > m_{2j+1}$.

Let (V_k, ϵ_k) be a symplectic space if $\epsilon_k = -1$, orthogonal if $\epsilon_k = 1$, $k = 0, \dots, 2p$. ϵ_0 is the same as the type of the Lie algebra, dim V_0 is the sum of the columns. Let (V_k, ϵ_k) be the space with dimension the sum of the lengths of the columns labelled $\geq k$, $\epsilon_{k+1} = -\epsilon_k$. Then

$$(V_k, V_{k+1})$$

gives rise to a dual pair. The main result will be that unipotent representations corresponding to $\mathcal{O}_k$ are obtained from the unipotent representations corresponding to $\mathcal{O}_{k+1}$.

Metaplectic Correspondence, continued

More precisely,

- The matching of infinitesimal characters applies.
- $\epsilon_0 = -1$. There is a $1-1$ correspondence between unipotent representations of $Sp(V_1)$ attached to $\mathcal{O}_1$ and unipotent representations attached to $\mathcal{O} = \mathcal{O}_0$.
- $\epsilon_0 = 1$. There is a $1-1$ correspondence between unipotent representations of $O(V_1)$ attached to $\mathcal{O}_1$ and unipotent representations of $Sp(V_0)$ attached to $\mathcal{O} = \mathcal{O}_0$.

The passage between unipotent representations of $SO(V)$ and $O(V)$ is done by pulling back and tensoring with the *sign* character of $O(V)$.

I use Weyl's conventions for parametrizing representations of $O(n)$. The proof is a straightforward application of [AB1].

Sketch of Proof

1. Adding a column longer than any existing columns changes the parity of the rows, and adds a number of rows of size 1. If we pass from $sp()$ to $so()$, another $\mathbb{Z}_2$ is added to $A(\mathcal{O})$. If we pass from $so()$ to $sp()$ the component group does not change.
2. $\epsilon_0 = -1$. If the pair is from type C to type D, $c_0, \ldots, c_{2p}$ are changed to $m_1, \ldots, m_{2p+1}$ and an m_0 is added. They are paired up $(m_0, c_{2p})(c_1, c_2) \ldots (c_{2p-2}, c_{2p-1})$. A parameter corresponding to a $(\eta_1, \ldots, \eta_p)$ goes to the corresponding one for type D.
   If the pair is from type C to type B, and $c_{2p} = 0$, then $c_0, \ldots, c_{2p-1}$ go to $m_1, \ldots, m_{2p}$ and an m_0 is added. If $c_{2p} \neq 0$, then $c_0, \ldots, c_{2p}$ go to $m_1, \ldots, m_{2p+1}$, a m_0 and a $m_{2p+2} = 0$ is added. The pairs are $(m_0)(c_0, c_1) \ldots (c_{2p-2}, c_{2p-1})(c_{2p}, 0)$ and $(\eta_1, \ldots, \eta_p)$ goes to the corresponding one for type B.
3. $\epsilon_0 = 1$. We have to use the more difficult matching in [AB1].

Example 1

Consider the nilpotent orbit in $so(8)$ with columns $\mathcal{O} \longleftrightarrow (4,3,1)$. The infinitesimal character is $(1,0,3/2,1/2)$. Then $(V_0,1)$ is of dimension 8, and $(V_1,-1)$ is of dimension 4. $\mathcal{O}_1 \longleftrightarrow (3,1)$ and the unipotent representations are the two metaplectic representations

$$\begin{pmatrix} 3/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 3/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}$$

They correspond to the two unipotent representations of $SO(8)$ with parameters

$$\begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 1 & 0 & 3/2 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 1 & 0 & 3/2 & -1/2 \end{pmatrix}$$

Example 2

Let $\mathcal{O}_1 \longleftrightarrow (4,2,2)$ in $so(8)$. It matches $\mathcal{O}_0 \longleftrightarrow (4,4,2,2)$ in $sp(12)$. the infinitesimal characters are $(1,1,0,0)$ and $(2,1,1,1,0,0)$. The parameters for $\mathcal{O}_1$ are

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The parameters for $\mathcal{O}$ are

$$\begin{pmatrix} 2 & 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 & -1 & 0 & -1 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & -2 & 0 & -1 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

The second column is obtained by applying the correspondence to the parameters for $O(8)$ tensored with *sgn*.

