SPHERICAL UNITARY REPRESENTATIONS FOR SPLIT REAL AND P-ADIC GROUPS

DAN BARBASCH

1. Introduction

1.1. **Positive definite functions.** We start with the classical notion of *positive definite functions.*

Definition. *A continuous function* $f:\mathbb{R}\longrightarrow\mathbb{C}$ *is called positive definite if it satisfies*

(1) $f(-x)=\overline{f(x)}$,
(2) *the matrix* $\big(f(x_i-x_j)\big)$ *for any* $x_1,\dots,x_n\in\mathbb{R}$ *is positive definite.*

For such a function $f(0)\ge 0$, and because

$$\begin{bmatrix} f(0) & f(x-y)\\ f(y-x) & f(0)\end{bmatrix} \tag{1.1.1}$$

is positive definite, we get $|f(x-y)|^2\le|f(0)|^2$. So if $f(0)=0$, then $f\equiv 0$. We assume $f\not\equiv 0$, so we normalize so that $f(0)=1$.

The main result about positive definite functions is Bochner's theorem.

Theorem. *A continuous function* f *is positive definite if and only if it is the Fourier transform of a positive finite measure.*

This notion generalizes to locally compact topological groups.

Definition. *Let* G *be a locally compact topological group. A continuous function* $f:G\longrightarrow\mathbb{C}$ *is positive definite if*

(1) $f(x^{-1})=\overline{f(x)}$,
(2) *the matrix* $\big(f(x_i^{-1}x_j)\big)$ *for any* $x_1,\dots,x_n\in G$ *is positive definite.*

The *only* natural way to construct such functions is via representation theory.

A representation is a group homomorphism $\pi:G\longrightarrow Aut(\mathcal{X})$ where $\mathcal{X}$ is a Hilbert space, which is continuous into bounded linear operators. It is called

irreducible if $\mathcal{X}$ has no nontrivial proper closed G-invariant subspaces,
completely reducible if any $G-$invariant closed subspace $\mathcal{W}\subset\mathcal{X}$ has a $G-$invariant complement,
unitarizable if there is an inner product $\langle\ ,\ \rangle$ for which $\pi(g)$ are unitary operators, *i.e.*

$$\langle\pi(g)v,\pi(g)w\rangle=\langle v,w\rangle,\qquad g\in G,\ v,w\in\mathcal{X}. \tag{1.1.2}$$

We will consider representations which are *algebraic* only, no topology involved. In that case the same definitions hold, one drops the condition of *closed*.

Suppose π is unitarizable. We can construct a positive definite function as follows. Let $0 \neq v \in \mathcal{X}$ be a vector of norm 1. Then

$$\Phi_v(g) := \langle v, \pi(g)v\rangle \tag{1.1.3}$$

is positive definite.

Conversely, let f be positive definite. Then define

$$\mathcal{V} := span_{\mathbb{C}}\ \{L_x f\ :\ x \in G\}. \tag{1.1.4}$$

This is a $G-$module with action $\pi(g)v(x) := (L_g v)(x) = v(g^{-1}x)$. The space

$$\mathcal{N} := \{v \in \mathcal{V}\ :\ |v| = 0\} \tag{1.1.5}$$

is the radical of the hermitian bilinear form

$$\langle L_g f, L_h f\rangle := f(g^{-1}h). \tag{1.1.6}$$

This is because if $v \in \mathcal{N}$,

$$\langle w, v\rangle \leq |v| \cdot |w| = 0, \quad \forall w \in \mathcal{V}. \tag{1.1.7}$$

Let $\mathcal{X}$ be the completion of $\mathcal{V}/\mathcal{N}$ with respect to the induced inner product $\langle\ ,\ \rangle$. This is a unitary representation of G such that $\Phi_f(g) = f(g)$.

1.2. **The General Problem.** Given a locally compact topological group G, find all irreducible unitarizable representations.

If G is compact, the Peter-Weyl theorem provides an answer. First every finite dimensional representation is unitarizable. Namely if (π, V) is a complex finite dimensional representation, choose any inner product $(\ ,\ )$. Then

$$< v, w >:= \int_G (\pi(g)v, \pi(g)w)\ dg \tag{1.2.1}$$

is an invariant inner product. If (σ, V_σ) is an irreducible representation, define the space of matrix entries of σ

$$\mathcal{F}_\sigma := span\{f_{v,w}(g) := \langle v, \pi(g)w\rangle\}. \tag{1.2.2}$$

This is a module for $G \times G$ by $\rho(g_1, g_2)f(x) := f(g_1^{-1}xg_2)$. Then $\mathcal{F}_\rho \cong V_\sigma^* \otimes V_\sigma$. The Peter-Weyl theorem states that the regular representation $L^2(G)$ decomposes as

$$L^2(G) = \bigoplus_{\sigma\in\widehat{K}} V_\sigma^* \otimes V_\sigma. \tag{1.2.3}$$

So in some sense the problem in this case is solved. But even an explicit answer can be very difficult to come by. For example for $G = S_n$ there is an entire field of combinatorics devoted to representation theory. For the kind of groups $G_{x,0}$ in the previous lectures the answer is quite intricate. For noncompact groups this gets even more difficult. The space $L^2(G)$, if it even makes sense to talk about a decomposition, will have a continuous and a discrete spectrum. Not all unitary representation will occur. For example the trivial representation is unitary, but unless G is compact, it will *not* occur in $L^2(G)$.

1.3. **An Example.** Let $G = SL(2,\mathbb{R})$. It contains a maximal compact subgroup $K = SO(2)$. The quotient space G/K is a very interesting space for analysis. It identifies with the upper half plane $\{z \ : \ Imz > 0\}$ by

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G/K \longleftrightarrow \frac{ai+b}{ci+d}. \tag{1.3.1}$$

We are interested in the spectral decomposition of $\Delta = y^2(\partial_x^2 + \partial_y^2)$ on $L^2(\mathcal{H})$ with the invariant measure $y^{-2}(dx^2 + dy^2)$. For this we must look for eigenfunctions $\Delta F = \alpha F$ satisfying the additional invariance condition $F(r(\theta)\cdot z) = F(z)$ where

$$F(r(\theta)\cdot z) := F\left(\frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}\right), \qquad r(\theta) := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1.3.2}$$

Lift such functions to G; they satisfy $F_\alpha(r(\theta_1)gr(\theta_2)) = F_\alpha(g)$, as well as a differential equation coming from Δ. They are of the form $\Phi_{v,\pi}$ for π the **spherical principal series**. Let

$$B := \{b := \begin{bmatrix} e^{r/2} & y \\ 0 & e^{-r/2} \end{bmatrix}\}, \quad A = \{a(r) := \begin{bmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{bmatrix}\}, \quad N = \{n(y) := \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\}$$
$$K := \{r(\theta)\}. \tag{1.3.3}$$

Note that G has the decompositions $G = BK$ and $G = KAK$. Then let

$$X(\nu) := \{f : G \longrightarrow \mathbb{C} \ : f(gb) = e^{-(\nu-1/2)r}f(g) \quad f\mid_K \in L^2(K)\} \tag{1.3.4}$$

