Relevant and Petite K-types

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The Unitarity Problem

- **NOTATION**
  - G is the real points of a linear connected reductive group.
  - $\mathfrak{g}_0 := Lie(G)$, θ Cartan involution, $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}}$, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, K maximal compact subgroup.
  - A representation $(\pi, \mathcal{H})$ on a Hilbert space is called unitary, if $\mathcal{H}$ admits a G invariant positive definite inner product.
- **PROBLEM**
  Classify all irreducible unitary representations of G.

  By results of Harish-Chandra, it is enough to solve the
- **ALGEBRAIC PROBLEM**
  Classify all irreducible admissible unitary $(\mathfrak{g}, K)$ modules.

Irreducible admissible representations of G

- $P = MAN$ a parabolic subgroup of G, $\mathfrak{a}_0 := Lie(A)$, $\mathfrak{a}$ its complexification,
- (δ, V_δ) an irreducible tempered unitary representation of M,
- $\nu \in \mathfrak{a}^*$, with real part in the open positive Weyl chamber,
- $X_P(\delta \otimes \nu)$ the corresponding Harish-Chandra induced (normalized induction) **standard module**,
- $\overline{X}_P(\delta \otimes \nu)$: **the unique irreducible quotient**.

Classification

Langlands, early 1970s:

- Every irreducible admissible representation of G is infinitesimally equivalent to a **Langlands quotient** $\overline{X}_P(\delta \otimes \nu)$.
- Two Langlands quotients $\overline{X}_P(\delta \otimes \nu)$ and $\overline{X}_{P'}(\delta' \otimes \nu')$ are infinitesimally equivalent if and only if there exists an element ω of K such that

$$\omega P \omega^{-1} = P' \quad \omega \cdot \delta \cong \delta', \quad \omega \cdot \nu = \nu'.$$

- $\overline{X}(\delta, \nu)$ is the image of an intertwining operator

$$A(\overline{P}, P, \delta, \nu) \ : \ X_P(\delta, \nu) \longrightarrow X_{\overline{P}}(\delta, \nu).$$

Hermitian Langlands Quotients

Knapp and Zuckerman, 1976:

$\overline{X}_P(\delta \otimes \nu)$ admits **a non-degenerate invariant Hermitian form** if and only if there exists $\omega \in K$ satisfying

$$\omega P \omega^{-1} = \bar{P} \qquad \omega \cdot \delta \simeq \delta \qquad \omega \cdot \nu = -\bar{\nu}$$

(because the Hermitian dual of $\overline{X}_P(\delta \otimes \nu)$ is $\overline{X}_{\bar{P}}(\delta \otimes -\bar{\nu})$).
Any non-degenerate invariant Hermitian form on $\overline{X}_P(\delta \otimes \nu)$ is a real multiple of the form induced by the Hermitian operator

$$\boxed{B = \delta(\omega) \circ R(\omega) \circ A(\bar{P} : P : \delta : \nu)}$$

from $X_P(\delta \otimes \nu)$ to $X_P(\delta \otimes -\bar{\nu})$.

$\overline{X}_P(\delta, \nu)$ is unitary if and only if B is positive semidefinite.

The signature of B

- For every K-type (μ, E_μ), we have a Hermitian operator

$$R_\mu(\omega, \nu) \colon \operatorname{Hom}_K(E_\mu, X_P(\delta \otimes \nu)) \to \operatorname{Hom}_K(E_\mu, X_P(\delta \otimes -\bar{\nu})).$$

- By Frobenius reciprocity:

$$R_\mu(\omega, \nu) \colon \operatorname{Hom}_{M\cap K}(E_\mu \mid_{M\cap K}, V^\delta) \to \operatorname{Hom}_{M\cap K}(E_\mu \mid_{M\cap K}, V^\delta).$$

If P is the minimal parabolic subgroup, and $\delta = Triv$, then

$$R_\mu(\omega, \nu) : (E_\mu^*)^M \longrightarrow (E_\mu^*)^M.$$

Spherical Representations

- G is **split**, in particular $SL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$, $SO(n, n)$, F_4, E_6, E_7, E_8.
- $P = MAN$ is a **minimal** parabolic subgroup of G.
- δ is the **trivial** representation of M.
- ν is a **real** character of A.

