Primitive Ideals and Unitarity

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The Unitary Dual

NOTATION. G is the rational points over $F = \mathbb{R}$ or a p-adic field, of a linear connected reductive group.

A representation $(\pi, \mathcal{H})$ on a Hilbert space is called unitary, if $\mathcal{H}$ admits a G invariant positive definite inner product.

PROBLEM. Classify the unitary dual of G.

ALGEBRAIC PROBLEM. By results of Harish-Chandra, and others, this is turned into an algebraic problem, about $(\mathfrak{g}, K)$ modules.

In this talk, we mostly deal with complex reductive group viewed as a real group. The answer is known for classical groups, but not for groups of exceptional type. For $Spin(n, \mathbb{C})$, the complete answer is also not known.

Admissible Modules

Notation

- $B = HN$ is a Borel subgroup, with $H = TA$, T the toral part, A the split part. The corresponding Lie algebras are $\mathfrak{b}$, $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{n}$.
- $\mathfrak{g} := Lie(G)$, $\mathfrak{g}_c \cong \mathfrak{g}^L \oplus \mathfrak{g}^R$, $U(\mathfrak{g}_c) \cong U(\mathfrak{g}^L) \otimes U(\mathfrak{g}^R)$.
- Δ the positive roots corresponding to $\mathfrak{b}$, Π the simple roots, W the Weyl group.
- K a maximal compact subgroup satisfying $K \cap B = T$, and $G = K \cdot B$.
- μ a weight of T, $V(\mu)$ the irreducible representation of K with extremal weight μ.
- $X(\mu, \nu)$ the principal series representation $Ind_B^G[\mathbb{C}_\mu \otimes \mathbb{C}_\nu]_{K-finite}$, where $\nu \in \widehat{A}$. Let $L(\mu, \nu)$ be the unique irreducible subquotient of $X(\mu, \nu)$ containing the K-type $V(\mu)$.

Theorem (P-R-V, Zh). *Any irreducible* $(\mathfrak{g}_c, K)$ *module is isomorphic to an* $L(\mu, \nu)$.
$L(\mu, \nu) \cong L(\mu', \nu')$ *if and only if there is* $w \in W$ *such that* $w\mu = \mu',\ w\nu = \nu'$.

We will parametrize representations by (λ^L, λ^R) such that

$$\mu = \lambda^L - \lambda^R, \qquad \nu = \lambda^L + \lambda^R.$$

Hermitian Structure. $U(\mathfrak{g}_c)$ has a $*$ operation which interchanges $\mathfrak{g}^L$ and $\mathfrak{g}^R$. A module $L(\mu, \nu)$ is hermitian if and only if there is $w \in W$ such that $w\mu = \mu,\ w\nu = -\overline{\nu}$.
Annihilator. The annihilator of an $L(\lambda^L, \lambda^R)$ is of the form $U(\mathfrak{g}) \otimes \check{I} + I \otimes U(\mathfrak{g})$, with I a primitive ideal (according to Duflo, the annihilattor of an irreducible highest weight module). When the module is hermitian, $I = \check{I}$.

In such a case, we write λ for the infinitesimal character. We can think of it as the W-orbit of λ^L, or equivalently λ^R.

Arthur Parameters

Let $\check{\mathfrak{g}}$ be the dual Lie algebra. According to Arthur's conjectures, attached to each ($\check{G}$ conjugacy class of)

$$\Psi : W_{\mathbb{R}} \times SL(2) \longrightarrow \check{\mathfrak{g}},$$

there should be a packet of representations which are unitary. Their characters should have nice properties with respect to endoscopic groups. The restriction of Ψ to $SL(2)$ determines a nilpotent orbit in the centralizer of $\Psi(W_{\mathbb{R}})$, which is a Levi component, say M_Ψ. $\Psi(W_{\mathbb{R}})$ is assumed to be bounded. $\Psi|_{W_{\mathbb{R}}}$ determines a unitary character ψ of M_Ψ. On the other hand $\Psi|SL(2)$ also determines an Arthur parameter for M_Ψ. The packet for Ψ is obtained by tensoring the representations associated to $\Psi_{SL(2)}$ with ψ, and inducing up to G.