The group G acts on $X(\nu)$ by $\pi_\nu(g)f(x) = f(g^{-1}x)$. The space $X(\nu)$ identifies with $L^2(K)$ so it is a Hilbert space. The $K-$finite functions have a basis $f_{2n}(r(\theta)a(r)n(y)) = e^{2ni\theta}e^{-(\nu-1/2)r}$. The usual inner product on $L^2(K)$ is unitary for the action of G only when $\nu \in \sqrt{-1}\mathbb{R}$. Let $v(kb) = |x|^{-\nu+1/2}$. The functions F_ν from before are $F_\nu(g) = \langle v, \pi(g)v\rangle$. Since $F(k_1gk_2) = F(g)$, it is enough to give the values on the elements of the form $\begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix}$:

$$F_\nu(r) = \frac{1}{2\pi}\int_0^{2\pi} (e^r\cos^2 u + e^{-r}\sin^2 u)^{\nu-1/2}\,du. \tag{1.3.5}$$

For the spectral decomposition of $L^2(\mathcal{H})$ we only need $\nu \in i\mathbb{R}$. It turns out that the function in (1.3.5) is positive definite precisely for $-1/2 \le \nu \le 1/2$. The relation $F_\nu = F_{-\nu}$ holds. F_ν is a special case of the hypergeometric function, but one would be hard pressed to get the positivity result without representation theory.

There are other problems coming from number theory/automorphic forms where one wants information about $L^2(\Gamma\backslash\mathcal{H})$ where $\Gamma \subset G$ is an arithmetic subgroup. In this case the $\nu \in \mathbb{R}$ play a role. For example since

$vol(G/\Gamma) < \infty$, the trivial representation occurs. Often all one has available is information about the nature of the unitary representations that occur.

1.4. **General Strategy.** The way one proceeds towards solving the problem of classifying the unitary dual is via several steps:

(1) Classify the irreducible admissible modules
(2) Classify the irreducible hermitian modules
(3) Find the unitary modules

In the rest of the notes I will spend considerable time on (1) and (2). The general framework will be a group G which is the rational points over a local field real or p-adic of a connected linear reductive group defined over $\mathbb{G}$. The group will be split. I will concentrate on *spherical* representations. The group G has a maximal compact subgroup $K = K_0$. This group is unique up to conjugation in the real case, a special choice in the $p-$ adic case. A representation is called *spherical* if it has a fixed vector under K. The question is to classify spherical irreducible representations. These are the ones who give rise to the $K-$biinvariant positive definite functions. But to do this for real groups I will study a wider class of representations for $p-$adic groups, namely the category of representations admitting fixed vectors under an Iwahori subgroup $\mathbb{I}$.

2. Split p-Adic Groups

2.1. **Iwahori subgroups.** First some notation; let $\mathbb{F}$ be a local field of characteristic zero. Then

$$\mathbb{F} \supset \mathcal{R} \supset \varpi\mathcal{R} = \mathcal{P}, \tag{2.1.1}$$

where $\mathcal{R}$ is the ring of integers and $\mathcal{P}$ the maximal ideal generated by the uniformizer ϖ. For a split group reductive connected group $\mathbb{G}$ defined over $\mathbb{Q}$, denote by $G := \mathbb{G}(\mathbb{F})$ its rational points, by $K_0 := \mathbb{G}(\mathcal{R})$ the open compact subgroup with entries in $\mathcal{R}$, and by

$$K_i := \{x \in \mathbb{G}(\mathcal{R}) \ : \ x \equiv Id \ mod \ (\varpi^i)\} \tag{2.1.2}$$

Then if $\mathbb{F}_q := \mathcal{R}/\mathcal{P}$ is the residue field, there is an exact sequence

$$1 \longrightarrow K_1 \longrightarrow K_0 \xrightarrow{\pi} \mathbb{G}(\mathbb{F}_q) \longrightarrow 1. \tag{2.1.3}$$

Let $B = AN$ be a Borel subgroup. Then $G = KB$. We are interested in $K-$ biinvariant functions. Note also that $G = KAK$, so such functions are determined by their restrictions to A.

An Iwahori subgroup is a group conjugate to $\mathbb{I} := \pi^{-1}(B(\mathbb{F}_q))$, where $B(\mathbb{F}_q)$ is a Borel subgroup of $\mathbb{G}(\mathbb{F}_q)$. This group is open compact, and will play an essential role in the analysis of the induced modules. A character χ of A is called *unramified* if it is trivial on $A \cap K$. The induced principal series is

$$X(\chi) := \mathrm{Ind}_B^G[\mathbb{C}_\chi]_{lc} \tag{2.1.4}$$

where lc refers to functions $f : G \longrightarrow \mathbb{C}$ which are locally constant in addition to $f(gb) = \chi(b^{-1})\delta_B(b)^{1/2}$. The action is the usual one. Using lc has the advantage that $X(\chi)$ is acted upon by the *Hecke algebra*

$$\mathcal{H}(G) := \{f : G \longrightarrow \mathbb{C} \ \mid f \text{ locally constant with compact support}\}. \tag{2.1.5}$$

The action of $\mathcal{H}$ is

$$\pi(f)v := \int_G f(g)\pi(g)v \ dg, \tag{2.1.6}$$

and the algebra structure is convolution

$$f * g(x) = \int_G f(yx^{-1})g(x) \ dx. \tag{2.1.7}$$

The proper framework to do representation theory is the notion of an *admissible representation.*

Definition. *A representation (π, V) is called admisible if*

1. *The stabilizer $Stab_G(v)$ of any vector $v \in$ is an open compact subgroup.*
2. *For any irreducible representation τ of an open compact subgroup $\mathcal{K}$,* $\dim \operatorname{Hom}_{\mathcal{K}}[\tau, V] < \infty$.

Irreducible modules, as well as induced modules of the form (2.1.4) are admissible. As in the example 1.3 the module $X(\nu)$ has a unique $K-$fixed vector $f(kb) = \chi(b^{-1})\delta_B(b)^{1/2}$, and we can use it to form a spherical function $\Phi_\nu(g) =< f, \pi(g)f >$. It will turn out that these are (essentially) all spherical functions.

2.2. **Hermitian Modules.** A group representation (π, V) is called hermitian if V admits a nonzero sesquilinear form $< \ , \ >$ satisfying

$$< \pi(g)v, \pi(g)w >=< v, w >, \quad \forall g \in G, \ v, w, \in V \tag{2.2.1}$$

For a representation of $\mathcal{H}$ this relation translates into the following:

$$< \pi(f)v, w >=< v, \pi(f^*)w > \tag{2.2.2}$$

where $f^*(x) := \overline{f(x^{-1})}$.