In this case we can regard ω as an element of $W := N_K(\mathfrak{a}_0)/M$. The operator $R_\mu(\omega, \nu)$ decomposes into a product of factors according to the decomposition of ω into a product of simple reflections (as in Gindikin-Karpelevic). These factors are induced from the corresponding intertwining operators on $SL(2, \mathbb{R})$.

Root SL(2)'s

For each $\alpha \in \Delta(\mathfrak{n}_0, \mathfrak{a}_0)$, choose a map $\psi_\alpha : sl(2, \mathbb{R}) \longrightarrow \mathfrak{g}_0$ which commutes with θ, and satisfies

$$\psi_\alpha\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = E_\alpha, \qquad \psi_\alpha\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = E_{-\alpha},$$

where $E_{\pm\alpha}$ are the root vectors, and $\theta(E_\alpha) = -E_{-\alpha}$. Then ψ_α determines a map

$$\Psi_\alpha \ : \ SL(2, \mathbb{R}) \longrightarrow G$$

with image G_α, a connected group with Lie algebra isomorphic to

$sl(2,\mathbb{R})$. Denote by

$$\sigma_\alpha := \Psi_\alpha\Big(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\Big), \qquad m_\alpha := \sigma_\alpha^2,$$

and let $Z_\alpha := E_\alpha - E_{-\alpha} \in \mathfrak{k}_0$.

Definition. *A K-type is called* ***petite,*** *if* $\mu(iZ_\alpha) = 0, \pm 1, \pm 2, \pm 3$.

The operators $R_\mu(\omega, \nu)$ have a simpler form for such K-types. The factors corresponding to the simple root reflections are

$$R_\mu(s_\beta,\, \nu) = \begin{cases} +1 & \textbf{on the } (+1)\textbf{-eigenspace of } \mu(\sigma_\beta) \\ \frac{1-\langle \nu, \check{\beta}\rangle}{1+\langle \nu, \check{\beta}\rangle} & \textbf{on the } (-1)\textbf{-eigenspace of } \mu(\sigma_\beta) \end{cases}$$

The operator $R_\mu(s_\beta, \nu)$ acts on $(E_\mu^*)^M$, and depends only on the W-module structure of this space.

P-adic Groups

The formula for $R_\mu(s_\beta, \nu)$ coincides with the formula for the similar operator for a split p-adic group. To be more precise, results of Barbasch-Moy reduce the problem of the determination of the Iwahori spherical dual of split p-adic group to the problem of determining the unitary dual of finite dimensional representations of the corresponding affine graded Hecke algebra. In this case, for each representation $\tau \in \widehat{W}$, there is an operator $R_\tau(\omega, \nu)$ with the same formula as the one for the real case. A spherical representation $\overline{X}(\nu)$ is unitary if and only if R_τ is positive definite for all τ.

Relevant K-types

Work of Barbasch for the classical groups, Ciubotaru for F_4, and Barbasch-Ciubotaru for E_6, E_7, and E_8, determine a set of W-representations, called **relevant** with the property that a spherical module $\overline{X}(\nu)$ is unitary, if and only if $R_\tau(\omega,\nu)$ is positive semidefinite for τ in the relevant set.

PROBLEM Find a set of petite K-types so that the $(E_\mu^*)^M$ realize all the relevant W-representations.

If we can solve this problem, then we get powerful necessary conditions for unitarity in the real case. Conjecturally the spherical unitary dual for a split reductive group should be independent of whether the field is real or p-adic. This is true for the classical groups, but a conjecture for the exceptional groups.