Example 3

Let $\mathcal{O} \longleftrightarrow (33)$ in $sp(6)$. Then $\mathcal{O}_1 \longleftrightarrow (3)$ in $so(3)$. The infinitesimal characters are $(3/2, 1/2, 1/2)$ and $(1/2)$ as given by the previous algorithms.

The rows of $\mathcal{O}$ are $(2, 2, 2)$ there is only one special unipotent representation, its infinitesimal character is $(1, 1, 0)$. By contrast infinitesimal character $(3/2, 1/2, 1/2)$ matches the $\Theta-$correspondence and there are two parameters.

Example 4

Let $\mathcal{O} \longleftrightarrow (4,2,2)$ in $sp(8)$. It corresponds to $\mathcal{O} \longleftrightarrow (2,2)$ in $so(4)$. There are two such nilpotent orbits if we use $SO(4)$, one if we use $O(4)$. We will use orbits of the orthogonal group. The infinitesimal character corresponding to $(2,2)$ is $(1/2,1/2)$. The representations corresponding to $(4,2,2)$ have infinitesimal character $(1,0,1/2,1/2)$. The Langlands parameters are spherical

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 1 & 0 & 1/2 & 1/2 \end{pmatrix}$$

We can go further and match $(2,2)$ in $so(8)$ with (2) in $sp(2)$. If we combine these steps we get infinitesimal characters $(1) \mapsto (0,1) \mapsto (2,1,0,1)$. There is nothing wrong with the correspondence of irreducible modules. But note that the infinitesimal character $(2,1,1,0)$ has maximal primitive ideal corresponding to the orbit $\mathcal{O} \longleftrightarrow (4,4)$, rows $(2,2,2,2)$.

Example 4, continued

This is one of the reasons for imposing the conditions in the previous slides, to be able to iterate and stay within the class of unipotent representations. In the absence of these restrictions one obtains induced modules with interesting composition series. In this example,

$$\mathrm{Ind}_{GL(2)xSp(4)}^{Sp(8)}[\chi \otimes \mathit{Triv}] =$$
$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

The first two parameters are unipotent, corresponding to $\mathcal{O} \longleftrightarrow (4, 4)$, the last one is bigger, the annihilator gives $(4, 2, 2)$. All these composition factors have nice character formulas analogous to those for the special unipotent representations; their annihilators are no longer maximal. Daniel Wong has made an extensive study of these representations in his thesis.

Example 4, continued

This example is tied up with the fact that nilpotent orbits are not always normal.

A nilpotent orbit is normal if and only if $R(\mathcal{O}) = R(\overline{\mathcal{O}})$. The orbit $(4, 2, 2)$ is **not normal**. From the previous slide, $R(\mathcal{O})$ is the full induced representation, $R(\overline{\mathcal{O}})$ is missing the middle representation.

These equalities are in the sense that the $K-$types of the representations match the $G-$types of the regular functions, I am using the identification $K_c \cong G$.

K-P Model

We follow [Bry]. Let $(\mathcal{G}, K) := (\mathfrak{g}_0, K_0) \times \cdots \times (\mathfrak{g}_\ell, K_\ell)$ be the algebras corresponding to removing a column at a time. Each pair $(\mathfrak{g}_i, K_i) \times (\mathfrak{g}_{i+1}, K_{i+1})$ is equipped with a metaplectic representation Ω_i. Form $\Omega := \otimes \Omega_i$. The representation we are interested in, is

$$\Pi = \Omega / (\mathfrak{g}_1 \times \ldots \mathfrak{g}_\ell)(\Omega).$$

Let $(\mathfrak{g}^1, K^1) := (\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_\ell, K_1 \times \cdots \times K_\ell)$, and let $\mathfrak{g}^1 = \mathfrak{k}^1 + \mathfrak{p}^1$ be the Cartan decomposition.