In general if V is a complex vector space, its hermitian dual is

$$V^h := \{\ell : V \longrightarrow \mathbb{C} \ : \ \ell(\lambda_1 v_1 + \lambda_2 v_2) = \overline{\lambda_1}\ell(v_1) + \overline{\lambda_2}\ell(v_2)\}. \tag{2.2.3}$$

The complex structure is $(\lambda \cdot \ell)(v) := \lambda\ell(v)$. If (π, V) is a representation of an algebra $\mathcal{M}$, then V^h will not in general have an $\mathcal{M}$ module structure. What is needed is a complex conjugate linear automorphism $^* : \mathcal{M} \longrightarrow \mathcal{M}$. Then we can define (π^h, V^h) by the formula $(\pi^h(m)\ell)(v) = \ell(\pi(m^*)v)$. A module is hermitian if and only if there is a nontrivial complex linear map $h : V \longrightarrow V^h$ which *intertwines* π and π^h, *i.e.*

$$\pi^h(m) \circ h = h \circ \pi(m). \tag{2.2.4}$$

When (π, V) is irreducible h must be an isomorphism.

2.3. **The Iwahori-Hecke Algebra.** One of the main tools for studying $X(\nu)$ is the *Jacquet functor.* Frobenius reciprocity states that

$$\mathrm{Hom}_G[V, \mathrm{Ind}_B^G[\sigma]] \cong \mathrm{Hom}_A[V_N, \sigma\delta_B^{1/2}], \tag{2.3.1}$$

where

$$V_N := V/span\{\pi(n)v - v\}_{n\in N}. \tag{2.3.2}$$

The module $span\{\pi(n)v - v\}_{n\in N}$ is also denoted $V(N)$, and V_N is called the Jacquet module of V. We note that

$$\begin{aligned} V(N) :=&\{v \in V \mid \int_{N'} \pi(n')v\; dn' = 0\} \\ &\text{for some open compact subgroup } N' \subset N. \end{aligned} \tag{2.3.3}$$

Unlike for reductive groups, unipotent groups have the property that they are the union of their open compact subgroups. The isomorphism in (2.3.1) is implemented by the formula

$$L \in \mathrm{Hom}_G[V, \mathrm{Ind}_B^G[\chi]] \longrightarrow (v \mapsto L(v)(1)). \tag{2.3.4}$$

It is clear that the map on the right takes vectors of the form $\pi(n)w - w$ to 0.

Let $\overline{B} = A\overline{N}$ be the opposite parabolic subgroup. An important property of $\mathbb{I}$ is that it has a *Bruhat* decomposition

$$\mathbb{I} = \mathbb{I}_+ \cdot \mathbb{I}_0 \cdot \mathbb{I}_- := (\mathbb{I} \cap N)(\mathbb{I} \cap A)(\mathbb{I} \cap \overline{N}). \tag{2.3.5}$$

One of the main tools will be the functor

$$V \mapsto V^{\mathbb{I}}. \tag{2.3.6}$$

This takes *admissible* modules of G with nontrivial $\mathbb{I}-$fixed vectors to nontrivial modules of the **Iwahori-Hecke algebra**

$$\mathcal{H}_{\mathbb{I}} := \mathcal{H}(\mathbb{I}\backslash G/\mathbb{I}),$$

the subalgebra of $\mathcal{H}$ of compact supported locally constant functions invariant under left and right translations by $\mathbb{I}$.

The main property of $\mathbb{I}$ is that $V^{\mathbb{I}}$ is isomorphic to V_N. The relevant result is the following.

Theorem (Borel-Casselman)**.** *The functor*

$$V \longrightarrow V^{\mathbb{I}}$$

is an equivalence of categories between $\mathcal{C}(\mathbb{I})$, formed of admissible modules such that each factor is generated by its $\mathbb{I}-$fixed vectors, and the category of finite dimensional modules of $\mathcal{H}(\mathbb{I}\backslash G/\mathbb{I})$.

Every irreducible module satisfying $V^{\mathbb{I}} \neq 0$ is a subquotient of an $X(\chi)$ with χ unramified. In particular his is true for spherical modules. Denote this subquotient by $L(\chi)$. Then $L(\chi) \cong L(\chi')$ if and only if there is $w \in W$ such that $\chi' = w\chi$.

The fact that every irreducible module with $\mathbb{I}-$fixed vectors is a subquotient of an $X(\chi)$ comes from the fact that $V^{\mathbb{I}} \cong V_N^{\mathbb{I}_0}$. This theorem is very powerful in that it reduces representation theory problems to problems about the algebra $\mathcal{H}(\mathbb{I}\backslash G/\mathbb{I})$.

The * operator of $\mathcal{H}$ induces one for $\mathcal{H}_{\mathbb{I}}$. If (π, V) is unitary, then so is $(\pi_{\mathcal{H}}, V^{\mathbb{I}})$. But the converse is not at all clear.

2.4. **Example 1.** We illustrate the ideas of the proofs of the facts in the previous section in the case of $SL(2,\mathbb{F})$. The general case is not all that different. Then

$$B = AN = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$
$$\mathbb{I} = \begin{bmatrix} \alpha & \beta \\ \varpi\gamma & \delta \end{bmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathcal{R}. \tag{2.4.1}$$

We will need to decompose G into double cosets under $\mathbb{I}$. The answer is

$$G = \bigcup \mathbb{I} \begin{bmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{bmatrix} \mathbb{I} \cup \mathbb{I} \begin{bmatrix} 0 & \varpi^m \\ -\varpi^{-m} & \end{bmatrix} \mathbb{I} \tag{2.4.2}$$

As an algebra $\mathcal{H}_{\mathbb{I}}$ is generated by two elements T_0 and T_1. They satisfy the relations

$$T_i^2 = (q-1)T_i + q. \tag{2.4.3}$$

It is better to consider the generators $\theta = q^{-1}T_0T_1,\ T = T_1$. The relations become

$$T^2 = (q-1)T + q, \qquad T\theta = \theta^{-1}T + (q-1)(\theta + 1). \tag{2.4.4}$$

The algebra has a basis over $\mathbb{C}$ given by $\theta^n,\ T\theta^m$. or alternatively $\theta^n, \theta^m T$. This new basis has the advantage that it is easy to determine the center.

Theorem. *The center of $\mathcal{H}_{\mathbb{I}}$ is generated by the expressions $\theta^n + \theta^{-n}$ (or $(\theta + \theta^{-1})^n$).*

So this algebra is a polynomial algebra with generator $\theta + \theta^{-1}$. Another way of saying this is that W acts on $\mathbb{C}[\theta, \theta^{-1}]$ by $w \cdot \theta = \theta^{-1}$. Then the center is $\mathbb{C}[\theta, \theta^{-1}]^W$.

Here is a sketch that $V^{\mathbb{I}} \cong V_N^{\mathbb{I}_0}$. For any compact open subgroup $\mathcal{K}$ let $e_{\mathcal{K}}$ be the delta function of $\mathcal{K}$ normalized so that the volume of $\mathcal{K}$ is 1. Let $J : V \longrightarrow V_N$ be the quotient map. There are three steps:

(1) The image of $V^{\mathbb{I}}$ and $V^{\mathbb{I}_0\mathbb{I}_-}$ under J are the same.
(2) J is injective when restricted to $V^{\mathbb{I}}$.
(3) J maps $V^{\mathbb{I}_0\mathbb{I}_-}$ onto $V_N^{\mathbb{I}_0}$.