Classical Groups

For type $\mathbf{A_{n-1}}$, $W = S_n$, and the relevant representations are

$$(n-k,k).$$

For types $\mathbf{B_n}$, and $\mathbf{C_n}$, the Weyl group W consists of permutations and sign changes of of the coordinates of $\mathbb{R}^n$, and the relevant W-types are

$$(n-k,k)\times(0), \qquad (n-k)\times(k).$$

Similarly for $\mathbf{D_n}$.

Exceptional Groups

The relevant W representations are

F_4 $1_1,\ 2_3,\ 8_1,\ 4_2,\ 9_1,$

E_6 $1_p,\ 6_p,\ 20_p,\ 30_p,\ 15_q,$

E_7 $1_a,\ 7'_a,\ 27_a,\ 56'_a,\ 21'_b,\ 35_b,\ 105_b,$

E_8 $1_x,\ 8_z,\ 35_x,\ 50_x,\ 84_x,\ 112_z,\ 400_z,\ 300_x,\ 210_x.$

The notation is as in Kondo's and Frame's work.

Weyl group representations

Let (μ_a, V_a) and (μ_b, V_b) be representations of K. Then $\mathrm{Hom}_M[V_a, V_b]$ is endowed with a representation of $N_K(M)$ via

$$n \cdot f(v) := \mu_b(n) f(\mu_a(n^{-1})v).$$

Under this action, $M \subset N_K(M)$ acts trivially, so we get a representation of W. Because

$$\mathrm{Hom}_M[V_a, V_b] \cong \mathrm{Hom}_M[V_a \otimes V_b^*, Triv],$$

this generalizes the action of W on $(E_\mu^*)^M$ from before.

Fine K-types

A K-type is called **fine** (Bernstein-Gelfand, Vogan), if $\mu(iZ_\alpha) = 0, \pm 1$.

These are the lowest K-types of principal series. A fine K-type has the property that its restriction to M is multiplicity free, and is a single $N_K(\mathfrak{a}_0)$-orbit of representations of M.

In the case of a linear group, M is abelian, so $\widehat{M}$ is formed of characters.

Fix a representative δ for each W-orbit, and a fine K-type μ_δ. Then

$\mu_\delta \otimes \mu_\delta^*$ is formed of petite K-types only.

We will use the previous formula to determine the Weyl group representation on $\mu_\delta \otimes \mu_\delta^*$.

Stabilizer of δ

- ${}^\vee\Delta^\delta := \{\check{\alpha} \mid \delta(m_\alpha) = 1\}$ is a root system.
- The Weyl group generated by the roots in ${}^\vee\Delta^\delta$ is called W_δ^0, and is a normal subgroup of the stabilizer W_δ of δ.
- The quotient $R_\delta := W_\delta / W_\delta^0$ is a product of $\mathbb{Z}_2$'s.
- $\widehat{R_\delta}$ acts simply transitively on the fine K-types containing δ.
- Inflate $\tau \in \widehat{R_\delta}$ to W_δ. Having fixed a μ_δ, there is a 1-1 correspondence

$$\{\tau \in \widehat{W_\delta} \mid \tau|_{W_\delta^0} = triv\} \longleftrightarrow \{\mu_{\delta,\tau}\}, \qquad triv \longleftrightarrow \mu_\delta.$$

Theorem. *As a W-module,*

$$\mathrm{Hom}_M[\mu_{\delta,1}, \mu_{\delta,\tau}] \cong Ind_{W_\delta}^W[\tau].$$

Example 1

G type E_8, $K = Spin(16)$. This is really the double cover of the rational points of the linear group. Let ω_i be the fundamental weights of K. In this case, $W_\delta = W_\delta^0$. The fine K-types are

K-type	M-type		W_δ
(0)	δ_1,	trivial representation,	$W(E_8)$
(ω_1)	δ_{16},	dimension 16,	$W(E_8)$
(ω_2)	δ_{120},	120 characters,	$W(E_7A_1)$
$(2\omega_1)$	δ_{135},	135 characters,	$W(D_8)$

In all cases, $W_\delta^0 = W_\delta$.