Let

$$h := d\Psi(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}), \qquad e := d\Psi(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), \qquad f := d\Psi(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}),$$

Definition (B-V). *The Arthur parameter is called special unipotent, if $\Psi|_{W_{\mathbb{R}}}$ is trivial. The packet is defined to be the set of irreducible representations with infinitesimal character $\lambda := \frac{1}{2}h$, and annihilator the unique maximal primitive ideal I with this infinitesimal character.*

Since such maps Ψ are in one-to-one correspondence with nilpotent orbits, we use $\check{\mathcal{O}}$ as parameters.

It is sufficient to consider the case when $\check{\mathcal{O}}$ is even; this means $\lambda_{\check{\mathcal{O}}}$ is integral. If not, there is a reduction to this via unitary induction.

Character Formulas of Unipotent Parameters

- $Unip(\check{\mathcal{O}})$ the set of unipotent represntations attached to $\check{\mathcal{O}}$.
- $\mathcal{G}(\check{\mathcal{O}})$ the Grothendieck group spanned by the characters of the $Unip(\check{\mathcal{O}})$.

When $\check{\mathcal{O}}$ is *special*, Lusztig defines a finite group $\overline{A(\check{\mathcal{O}})}$ with the following properties:

- $\overline{A(\check{\mathcal{O}})}$ is a quotient of $A(\check{\mathcal{O}})$, the component group of the centralizer of an element in $\check{\mathcal{O}}$.
- There is a bijection

  $$[x] \in [\overline{A(\check{\mathcal{O}})}] \longleftrightarrow \sigma_x,$$

  between conjugacy classes in $\overline{A(\check{\mathcal{O}})}$ and representations in the left cell attached to $I(\check{\mathcal{O}})$.

- There is a bijection

$$\chi \in \widehat{\overline{A(\check{\mathcal{O}})}} \longleftrightarrow L_\chi \in Unip(\check{\mathcal{O}}).$$

- There are virtual characters R_{σ_x} (which are computable in terms of $X(\lambda^L, \lambda^R)$) so that the change of bases between the R_x and L_χ is the same as between the class functions χ_x, and characters χ of $\overline{A(\check{\mathcal{O}})}$.

These results can be found in [B-V]. Lusztig's results on the character theory of the finite Chevalley groups, and their consequences for primitive ideals play a prominent role.

These formulas imply the unitarity of the special unipotent representations in the classical cases.

Unitary Dual

The general strategy to classify the unitary dual is to find a finite set of representations which we will call basic (with nice properties) so that the full dual is obtained from them by unitary induction and complementary series. To make this idea more precise, we need the notion of induced nilpotent, of Lusztig-Spaltenstein. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} \subset \mathfrak{g}$ be a parabolic subalgebra, and $\mathcal{O}_{\mathfrak{m}}$ a nilpotent orbit in $\mathfrak{m}$. Then there is a unique nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ with the property that its intersection with $\mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$ is a dense subset. We write

$$\mathcal{O} = Ind_{\mathfrak{m}}^{\mathfrak{g}}[\mathcal{O}_{\mathfrak{m}}].$$

The orbit $\mathcal{O}$ only depends on $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m}$, not on the particular parabolic subalgebra. A representation π has a wave front set denoted $WF(\pi)$, which in the case of complex groups is the closure of a single nilpotent orbit.

If $\pi = Ind_M^G[\pi_M]$, then $WF(\pi) = Ind_{\mathfrak{m}}^{\mathfrak{g}}[WF(\pi_M)]$.

So the basic set should consist of representations of G with WF-set cuspidal, *i.e.* not induced from any proper Levi component. It is natural to expect that this set should consist of special unipotent representations. There aren't enough of them to describe the unitary dual in this way. For example the oscillator/metaplectic/Segal-Shale-Weil representation must be one of the basic representations, but is not special unipotent.

Basic Representations

For each classical Lie algebra, we give a set of nilpotent orbits. For each such orbit we describe an infinitesimal character $\lambda(\mathcal{O})$. The basic representations $Unip(\mathcal{O})$ are the ones with this infinitesimal character, and annihilator the maximal primitive ideal. The WF-set is the closure of $\mathcal{O}$.