For (1), let $v \in V^{\mathbb{I}_0\mathbb{I}_-}$. Then because v is fixed by $\mathbb{I}_0$, $\mathbb{I}_-$, and some compact open $N' \subset \mathbb{I}_+$,

$$\pi(e_{\mathbb{I}})v = \int_{\mathbb{I}_+} \pi(i_+)v\ di_+ = \frac{1}{N}\pi(n_i)v \tag{2.4.5}$$

where $N=[\mathbb{I}_+/N']$. Thus $v-\pi(e_{\mathbb{I}})v\in V(N)$, *i.e.* $J(v)=J(\pi(e_{\mathbb{I}})v)$. For (2) and (3), let

$$\Lambda^+ := \left\{ a^n = \begin{bmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{bmatrix} \right\}_{n\ge 0}. \tag{2.4.6}$$

Note that Λ^+ preserves $V^{\mathbb{I}_0\mathbb{I}_-}$. Also note that A acts on V_N, and $J(\pi(a)v)=\pi_N(a)J(v)$. Suppose $\overline{U}\subset V_N^{\mathbb{I}_0}$ is any finite dimensional subspace. Let U be a finite dimensional subspace in $V^{\mathbb{I}_0}$ such that $J(U)=\overline{U}$. There is an open compact subgroup $\overline{N}'\subset\overline{N}$. which fixes U. Then there is $n>>0$ such that $a^{-n}\mathbb{I}_-a^n\subset\overline{N}'$. Thus $\pi(a^n)U\subset V^{\mathbb{I}_0\mathbb{I}_-}$. and $J(\pi(a^n)U)=\pi(a^n)\overline{U}$. In particular,

$$\dim\overline{U}=\dim\pi(a^n)\overline{U}\le\dim V^{\mathbb{I}}<\infty. \tag{2.4.7}$$

To finish the proof of (2) observe that if $\overline{U}$ contains a basis of $V_N^{\mathbb{I}_0}$, then so does $\pi(a^n)\overline{U}$. This also proves that $V_N^{\mathbb{I}_0}$ is finite dimensional.

For (3), suppose $v\in\ker J\cap V^{\mathbb{I}}$. Then $v=\sum\pi(n_i)w_i-w_i$. There is $n>>0$ such that $a^nn_ia^{-n}\in\mathbb{I}_+$ for all i. Then

$$\pi(e_{\mathbb{I}}a^ne_{\mathbb{I}})v=\pi(e_{\mathbb{I}})\pi(a^n)v=\sum\pi(e_{\mathbb{I}})\pi(a^nn_ia^{-n})\pi(a^n)w_i-\pi(e_{\mathbb{I}})\pi(a^n)w_i=0. \tag{2.4.8}$$

The proof is complete once we esatblish that $e_{\mathbb{I}}a^ne_{\mathbb{I}}\in\mathcal{H}_{\mathbb{I}}$ is invertible. This is an exercise using the generators and relations of $\mathbb{H}_{\mathbb{I}}$.

We can determine the finite dimensional irreducible modules of $\mathcal{H}_{\mathbb{I}}$. Let (π,V) be such a module. Then $\pi(\theta)$ has an eigenvector v_ν, $\pi(\theta)v=q^\nu v_\nu$. Consider Tv_ν. The subspace $\{v_\nu,\ Tv_\nu\}$ is nonzero and invariant, so equals V. There are two possibilities:

(1) $Tv=\lambda v$,
(2) Tv is linearly independent of v.

In case (1) we find that $q^{2\nu}=\pm q^{\pm1/2}$. In case (2) the irreducible module is 2-dimensional, and we will see explicit realizations later.

For $\mathcal{H}(\mathbb{I}\backslash G/\mathbb{I})$, the star operator * is determined by the formulas

$$T_i^*=T_i. \tag{2.4.9}$$

Then

$$\theta^*=(T_0T_1)^*=T_1T_0=T_0\theta T_0^{-1}. \tag{2.4.10}$$

A more systematic way to determine the irreducible modules is the following. Note that

$$\mathcal{H}_{\mathbb{I}}=\mathbb{C}[T]\otimes\mathcal{A},\qquad\mathcal{A}=\mathbb{C}[\theta,\theta^{-1}]. \tag{2.4.11}$$

Let (π,V) be an irreducible module. Since $\mathcal{A}$ is abelian, it must have a common eigenvector $v\in V$. Thus there is a representation $\mathbb{C}_\chi$ of $\mathcal{A}$ such that $\pi(a)v=\chi(a)v$. Of course such a representation is determined by the scalar value it takes on θ. Let

$$X(\chi):=\mathcal{H}_{\mathbb{I}}\otimes_{\mathcal{A}}\mathbb{C}_\chi.$$

Then there is a nonzero map

$$\Phi : X(\chi) \longrightarrow V, \qquad \Phi(x \otimes 1\!\!1_\chi) = \pi(x)v \tag{2.4.12}$$

which is onto, since the representation is irreducible. This means that in order to classify irreducible representations of $\mathcal{H}_{\mathbb{I}}$, we must study $X(\chi)$ in particular its irreducible quotients.

2.5. **Example 2.** Consider the case of $\mathbb{G} = PSL(2) \cong SL(2)/\{\pm Id\}$. Then the algebra $\mathcal{H}_{\mathbb{I}}$ is still generated by $\{T, \theta\}$ but the relations are

$$T^2 = (q-1)T + q, \qquad T\theta = \theta^{-1}T + (q-1)\theta. \tag{2.5.1}$$

2.6. **The General Case.** We consider a linear algebraic reductive group $\mathbb{G}$ defined over $\mathbb{Q}$. Such groups are classified by their root datums $\mathcal{R} = (X, R, Y, \check{R})$ where R is a reduced roots system. We associate a Hecke algebra to this datum.

Let W_0 be the (finite) Weyl group associated to R, and fix $\Pi \subseteq R$ a set of simple roots. Then let $\mathcal{H}_K$ be the Hecke algebra associated to this group, generated by $T_1, \cdots, T_n$ subject to

- $T_i^2 = (q-1)T_i + q$
- $\underbrace{T_iT_j \cdots T_iT_j}_{m(i,j)} = \underbrace{T_jT_i \cdots T_jT_i}_{m(i,j)}$

Here the $m_{i,j}$ is the order of the element $s_is_j \in W$, where the s_i are the simple root reflections in the Weyl group W. Let $\mathcal{A} := \mathbb{C}[X]$ be the group algebra of X generated by $\{\theta_x\}_{x\in X}$ subject to

$$\theta_x \cdot \theta_y = \theta_{x+y}, \quad \theta_0 = 1\!\!1.$$

The Iwahori-Hecke algebra $\mathcal{H}$ is the algebra generated by $\mathcal{H}_K$ and $\mathcal{A}$ subject to

$$T_s\theta_x = \theta_{s(x)}T_s + (q-1)\frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}}$$

where $s = s_\alpha$ is the reflection corresponding to the simple root $\alpha \in \Pi$ (recall that W_0 acts on X and Y).