$$\mathrm{Hom}_M[\mu_{\delta_{120}}, \mu_{\delta_{120}}] \cong Ind_{W(E7A1)}^{W(E8)}[triv] = 1_x + 35_x + 84_x,$$

$$\mathrm{Hom}_M[\mu_{\delta_{135}}, \mu_{\delta_{135}}] \cong Ind_{W(D8)}^{W(E8)}[triv] = 1_x + 84_x + 50_x.$$

It is straightforward that the reflection representation 8_z corresponds to the representation of K on $\mathfrak{s}_0$:

$$\omega_8 = 8_z\delta_1 + \delta_{120}. \qquad (1)$$

Quite a few relevant Weyl group representations do not occur in these two formulas.

The next tables give the Weyl representations on $(E_\mu^*)^M$ for μ petite.

Petite K-types for E8

K-type	W-type on $(E_\mu^*)^M$
(0)	$1_x,$
ω_8	$8_z,$
ω_4	$35_x,$
$2\omega_2$	$84_x,$
$\omega_2+\omega_8$	$112_z,$
$4\omega_1$	$50_x,$
$3\omega_1+\omega_7$	$400_z,$
$2\omega_3$	$300_x,$

K-type	W-type on $(E^*)^M$
$\omega_3 + \omega_7$	160_z,
ω_6	28_x,
$\omega_1 + \omega_5$	210_x,
$\omega_1 + \omega_2 + \omega_7$	560_z,
$\omega_2 + \omega_4$	567_x,
$2\omega_1 + \omega_4$	700_x,
$3\omega_1 + \omega_3$	1050_x,
$\omega_1 + \omega_2 + \omega_3$	1344_x,
$3\omega_2$	525_x,
$2\omega_1 + 2\omega_2$	972_x,
$4\omega_1 + \omega_2$	700_{xx},
$6\omega_1$	168_y.

Some Proofs

Only ω_1 is genuine for $K = Spin(16)$, the others factor to a quotient group. In particular, genuine representations of $Spin(16)$ restrict to multiples of δ_{16}. All representations are self dual. We compute

$$\begin{aligned} &\omega_2 \otimes \omega_2 = (2\omega_2) + (\omega_1 + \omega_3) + (2\omega_1) + (\omega_2) + (\omega_4) + (0), \\ &(2\omega_1) \otimes (2\omega_1) = (4\omega_1) + (2\omega_1 + \omega_2) + (2\omega_1) + (2\omega_2) + (\omega_2) + (0). \end{aligned} \tag{2}$$

Furthermore, ω_3 restricts to $35\delta_{16}$, and

$$\omega_1 \otimes \omega_3 = (\omega_1 + \omega_3) + (\omega_2) + (\omega_4). \tag{3}$$

Thus the multiplicity of δ_1 in $(\omega_1 + \omega_3) + (\omega_4)$ is 35. On the other hand, $\dim \omega_4 = 1820$, so the multiplicity of δ_1 in ω_4 is nonzero.

From (2) it follows that the multiplicity is exactly 35, and so

$$\omega_4 \longleftrightarrow 35_x. \qquad (4)$$

We also conclude that the multiplicity of δ_1 in $\omega_1 + \omega_3$ is zero.
From the first equation in (2) we also conclude that $(2\omega_2)$ contains δ_1 84 times, so

$$(2\omega_2) \longleftrightarrow 84_x.$$

Consider $(\omega_1 + \omega_2)$ which restricts to $84\delta_{16}$. Then

$$(\omega_1 + \omega_2) \otimes \omega_1 = (2\omega_1 + \omega_2) + (2\omega_1) + (\omega_1 + \omega_3) + (2\omega_2) + (\omega_2). \qquad (5)$$

Thus only $2\omega_2$ contains δ_1.