- $Unip(\mathcal{O})$ is in one-to-one correspondence with $\widehat{A(\mathcal{O})}$.
- L_1 has K-structure equivalent to $R(\overline{\mathcal{O}})$, the ring of regular functions on $\overline{\mathcal{O}}$.
- Character formulas similar to the ones for special unipotent represenions hold.
- The K-spectrum of L_χ has a structure of $S(\mathfrak{g})$ module, it is essentially $\dim \chi$ times $R(\overline{\mathcal{O}})$.

Remark

This parametrization is along the lines of the orbit method. The combinatorics defining $\lambda(\mathcal{O})$ are in the spirit of Lusztig's symbols for primitive ideal cells.

The list of nilpotents is larger than all cuspidal ones. Also some special unipotent parameters are missing, for example the orbit with Jordan blocks (22) in $Sp(2)$ appears as WF-set for special unipotent parameters $(1, 0)$ as well as $(1/2, 1/2)$. Only the first one is listed as basic. These two parameters are part of a complementary series.

Type B

Let

$$(c_0)(c_1c_2)\dots(c_{2m}c_{2m+1}) \qquad c_{2i} > c_{2i+1}$$

be the sizes of the columns of the Jordan blocks of $\mathcal{O}$ in decreasing order. Then the coordinates of $\lambda(\mathcal{O})$ are the union of the coordinates

$$\begin{aligned} (c_0) &\quad\longleftrightarrow\quad (\frac{c_0}{2}-1,\dots,1/2) \\ (c_{2i-1}c_{2i}) &\quad\longleftrightarrow\quad (\frac{c_{2i-1}}{2},\dots,-\frac{c_{2i}}{2}) \end{aligned} \tag{1}$$

Type C

Let

$$(c_0c_1)\dots(c_{2m-1}c_{2m})(c_{2m+1}) \qquad c_{2i-1} > c_{2i}$$

be the sizes of the columns of the Jordan blocks of $\mathcal{O}$. Then the coordinates of $\lambda(\mathcal{O})$ are the union of the coordinates

$$\begin{aligned} (c_{2m+1}) \quad &\longleftrightarrow \quad (\frac{c_{2m+1}}{2}-1,\dots,0) \\ (c_{2i}c_{2i+1}) \quad &\longleftrightarrow \quad (\frac{c_{2i}}{2},\dots,-\frac{c_{2i+1}}{2}) \end{aligned} \tag{2}$$

Type D

Let

$$(c_0 c_{2m-1})(c_1 c_2)\dots(c_{2m-3} c_{2m-2}) \qquad c_{2i} > c_{2i+1}$$

be the sizes of the columns of the Jordan blocks of $\mathcal{O}$. Then the coordinates of $\lambda(\mathcal{O})$ are the union of the coordinates

$$\begin{aligned} (c_0 c_{2m-1}) &\quad\longleftrightarrow\quad (\frac{c_0}{2} - 1, \dots, -\frac{c_{2m-1}}{2}) \\ (c_{2i-1} c_{2i}) &\quad\longleftrightarrow\quad (\frac{c_{2i-1}}{2}, \dots, -\frac{c_{2i}}{2}) \end{aligned} \qquad (3)$$

Lie Algebras of Exceptional Type

We will concentrate on nilpotent orbits related to what Lusztig calls triangular nilpotent orbits.

Type	*Label*	*Component Group*
G_2	$G_2(a_1)$	S_3
F_4	$F_4(a_3)$	S_4
E_6	$D_4(a_1)$	S_3
E_8	$E_8(a_7)$	S_5

These are interesting nilpotents in many ways. In particular the component groups $A(\mathcal{O})$ are not powers of $\mathbb{Z}_2$. These nilpotent orbits are even. To each of them corresponds an Arthur parameter, which is $h/2$, but computed in terms of the dual roots.

The Arthur packets, and their character theory are computed in [BV].

In this talk I am interested in unitary parameters for a set of nilpotent orbits which are related to the triangular ones. They are not special unipotent. The next slides will describe them in detail.