2.7. **Connection to p-adic groups.** If G is a split $p-$adic group, its algebra $\mathcal{H}_{\mathbb{I}}$ is isomorphic to the above for the dual root system, $\check{\mathcal{R}} = (Y, \check{R}, X, R)$

2.8. **Example 1.** $G = Sp(2)$ the symplectic group of rank 2. Then $\check{G}$ is $SO(5)$.

$$\begin{aligned} &X = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2,\ R = \pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}, \\ &Y = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2\ \check{R} = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}. \end{aligned} \tag{2.8.1}$$

Let $\theta_1 \longleftrightarrow \varepsilon_1, \quad \theta_2 \longleftrightarrow \varepsilon_2$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2$. Let T_1 and T_2 be the corresponding elements in $\mathcal{H}_K$.

$$\begin{aligned}
&T_1\theta_1 = \theta_2 T_1 + (q-1) \quad \frac{\theta_1 - \theta_2}{1 - \theta_2\theta_1^{-1}} = \theta_2 T_1 + (q-1)\theta_1\\
&T_2\theta_1 = \theta_1 T_2 \qquad\qquad T_2\theta_2 = \theta_2^{-1}T_2 + (q-1)\frac{\theta_2 - \theta_2^{-1}}{1-\theta_2^{-1}} = \theta_2^{-1}T_2 + (q-1)(\theta_2+1)\\
&T_1\theta_2 = \theta_1 T_1 + (q-1) \quad \frac{\theta_2 - \theta_1}{1 - \theta_2\theta_1^{-1}} = \theta_1 T_1 - (q-1)\theta_1
\end{aligned} \tag{2.8.2}$$

2.9. **Example 2.** Let $G = SO(5)$. Then $\check{G}$ is $Sp(2,\mathbb{C})$.

$$\begin{aligned}
&X = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2, R = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}\\
&Y = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2, \check{R} = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}.
\end{aligned} \tag{2.9.1}$$

Again let $\theta_1 \longleftrightarrow \varepsilon_1 \quad \theta_2 \longleftrightarrow \varepsilon_2$

$$T_1\theta_1 = \theta_2 T_1 + (q-1)\frac{\theta_1 - \theta_2}{1 - \theta_2\theta_1^{-1}} = \theta_2 T_1 + (q-1)\theta_1.$$

The other relations are the same, except

$$T_2\theta_2 = \theta_2^{-1}T_2 + (q-1)\frac{\theta_2 - \theta_2^{-1}}{1-\theta_2^{-2}} = \theta_2^{-1}T_2 + (q-1)\theta_2.$$

Even though $sp(2) \simeq so(5)$, the algebras are not the same.

2.10. **The Generic and Graded Hecke Algebras.** The Hecke algebra $\mathcal{H}_{\mathbb{I}}$ can be described as the specialization of a generic Hecke algebra. We describe this generic algebra in terms of the root datum of the *complex* dual group $\check{G}$, with maximal torus $\check{A}$ and Borel subgroup $\check{B}$ containing $\check{A}$. Let z be an indeterminate (which we will specialize to $q^{1/2}$ to recover the Hecke algebra $\mathcal{H}$ of the p–adic group). Let $\Pi \subset R^+ \subset R$ be the simple roots, positive roots and roots corresponding to $(\check{B}, \check{A})$ and S be the simple root reflections. Let $\mathcal{Y} = Hom(G_m, \check{A})$ and $\mathcal{X} = Hom(\check{A}, G_m)$. Then the generic Hecke algebra $\mathcal{H}(z)$ is an algebra over $\mathbb{C}[z, z^{-1}]$ described in terms of the root datum $\mathcal{R} = (\mathcal{X}, \mathcal{Y}, R, \check{R}, \Pi)$. (This set is the dual data to the p–adic group G in section 1.1.) The set of generators we will use is the one first introduced by Bernstein. Write

$$\mathcal{A}(z) := \text{ regular functions on } \mathbb{C}^* \times \check{A}. \tag{2.10.1}$$

This can be viewed as the algebra generated by $\{\theta_x\}_{x\in\mathcal{X}}$ with coefficients in $\mathbb{C}[z, z^{-1}]$, Laurent polynomials in z. Then $\mathcal{H}(z)$ is generated (over $\mathbb{C}[z, z^{-1}]$)

by $\{T_w\}_{w\in W}$ and $\mathcal{A}(z)$ subject to the relations

$$\begin{aligned} T_wT_{w'} =&T_{ww'} \quad (l(w)+l(w')=l(ww')),\\ T_s^2 =&(z^2-1)T_s+z^2,\\ \theta_xT_s =&T_s\theta_{sx}+(z^2-1)\frac{\theta_x-\theta_{sx}}{1-\theta_{-\alpha}}. \end{aligned} \tag{2.10.2}$$

Specialized at $z=q^{1/2}$, $\mathcal{A}(z)$ gives an algebra isomorphic to $\mathcal{A}$ in 2.6. This realization is very convenient for detemining the center of $\mathcal{H}$ and thus parametrizing infinitesimal characters of representations. Note that W acts on $\mathcal{A}$ via the formula $w\cdot\theta_x=\theta_{wx}$.

Theorem (Bernstein-Lusztig)**.** *The center of $\mathcal{H}(z)$ is given by the Weyl group invariants in $\mathcal{A}(z)$.*

In particular, infinitesimal characters are parametrized by $W-$orbits $\chi=(q,t)\in\mathbb{C}^*\times\check{A}$. Such an infinitesimal character is called *real* if t has no elliptic part.

The study is simplified by using the graded Hecke algebra introduced by Lusztig in [L2]. Given an arbitrary infinitesimal character (q,t), decompose t into its elliptic and hyperbolic part $t=t_et_h$. In [L2] the graded algebra $\mathbb{H}_{t_e}$ is introduced. This is done by considering the ideal $\mathcal{H}^1$ generated by $z-1$ and $\theta_x-\theta_x(t_e)$. Then $\mathcal{H}^i$ is a filtration of ideals of $\mathcal{H}$ and one can consider the graded algebra $\mathbb{H}_{t_e}$. This algebra is generated by $t_w=T_w$ in $\mathcal{H}/\mathcal{H}^1$ and $\omega_x=\theta_x-\theta_x(t_e)$ in $\mathcal{H}/\mathcal{H}^2$ with coefficients in $\mathbb{C}[\mathbf{r}]$ where $\mathbf{r}=z-1$ in $\mathcal{H}/CH^1$.

Remark. In [L2] the grading is done at an element t such that t_e is central or only fixed by the Weyl group. For the unitarity questions one needs to also greade when t_e is not central. This is done in [BM2].