These arguments also imply

$$\mathrm{Hom}_M[\omega_1, \omega_3] \simeq 35_x. \qquad (6)$$

Combined with the second equation in (2) we get

$$4\omega_1 \longleftrightarrow 50_x. \qquad (7)$$

We illustrate another aspect of the calculation. We know that $8_z \otimes 50_x = 400_z$. Furthermore, assume that we have done some earlier calculations, and found that

$$\begin{aligned}
&\mathrm{Hom}_M[\omega_1, (3\omega_1)] \cong 50_x, \\
&\mathrm{Hom}_M[\omega_1, \omega_7] \cong 8_z, \\
&\omega_2 + \omega_8 \longleftrightarrow 112_z.
\end{aligned}$$

Then,

$$\begin{aligned}
&(3\omega_1) \otimes (\omega_7) = (3\omega_1 + \omega_7) + (2\omega_1 + \omega_8), \\
&(\omega_1 + \omega_8) \otimes \omega_1 = (2\omega_1 + \omega_8) + (\omega_1 + \omega_7) + (\omega_2 + \omega_8) + (\omega_8).
\end{aligned} \tag{8}$$

Since $\omega_1 + \omega_8 = 120\delta_{16}$, and ω_8 contains eight copies of δ_1, it follows that δ_1 does not occur in $(2\omega_1 + \omega_8) + (\omega_1 + \omega_7)$. We conclude that

$$(3\omega_1 + \omega_7) \longleftrightarrow 400_z. \tag{9}$$

Petite K-types for C_n

$G = Sp(n,\mathbb{R})$, $K = U(n)$.

$$\mu_+(k) := (\underbrace{1,\dots,1}_{k},0,\dots,0), \qquad \mu_-(k) := (0,\dots,0,\underbrace{-1,\dots,-1}_{k})$$

are fine K-types containing the same orbit of a character $\delta \in \widehat{M}$. The stabilizers are $W^0_\delta \cong W(D_k) \times W(C_{n-k})$, and $W_\delta \cong W(C_k) \times W(C_{n-k})$. Then

$$Ind^W_{W_\delta}[triv] = \sum (n-\ell,\ell) \times (0), \quad 0 \le \ell \le min(k, n-k),$$
$$Ind^W_{W_\delta}[\tau] = (n-k) \times (k).$$

The tensor products are

$$\mu_+(k) \otimes \mu_-(k) = \sum_{2a+b=2k} (\underbrace{1,\dots,1}_{a}, \underbrace{0,\dots,0}_{b}, -1,\dots,-1),$$

$$\mu_+(k) \otimes \mu_-(k) = \sum_{2a+b=2k} (\underbrace{2,\dots,2}_{a}, \underbrace{1,\dots,1}_{b}, 0,\dots,0).$$

These K-types are automatically petite, and in fact satisfy $\mu(iZ_\alpha) = 0, \pm 1, \pm 2$.

The precise correspondence is

K-type	W-representation on $(E_\mu^*)^M$
$(\underbrace{2,\ldots,2}_{\ell},0,\ldots,0)$	$(n-\ell)\times(\ell)$
$(\underbrace{1,\ldots,1}_{k},0,\ldots,0,\underbrace{-1,\ldots,-1}_{k})$	$(n-k,k)\times(0).$

Level 2 Petite K-Types

The petite K-types with the property that $\mu(iZ_\alpha) = 0, \pm 1, \pm 2$, have some very nice properties. They are sufficient to determine unitarity in the classical cases, but not the exceptional ones.

Springer Correspondence

- $\mathfrak{g}$ complex semisimple Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ Borel subalgebra,
- $\mathcal{O} \subset \mathfrak{g}$ nilpotent orbit, $\{e, H, f\}$ Lie triple,
- $A(e)$ component group of the centralizer of e,
- $\mathcal{B}_e := \{\mathfrak{b} \mid e \in \mathfrak{b}\}$, the incidence variety.

The Springer correspondence attaches to each $(\mathcal{O}, \psi \in \widehat{A(e)})$ a representation $\sigma(\mathcal{O}, \psi)$ of W which is irreducible or zero. It is the representation of W on $H^{top}(\mathcal{B}_e)^\psi$, (maybe tensored with sgn in this case so that $\sigma((0), triv) = triv \in \widehat{W}$).