$G_2(a_1)$

Lusztig's matching of Weyl group representations is as follows (Carter's notation):

Conjugacy Class	*Weyl representation*	*Springer correpondence*
(1)	$(2,1)$	$G_2(a_1)$
(g_2)	$(1,3)''$	A_1^l
(g_3)	$(2,2)$	A_1^s.

A_1^l and A_1^s are the nilpotent orbits for which we want to find unitary parameters. The label A_1^l indicates that one should consider the unitarily induced module from the trivial representation on the Levi component of this type. The WF-set is $G_2(a_1)$. The induced representation is irreducible, and we can form a complementary series. In terms of the realization of the root system in Bourbaki, the parameter is

$$(1/2, -1, -1/2) + \eta(1, 0, -1).$$

The parameter is the special unipotent one for $G_2(a_1)$. The representation is reducible, there are two factors which are unipotent. Being an endpoint of a complementary series, the two factors are unitary. The next reducibility point is at $\eta = 5/6$. The WF-set of the spherical representation is A_1^l. The integral system of this infinitesimal character is A_2. This is the right representation, provided it is unitary. The answer is yes, the two factors do not change sign.

In the case of A_1^s, the situation is simpler. The parameter is $(0, 1/2, -1/2) + \nu(2, -1 - 1)$. It is special unipotent at $\nu = 1/2$, but the induced module is irreducible. The next reducibility gives a parameter integral for $A_1 \times A_1$, the spherical representation is unitary and has WF-set A_1^s.

In fact for G_2, every nilpotent orbit has a family (some several) of unitary representations with the desired properties.

$D_4(a_1)$

Again, in Carter's notation,

Conjugacy Class	*Weyl representation*	*Springer correpondence*
(1)	$(80, 7)$	$D_4(a_2)$
(g_2)	$(60, 8)$	A_3A_1
(g_3)	$(10, 9)$	$2A_2A_1$.

As for G_2, the induced from the trivial representation on A_3A_1 has WF-set $D_4(a_1)$, and a complementary series; the parameter is

$$\begin{aligned}&(-5/4, -1/4, 3/4, -5/4, -1/4, -3/4, -3/4, 3/4)\\&+\nu(1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2).\end{aligned} \tag{4}$$

The special unipotent parameter for $D_4(a_1)$ is a $\nu = 1/2$. There are two factors, but the signature does not change. So at the next reducibility point, $\nu = 1$, there is a spherical unitary representation with WF-set A_3A_1.

The induced from the trivial representation on $2A_2A_1$ has similar properties. The special unipotent parameter for $D_4(a_1)$, arises at $\nu = 1/2$. The representation is irreducible. The next reducibility point is at $\nu = 2/3$. The spherical irreducible module is unitary, and has WF-set $2A_22A_1$.

$F_4(a_3)$

Conjugacy Class	*Weyl*	*Springer*	*Unitary*
(1)	$(12,4)$	$F_4(a_3)$	$(2,1,1,0)$
(g_2')	$(9,6)''$	B_2	$(2,2,1,0)$
(g_4)	$(4,7)''$	$A_2\widetilde{A_1}$	$(11/4,5/4,1/4,-3/4)$
(g_3)	$(6,6)'$	$\widetilde{A_2}A_1$	$(8/3,1/3,1/3,1)$
(g_4)	$(16,5)$	$C_3(a_1)$	$(5/2,-1/2,1,0)$.

For each of these nilpotent orbits, there is an induced nilpotent, and a complementary series with the same properties as in the previous cases.

Remark

The nilpotent orbits B_2 and $C_3(a_1)$ are also unitarily induced, one from the cuspidal orbit 21^4 in C_3, the other from the orbit $2^2 1^3$ in B_3. The unitarily induced parameters are $(0, 5/2, 3/2, 1/2)$ and $(3/2, -3/2, 3/2, , 1/2)$. These are *different* from the endpoints of complementary series above.

Bibliography

References

[BV] D. Barbasch, D. Vogan *Unipotent representations of complex semisimple groups* Ann. of Math., 121, (1985), 41-110

[V] D. Vogan *The unitary dual of GL(n) over an archimedean field*, Inv. Math., 83 (1986), 449-505.