2.11. **Example.** Suppose we take the case of the example in 2.4. Then $\mathcal{A}$ is generated by the character θ corresponding to the root. The last relation in 2.10.2 is

$$\theta T=T\theta^{-1}+(z^2-1)\frac{\theta-\theta^{-1}}{1-\theta^{-1}}=T\theta^{-1}-(z^2-1)(\theta+1). \tag{2.11.1}$$

For example let us grade relation 2.11.1 at $(1,1)$. We rewrite

$$(\theta-1)T=T(\theta^{-1}-1)+((z-1)^2+2(z-1)))(\theta-1+2). \tag{2.11.2}$$

Throwing away the terms in $\mathcal{H}^2$, we get $\omega t=-t\omega+4\mathbf{r}$. Here we have used

$$\theta^{-1}-1=-(\theta-1)\frac{1}{1-(1-\theta)}=-(\theta-1)-(\theta-1)^2-\dots \tag{2.11.3}$$

to get $-\omega$ in the formula.

If on the other hand we grade at $(-1,1)$, we find $\omega t=-t\omega$ because $\theta+1\in\mathcal{H}^1$ already. So in this case we get the group algebra of the affine Weyl group. This gives a very different algebra from before; namely the group algebra of the affine Weyl group. It accounts for the reducibility of a certain tempered induced module for $SL(2)$, the ones where θ acted by $-q, -q^{-1}$.

Remark. If we think in terms of the algebraic groups involved, then the dual group to $SL(2) \cong Sp(2)$ is $\check{G} = PSL(2) \cong SO(3)$. In the first case we are grading at $t_e = Id$, in the second case at $t_e = \begin{bmatrix} i & \\ 0 & -i \end{bmatrix}$ The t_e should be thought of as elements in the diagonal torus. The second one is not central, but is fixed by the Weyl group. □

In Example 2.5, $\mathcal{A}$ is generated by a θ such that θ^2 is the character corresponding to the root. The third relation in 2.10.2 is

$$\theta T = T\theta^{-1} + (z^2-1)\frac{\theta - \theta^{-1}}{1-\theta^{-2}} = T\theta^{-1} + (z^2-1)\theta. \tag{2.11.4}$$

We summarize some properties when the grading is done at $(1,1)$. Let

$$\mathcal{J} = \{f \in \mathcal{A} \ : \ f(1,1) = 0\}. \tag{2.11.5}$$

This ideal satisfies $\mathcal{HJ} = \mathcal{JH}$, so we can introduce a filtration

$$\mathcal{H} = \mathcal{H}^0 \supset \dots \supset \mathcal{H}^i \supset \mathcal{H}^{i+1} \supset \dots, \tag{2.11.6}$$

and form the graded object $\mathbb{H}$. As a vector space it can be written as

$$\mathbb{H} = \mathbb{C}[\mathbf{r}]W \otimes_{\mathbb{C}} \mathbb{A}, \tag{2.11.7}$$

where $\mathbf{r} \equiv z - 1 (mod\ \mathcal{J})$, and $\mathbb{A}$ is the symmetric algebra over $\check{\mathfrak{t}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$. The generators satisfy the relations

$$\begin{aligned} t_w t_{w'} =& t_{ww'}, \\ t_\alpha^2 =& 1, \\ t_\alpha \omega =& s_\alpha(\omega) t_s + 2\mathbf{r} < \omega, \check{\alpha} >, \quad (t_\alpha = t_{s_\alpha},\ \omega \in \check{\mathfrak{t}}). \end{aligned} \tag{2.11.8}$$

Then W acts on $\mathbb{A}$ in the usual way and the center of $\mathbb{H}$ is $\mathbb{A}^W$. In particular, infinitesimal characters are parametrized by W–orbits of elements $\overline{\chi} = (r,s) \in \mathbb{C} \times \check{\mathfrak{t}}$. Such an infinitesimal character is called *real* if s is hyperbolic.

We can specialize z to $q^{1/2}$ in the generic algebra and $\mathbf{r}$ to r in the graded Hecke algebra. We fix a choice of q which is not a root of unity (in fact a power of a prime in the case of a p–adic group) and an r such that $e^r = q$. The study of representations of $\mathcal{H}$ with infinitesimal character (q,t) having elliptic part t_e, is reduced to the study of the representation theory of $\mathbb{H}_{t_e}$ where $\mathbf{r}$ is specialized to r satisfying $q = e^r$. Furthermore $\mathbb{H}_{q,t_e}$ is Morita equivalent to an $\mathbb{H}$ as above, which is the graded version of some related Hecke algebra at $(1,1)$. The subject of [BM2] is to show that in fact all questions of unitarity can be reduced to the case of a Hecke algebra obtained by grading at $t_e = 1$. As a consequence, we only need to study the representation theory and unitary spectrum of $\mathbb{H}$, and at that we only need to consider real infinitesimal character (*i.e.* $t_e = 1$). The next theorem is a summary of what we need from [L2]. More details and some facts that we need from [BM2] are in 1.3.

Theorem ([L2]). *There is a matching* $\chi \longleftrightarrow \overline{\chi}$ *between real infinitesimal characters* χ *of* $\mathcal{H}$ *and real infinitesimal characters* $\overline{\chi}$ *of* $\mathbb{H}$ *so that if* $\mathcal{H}_\chi$ *and* $\mathbb{H}_{\overline{\chi}}$ *are the quotients by the corresponding ideals, then*

$$\mathcal{H}_\chi \cong \mathbb{H}_{\overline{\chi}}.$$

2.12. **The case of SL(2).** We take up the case of $\mathbb{H}$ generated by ω and t subject to the relations

$$t\omega = -\omega t + 1, \qquad t^2 = 1. \tag{2.12.1}$$

Denote by $\mathbb{A} := S[\omega]$ the polynomial algebra generated by ω. To classify irreducible modules we need to analyze

$$X(\nu) := \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu. \tag{2.12.2}$$

This has a basis of vectors $1 \times 1\!\!1_\nu,\ t \otimes 1\!\!1_\nu$. We would like a basis of (generalized) eigenvectors for ω. This can be achieved by using the general element $r'_\alpha := t_\alpha \alpha - 1$ which specializes to $2t\omega - 1$ for $SL(2)$. The following relations hold in general [L2]:

$$\begin{aligned} &\eta r'_\alpha = r'_\alpha s_\alpha(\eta),\\ &\underbrace{r'_i r'_j \dots r'_i r'_j}_{m_{ij}} = \underbrace{r'_j r'_i \dots r'_j r'_i}_{m_{ij}},\\ &{r'_\alpha}^2 = 1 - \alpha^2. \end{aligned} \tag{2.12.3}$$

For $\nu \neq 0$ a basis of eigenvectors for $\mathbb{A}$ is then $v_\nu := 1 \otimes 1\!\!1_\nu$, $v_{-\nu} := (2\nu t - 1) \otimes 1\!\!1_\nu$. Consider the case $Re\nu > 0$. Since the dual of $X(\nu)$ is $X(-\nu)$, the case $Re\nu < 0$ is dual to this. The eigenvector $1 \otimes 1\!\!1_\nu$ generates $X(\nu)$. Thus $X(\nu)$ has a unique irreducible quotient; there is a unique maximal proper submodule namely the largest submodule that does not contain v_ν. This module could be zero. We also have an intertwining operator $A'_s(\nu) : X(\nu) \longrightarrow X(-\nu)$ given by $x \otimes 1\!\!1_\nu \mapsto x r'_s \otimes 1\!\!1_{-\nu}$. To be an intertwining operator means

$$\pi_{-\nu}(x) A'_s(\nu) v = A'_s(\nu) \pi_\nu(x) v. \tag{2.12.4}$$

The image of $A_s(\nu)$ is the unique irreducible quotient of $X(\nu)$. This is because $X(-\nu)$ is dual to $X(\nu)$ so it has a unique irreducible submodule. This submodule is characterized by the fact that it contains the eigenvector of $X(\nu)$ with eigenvalue ν, $(2\nu t - 1) \otimes 1\!\!1_{-\nu}$. So the image of $A_s(\nu)$ is the unique irreducible quotient of $X(\nu)$. Denote this module by $L(\nu)$.