We have the following two assertions for μ petite, level 2.

- $\sigma \cong (E^*_\mu)^M$ if and only if the restriction of σ to any rank two Levi does not contain sgn,
- $\sigma \cong (E^*_\mu)^M$ if and only if $\sigma = \sigma(\mathcal{O}, \psi)$ where $\mathcal{O}$ meets a Levi component with factors of type A_1 only.

Joint work with A. Pantano

Principal Series, Classical Groups

Type C

Consider

$$\delta_k = (\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{n-k}) \longleftrightarrow \begin{aligned} \mu_k^+ &= (\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{n-k}) \\ \mu_k^- &= (\underbrace{0,\dots,0}_{n-k},\underbrace{-1,\dots,-1}_{k}) \end{aligned} \tag{10}$$

Then the relevant K-types are

$$
\begin{array}{ll}
\text{K-type} & W(C_k \times C_{n-k})\text{-type} \\
(\underbrace{1,\dots,1}_{a+k},0,\dots,0,\underbrace{-1,\dots,-1}_{a}) & (triv)\otimes[(a,n-k-a)\times(0)] \\
(\underbrace{1,\dots,1}_{k-b},0,\dots,0,\underbrace{-1,\dots,-1}_{b}) & [(k-b)\times(b)]\otimes(triv) \\
(\underbrace{2,\dots,2}_{b},\underbrace{1,\dots,1}_{k},0,\dots,0) & (triv)\otimes[(n-k-b)\times(b)] \\
(\underbrace{2,\dots,2}_{b},\underbrace{1,\dots,1}_{k-2b},\underbrace{0,\dots,0}_{n-k+b}) & [(b,k-b)\times(0)]\otimes(triv).
\end{array}
\tag{11}
$$

We get another set of K-types by changing all the signs to minuses.

These K-types are petite because they are factors of the tensor products

$$\Lambda^r(\mathbb{C}^n) \otimes \Lambda^s(\mathbb{C}^n), \quad \text{or} \quad \Lambda^r(\mathbb{C}^n) \otimes \Lambda^s((\mathbb{C}^*)^n). \tag{12}$$

Type D

These are the cases $SO(2n, 2n)$ and $SO(2n+1, 2n+1)$. For simplicity, just use $2n$, the other case is equivalent. Consider

$$\delta_k = (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k}) \longleftrightarrow \quad \begin{aligned} \mu_k^+ &= (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k}) \otimes (0, \dots, 0) \\ \mu_k^- &= (0, \dots, 0) \otimes (\underbrace{0, \dots, 0}_{n-k}, \underbrace{-1, \dots, -1}_{k}) \end{aligned} \tag{13}$$

Then the relevant K-types are

$$
\begin{aligned}
&(\underbrace{1,\dots,1}_{a+k},0,\dots,0)\otimes(\underbrace{1.\dots,1}_{a},0,\dots,0),\\
&(\underbrace{1,\dots,1}_{k-b},0,\dots,0)\otimes(\underbrace{1,\dots,1}_{b},0,\dots,0),\\
&(\underbrace{2,\dots,2}_{b},\underbrace{1,\dots,1}_{k},0,\dots,0)\otimes(0,\dots,0),\\
&(\underbrace{2,\dots,2}_{b},\underbrace{1,\dots,1}_{k-2b},\underbrace{0,\dots,0}_{n-k+b})\otimes(0,\dots,0).
\end{aligned}
\tag{14}
$$

We get another set of K-types with the same properties by interchanging the factors.