Π is an admissible $(\mathfrak{g}_0, K_0)-$ module. It has an infinitesimal character compatible with the $\Theta-$correspondence. Furthermore the K_i which are orthogonal groups are disconnected, so the component group $\mathcal{K}^1 := K^1/(K^1)^0$ still acts, and commutes with the action of $(\mathfrak{g}_0, K_0)$. Thus Π decomposes

$$\Pi = \bigoplus \Pi_\Psi$$

where $\Pi_\Psi := \mathrm{Hom}_{\mathcal{K}^1}[\Pi, \Psi]$. The main result in [Bry] is that

$$\Pi_{Triv} \mid_{K_0} = R(\overline{\mathcal{O}}).$$

K-P Model, continued

We would like to conclude that the representations correponding to the other Ψ are the unipotent ones, and they satisfy an analogous relation to the above. The problem is that the component group of the centralizer of an element in the orbit does not act on $\overline{\mathcal{O}}$, so $R(\overline{\mathcal{O}}, \psi)$ does not make sense.

Let $\mathcal{V} := \prod \mathrm{Hom}[V_i, V_{i+1}]$. This can be identified with a Lagrangian. Consider the variety $\mathcal{Z} = \{(A_0, \dots, A_\ell)\} \subset \mathcal{V}$ given by the equations $A_i^\star \circ A_i - A_{i+1} \circ A_i^\star = 0, \dots, A_{\ell-1} \circ A_\ell^* = 0$. The detailed statement in [Bry] is that

$$Gr[\Omega/\mathfrak{p}^1\Omega] = R(\mathcal{Z}).$$

This is compatible with taking (co)invariants for $\mathfrak{k}^1$. To each character ψ of the component group of the centralizer of an element $e \in \mathcal{O}$, there is attached a character $\Psi \in \widehat{\mathcal{K}^1}$. Then

$$\Pi_\Psi \mid_{K_0} = R(\mathcal{Z}, \Psi).$$

Main Result

Theorem

Let $\mathcal{O}$ be a stably trivial orbit. There is a matchup

$$Unip(\mathcal{O}) \longleftrightarrow \widehat{A(\mathcal{O})}, \qquad \psi \longleftrightarrow X_\psi,$$

such that

$$X_\psi \mid_K \cong R(\mathcal{O}, \psi).$$

There are various compatibilities with induction.

For the rest of the orbits, the best I can do is to show that there is a matchup such that

$$X_\psi \cong R(\mathcal{O}, \psi) - Y_\psi$$

where Y_ψ is a genuine $K-$character supported on smaller orbits *cf* [V]. As in [V], the conjecture is that $Y_\nu = 0$.

Let $\mathcal{S}_0$ be a (real) symplectic space, $\mathcal{S}_0 = \mathcal{L}_0 + \mathcal{L}_0^{\perp}$, a decomposition into transverse Lagrangians. Let $\mathcal{R}_0$ be an orthogonal space. The real form of the orthogonal group gives a decomposition $\mathcal{R}_0 = V_0 + W_0$ where the form is positive definite on V_0, negative definite on W_0. The complexifications of the Cartan decompositions of $sp(\mathcal{S}_0)$ and $so(\mathcal{R}_0)$ are given by

$$\begin{aligned} sp(\mathcal{S}) &= \mathfrak{k} + (\mathfrak{s}^+ + \mathfrak{s}^-) = \mathrm{Hom}[\mathcal{L},\mathcal{L}] + (\mathrm{Hom}[\mathcal{L}^{\perp},\mathcal{L}] + \mathrm{Hom}[\mathcal{L},\mathcal{L}^{\perp}]),\\ so(\mathcal{R}) &= \mathfrak{k} + \mathfrak{s} = (\mathrm{Hom}[\mathcal{V},\mathcal{V}] + \mathrm{Hom}[\mathcal{W},\mathcal{W}]) + \mathrm{Hom}[\mathcal{V},\mathcal{W}]. \end{aligned} \tag{10}$$

Note that due to the presence of the nondegenerate forms, $\mathrm{Hom}[\mathcal{L},\mathcal{L}] \cong \mathrm{Hom}[\mathcal{L}^{\perp},\mathcal{L}^{\perp}]$ and $\mathrm{Hom}[\mathcal{V},\mathcal{W}] \cong \mathrm{Hom}[\mathcal{W},\mathcal{V}]$. The canonical isomorphisms are denoted by *.