This takes care of the cases when V has an eigenvector with eigenvalue ν such that $Re\nu > 0$. If there is no such vector, there are two cases,

1. V is a subquotient of an $X(\nu)$ with $Re\nu = 0$, but $Im\nu \neq 0$. This is taken care of by the above discussion.
2. V has only one weight with eigenvalue $-\nu > 0$.

(3) V has only generalized eigenvalues equal to 0.

In case (2), the only choice is that $v_{-\nu}$ is a subrepresentation of $X(\nu)$ for some $\nu > 0$. But

$$t(2\nu t - 1) \otimes 1\!\!1_\nu = (2\nu - t) \otimes 1\!\!1_\nu = c(2\nu t - 1) \times 1\!\!1_\nu \tag{2.12.5}$$

which gives the equations $-c = 2\nu,\ -1 = 2c\nu$ which for $\nu > 0$ has the only solution $c = -1,\ \nu = 1/2$. This module is called the *Steinberg module.*

In case (3) note that $\omega t \otimes 1\!\!1_0 = (-t\omega + 1)1\!\!1_0 = 1 \otimes 1\!\!1_0$, and $\omega \otimes 1\!\!1_0 = 0$. This vector also generates $X(0)$. So $X(0)$ has a unique eigenvector with eigenvalue 0. Any nonzero submodule $V \subset X(0)$ must contain an eigenvector, so it must equal $X(0)$.

The two modules $X(0)$ and St (as well as all $X(i\nu)$ with ν real are called *tempered.* They are modules of the form $V^{\mathbb{I}}$ for V which figure in the decomposition of $L^2(G)$, so are always unitary.

The spherical vector is $(1 + t) \otimes 1\!\!1_\nu$. Its image under the intertwining operator $A'_s(\nu)$ is

$$(1 + t)(-2\nu t - 1)1\!\!1_{-\nu} = (-2\nu - 1)(1 + t) \otimes 1\!\!1_{-\nu}. \tag{2.12.6}$$

We normalize the intertwining operator to be multiplication on the right by

$$r_\alpha = (t_\alpha \alpha - 1)(\alpha - 1)^{-1}. \tag{2.12.7}$$

Applied to a principal series it is $(-2\nu - 1)^{-1}A'_s$, and it is the identity on the spherical vector $(1 + t) \otimes 1\!\!1_\nu$. It satisfies

$$A_s(\nu)^2 = Id. \tag{2.12.8}$$

To determine when $L(\nu)$ is unitary, we need to determine first when it is hermitian. The hermitian dual of $X(\nu)$ is $X(-\overline{\nu})$. This is true in general, the pairing is

$$< t_x \otimes 1\!\!1_\nu, t_y \otimes 1\!\!1_{-\overline{\nu}} >= \delta_{y^{-1}x,1}. \tag{2.12.9}$$

If $Re\nu > 0$ then $Re(-\overline{\nu}) < 0$, so $Re(-\overline{s_\alpha \nu}) > 0$. So the irreducible quotients are $L(\nu)$ and $L(-s_\alpha(\overline{\nu}))$. The only way $L(\nu)$ is hermitian is if the two parameters are conjugate. It comes down to ν and $-\overline{\nu}$ must be conjugate. Thus either ν is real or it is imaginary. The imaginary case always gives unitary irreducible modules $X(\nu)$. In the real case, we can define a hermitian pairing on $X(\nu)$ by the formula

$$(a \otimes 1\!\!1_\nu, b \otimes 1\!\!1_\nu) =< a \otimes 1\!\!1_\nu, A_s(b \otimes 1\!\!1_\nu) > . \tag{2.12.10}$$

The radical of this form is $X'(\nu)$. We compute on the basis $(1+t) \otimes 1\!\!1_\nu,\ (1-t) \otimes 1\!\!1_\nu$. The matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1-2\nu}{1+2\nu}. \end{bmatrix} \tag{2.12.11}$$

This is positive (semi)definite in the interval $-1/2 \leq \nu \leq 1/2$.

We will describe the generalization of this result to the classical and exceptional split groups.

2.13. **The Langlands Classification.** So for any element $w \in W$ we take a reduced decomposition $w = s_1 \dots s_k$, and form $r'_w = \prod r'_i$. Then w gives rise to an *intertwining operator*

$$A'_w : X(\nu) \longrightarrow X(w\nu), \qquad x \otimes 1\!\!1_\nu \mapsto x r'_w \otimes 1\!\!1_{w\nu}. \tag{2.13.1}$$

Theorem. *Assume that $\langle \nu, \alpha \rangle \geq 0$ for all positive roots. Then $X(\nu)$ has a unique irreducible quotient $L(\nu)$. This module is spherical and the image of the intertwining operator A'_w where w is the shortest Weyl group element such that $\langle w\nu, \alpha \rangle \leq 0$ for all positive roots α. Two modules $L(\nu)$ and $L(\nu')$ are isomorphic if and only if there is $w \in W$ such that $w\nu = \nu'$.*

Proof. Consider the case of $\langle \nu, \alpha \rangle > 0$ for all roots. Then $\{v_w(\nu) := r'_w \otimes 1\!\!1_\nu\}$ forms a basis for $X(\nu)$. It is a basis of eigenvectors for $\mathcal{A}$, each $r'_w \otimes 1\!\!1_\nu$ has eigenvalue $w\nu$. These facts also imply that if a submodule W of $X(\nu)$ contains an eigenvector with eigenvalue ν, then it is equal to $X(\nu)$ because it contains all the basis vectors $v_w(\nu)$. Thus $X(\nu)$ has a unique maximal proper submodule $X'(\nu)$ characterized by the property that it doesn't contain any eigenvector with eigenvalue ν. Consider now the operator A_{w_0} where w_0 is the long Weyl group element. The image is in the module $X(w_0\nu)$ which is dual to $X(-w_0\nu)$. Since $\langle -w_0\nu, \alpha \rangle > 0$ for all roots, $X(w_0\nu)$ has a unique irreducible submodule generated by the eigenvector with eigenvalue ν. Consider the intertwining operator $A_{w_0}(\nu) : X(\nu) \longrightarrow X(w_0\nu)$. Then $A_{w_0}(x \otimes 1\!\!1_\nu) = \pi_\nu(x) A_{w_0}(1 \otimes 1\!\!1_\nu) = \pi_\nu(x) r'_{w_0} \otimes 1\!\!1_{w_0\nu}$. So the image is the unique irreducible module $L(\nu)$. It follows that the kernel of A_{w_0} is $X'(\nu)$. The fact that the module $L(\nu)$ is spherical follows from showing that

$$\sum_w (t_w) r'_{w_0} \otimes 1\!\!1_{w_0\nu} \neq 0. \tag{2.13.2}$$

In fact note that this is essentially the spherical function.