Principal Series, Type E8

δ_{16}

In the case of genuine principal series, only level $\leq 3/2$ K-types are petite. This has to do with the representation theory of the double cover $\widetilde{SL(2,\mathbb{R})}$. The matchings are

$$\begin{array}{rl} \textit{K-type} & \textit{M-type} \\ (\omega_1) & 1_x, \\ (\omega_1+\omega_2) & 84_x, \\ (\omega_3) & 35_x, \\ (\omega_7) & 8_z, \\ (3\omega_1) & 50_x. \end{array} \tag{15}$$

The missing ones are

$$\begin{array}{rl} \text{K-type} & \text{M-type} \\ (\omega_1+\omega_8) & 8_z + 112_z, \\ (2\omega_1+\omega_7) & 400_z + \dots, \\ (\omega_2+\omega_3) & 300_x + \dots, \\ (\omega_1+\omega_4) & 210_x + \dots, \\ (\omega_5) & 210_x + \dots\ . \end{array} \tag{16}$$

δ_{135}

$$\begin{array}{rl} \text{K-type} & \text{W-type} \\ (2\omega_1) & 8 \times 0 \\ (\omega_4) & 71 \times 0 \\ (4\omega_1) & 44 \times 0 \\ (\omega_1+\omega_7) & 7 \times 1 \\ (2\omega_2) & 62 \times 0 \\ (2\omega_1+\omega_4) & 6 \times 2 \\ (2\omega_1+\omega_2) & 4 \times 4_- \\ (\omega_2+\omega_8) & 61 \times 1 \\ (\omega_1+\omega_3) & 6 \times 2 \\ (2\omega_1+\omega_8) & 5 \times 3 + 7 \times 1. \end{array} \tag{17}$$

K-type	W-type
(ω_6)	6×11
$(2\omega_1 + \omega_4)$	$53 \times 0 + 4 \times 4_+ + 71 \times 0 + 51 \times 2 + 31 \times 4 + 42 \times 2$
$(\omega_1 + \omega_5)$	$4 \times 4_+ + 51 \times 2 + 6 \times 11 + 71 \times 0 + 6 \times 2.$

Theorem. *A parameter (δ_{135}, ν) is unitary only if the corresponding spherical parameter (δ_1, ν) is unitary for D_8.*

δ_{120}

$$
\begin{array}{rl}
\text{K-type} & \text{W-type} \\
(\omega_2) & 1_a \otimes 2 \\
(2\omega_2) & 21'_b \otimes 11 \\
(\omega_1+\omega_3) & 27_a \otimes 2 \\
(\omega_8) & 1_a \otimes 11 \\
(\omega_4) & 7'_a \otimes 11 \\
(\omega_1+\omega_7) & 7'_a \otimes 2 \\
(\omega_2+\omega_8) & 27_a \otimes 11 + 21'_b \otimes 2 + \dots \\
(\omega_1+\omega_5) & 56'_a \otimes 11 \\
(2\omega_1+\omega_8) & 56'_a \otimes 2 \\
(\omega_1+\omega_2+\omega_7) & 35_b \otimes 11 + \dots\ .
\end{array}
\tag{18}
$$

K-type	$M-type$
$(2\omega_1+\omega_2)$	$35_b\otimes 11+\dots$
$(3\omega_1+\omega_7)$	$105_b\otimes 11+\dots$
$(2\omega_1+\omega_4)$	$105_b\otimes 2+\dots$.

Theorem. *A parameter* (δ_{120},ν) *is unitary only if the corresponding spherical parameter* (δ_1,ν) *is unitary for* E_7A_1*.*

Useful Identities

Let $\mu,\ \mu_1,\ \mu_2$ be genuine representations. The main point is that δ_{16} is the unique genuine representation of M, and it IS the irreducible K-module ω_1.

- As a W-representation,

$$\mathrm{Hom}_M[\mu_1,\mu_2] \cong \mathrm{Hom}_M[\mu_1,\omega_1] \otimes \mathrm{Hom}_M[\mu_2,\omega_1].$$

  Decompose LHS as a K-module, RHS as a W-module.

- For $\delta = \overline{\delta}_{120}$ or $\delta = \overline{\delta}_{135}$, (irreducible representation of M)

$$\mathrm{Hom}_M[\delta,\omega_1 \otimes \mu] = Res_{W_\delta}\,\mathrm{Hom}_M[\omega_1,\mu].$$

  Decompose $\omega_1 \otimes \mu$ as a K-module, the RHS as a W_δ-module.