Real Groups, continued

Consider the pair $sp(\mathcal{S}) \times o(\mathcal{R}) \subset sp(\mathrm{Hom}[\mathcal{S}, \mathcal{R}])$. The space $\mathcal{X} := \mathrm{Hom}[\mathcal{S}, \mathcal{R}]$ has symplectic nondegenerate form

$$\langle A, B \rangle := \mathrm{Tr}(A \circ B^*) = \mathrm{Tr}(B^* \circ A).$$

A Lagrangian subspace is provided by

$$\mathfrak{L} := \mathrm{Hom}[\mathcal{L}, \mathcal{V}] + \mathrm{Hom}[\mathcal{L}^{\perp}, \mathcal{W}]. \tag{11}$$

The moment map

$$\begin{aligned} m &= (m_{sp}, m_{so}) : \mathfrak{L} \longrightarrow \mathfrak{sp}(\mathcal{S}) \times \mathfrak{o}(\mathcal{R}) \\ m(A, B) &= (A^*A + B^*B, BA^* \cong AB^*) \end{aligned} \tag{12}$$

maps $\mathfrak{L}$ to $\mathfrak{s}(sp) \times \mathfrak{s}(so)$. It is standard that $m_{sp} \circ m_{so}^{-1}$ (and symmetrically $m_{so} \circ m_{sp}^{-1}$) take nilpotent orbits to nilpotent orbits. In this special case, the moment maps take nilpotent K_c-orbits on $\mathfrak{s}$ to nilpotent K_c-orbits on $\mathfrak{s}$ of the other group.

Real Groups, continued

The K-P model has a straightforward generalization, one considers the columns of K_c-orbits on $\mathfrak{p}_c$. Things get rather interesting. Some unipotent representations won't have support just a single orbit; one has to consider other see-saw pairs in the $\Theta-$correspondence than those coming from the columns.

An Example

Let $\mathcal{O} \longleftrightarrow (2n, 2n-1, 1)$ in $so(4n)$. Then $\lambda_{\mathcal{O}} = (n-1/2, \ldots, 1/2, n-1, \ldots, 1, 0)$. There are two complex representations:

$$\begin{pmatrix} n-1/2 & \ldots & 1/2 & n-1 & \ldots & 0 \\ n-1/2 & \ldots & 1/2 & n-1 & \ldots & 0 \end{pmatrix}$$
$$\begin{pmatrix} n-1/2 & \ldots & 1/2 & n-1 & \ldots & 0 \\ n-1/2 & \ldots & -1/2 & n-1 & \ldots & 0 \end{pmatrix}$$

Example, continued

However if we consider $Spin(4n)$ there are two more,

$$\begin{pmatrix} n-1/2 & \dots & 1/2 & 0 & \dots & -n+1 \\ n-1 & \dots & 0 & -1/2 & \dots & -n+1/2 \end{pmatrix}$$
$$\begin{pmatrix} n-1/2 & \dots & -1/2 & 0 & \dots & -n+1 \\ n-1 & \dots & 0 & -1/2 & \dots & -n+1/2 \end{pmatrix}$$

They have analogous relations to the corresponding $R(\mathcal{O}, \psi)$. They cannot come from the $\Theta-$correspondence. For the real case, Wan-Yu Tsai has made an extensive study of the analogues of these repreentations. They cannot come from the $\Theta-$correspondence either. In work in progress we are attempting to show that they satisfy the desired relations to sections on orbits.

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