Now suppose ν is singular. Then there is a parabolic subgroup $P = MN$ with roots

$$\Delta(\mathfrak{m}, \mathfrak{a}) = \{\alpha \in \Delta \ : \ \langle \alpha, \nu \rangle = 0\}, \qquad \Delta(\mathfrak{n}, \mathfrak{a}) = \{\alpha \in \Delta \ : \ \langle \alpha, \nu \rangle > 0\}. \tag{2.13.3}$$

Then $\mathbb{H}$ has $\mathbb{H}_M$ corresponding to $(X, R_M, Y, \check{R}_M)$ as a subalgebra. Let $X_M(\nu)$ be the standard module for $\mathbb{H}_M$ corresponding to ν. We can write $\mathfrak{a} = \mathfrak{a}_M + \mathfrak{z}_M$ where

$$\mathfrak{z}_M = \{x \in \check{\mathfrak{a}} \ : \ \alpha(x) = 0 \text{ for all } \alpha \in \Delta(\mathfrak{m}, \mathfrak{a}), \quad \mathfrak{a}_M = span\{\check{\alpha} \ : \alpha \in \Delta(\mathfrak{m}, \mathfrak{a})\}. \tag{2.13.4}$$

Let also $\mathfrak{m}^0$ be the semisimple Lie algebra with root system $\mathcal{R}_M$. Then $\mathbb{H}_M = \mathbb{H}_{M^0} \otimes S(\mathfrak{z}_M)$. The module $X_M(\nu)$ decomposes as $X_{M^0}(0) \otimes \mathbb{C}_\nu$. In fact $X_{M^0}(0)$ is irreducible. Indeed it is enough to prove this for $X(0)$. This module has a basis $t_w \otimes 1\!\!1_0$ which is a basis of generalized eigenvectors for

$\mathbb{A}$. In fact

$$\omega : \bigoplus_{\ell(w)=i} t_w \otimes 1\!\!1_0 \longrightarrow \bigoplus_{\ell(w)\le i-1} t_w \otimes 1\!\!1_0. \tag{2.13.5}$$

For every w there is ω such that $\omega t_w \otimes 1\!\!1_0 \neq 0$. So the only eigenvector is $1 \otimes 1\!\!1_0$. Then any nonzero submodule $X' \subset X(0)$ must contain $1 \otimes 1\!\!1_0$. Since this vector also generate $X(0)$, the claim follows.

As before, if $w_{M,0}$ is the shortest element mapping ν to an antidominant element, then $X(L_M(0), \nu) := \mathbb{H} \otimes_{\mathbb{H}_M} L_M(\nu) \otimes \mathbb{C}_\nu$ has a unique irreducible subquotient $L(\nu)$ which is the image of the intertwining operator $A_{w_{M,0}}$.

□

2.14. **Kazhdan-Lusztig Classification.** The classification of irreducible representations is given by the work of Kazhdan-Lusztig for $\mathcal{H}$ and Lusztig for $\mathbb{H}$:

Theorem ([KL], [L4], [L5])**.** *The irreducible representations of $\mathbb{H}$ are parametrized by $\check{G}$ conjugacy classes (s, e, ψ), where $s \in \check{\mathfrak{g}}$ is semisimple, $e \in \check{\mathfrak{g}}$ is nilpotent such that $[s, e] = re$ and $\psi \in \widehat{A(s, e)}$ is an irreducible representation of the component group of the centralizer of s and e. The characters ψ that appear are the same ones that occur in the Springer correspondence.*

2.15. **Hermitian Modules.** The * operation also transfers to the graded version. Here is a summary of what we need. We refer to §5 of [BM2] for the details.

Let $w_0 \in W$ be the longest element, t_0 be the corresponding element in $\mathbb{C}W$. Since $\check{\mathfrak{t}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$, it has a conjugation coming from the complex conjugation on $\mathbb{C}$. We denote it by $^-$. Let $\iota(\omega) = (-1)^{deg \omega}\overline{\omega}$ and $\tilde{\omega} = w_0\iota(\omega)$.

Theorem ([BM2])**.** *Let $\omega \in \mathbb{A}$. Then*

$$t_w^* = t_{w^{-1}},$$
$$\omega^* = t_0 \cdot \tilde{\omega} \cdot t_0.$$

In particular, if $\omega \in \check{\mathfrak{t}}$, then

$$\omega^* = -\overline{\omega} + 2r \sum_{\beta \in R^+} < \overline{\omega}, \check{\beta} > t_\beta, \tag{2.15.1}$$

where $t_\beta \in \mathbb{C}[W]$ is the reflection about β.

The main result of [BM2] can be summarized as follows. Let (e, H, f) be a Lie triple corresponding to e. Write $t = t_0 t_H$, where t_0 is an element centralizing the triple and $t_H = exp(1/2 \log qH)$. Let $t_0 = t_e t_h$ be the decomposition of t_0 into elliptic and hyperbolic parts. Denote by $\overline{t} := t_e t_h^{-1}$. An irreducible representation admits a hermitian form if and only if (t, e, ψ) is conjugate to $(\overline{t_0} t_H, e, \psi)$. An infinitesimal character (or parameter) will be called *real* if $t_e = 1$.

Corollary. *The classification of the unramified unitary dual of a split $p-$adic group having infinitesimal character with a given elliptic part t_e reduces to the classification of the unitary dual of the corresponding graded Hecke algebra $\mathbb{H}_{q,t_e}$. Furthermore, this is equivalent to the classification of the unitary dual of an algebra $\mathbb{H}$ obtained from a Hecke algebra by grading at $(e^r, 1)$. In other words, it is sufficient to consider the case of parameters with* real *infinitesimal character.*

The relation between s in the theorem in 1.2 and the t in this corollary is $e^s = t_h t_H$.

2.16. **Parameters for irreducible modules.** We summarize some of the basic results about modules of $\mathbb{H}$ related to theorem 2.14. We may as well take $r = 1$.

Let (s, e, ψ) be a parameter as in theorem 2.14, and let $\{e, h, f\}$ be a Lie triple such that $s = s_0 + \frac{1}{2}h$ with s_0 in the centralizer of $\{e, h, f\}$. Denote by $\mathcal{O}$ the $\check{G}$ orbit of e. To each such parameter is associated a standard module $X(s, \mathcal{O})$ which decomposes into a direct sum of standard modules $X(s, \mathcal{O}, \psi)$ where ψ ranges over the characters of $A(s, e)$. Each $X(s, \mathcal{O}, \psi)$ has a unique irreducible quotient $L(s, \mathcal{O}, \psi)$. Every irreducible module is isomorphic to an $L(s, \mathcal{O}, \psi)$, and the factors of $X(s, \mathcal{O}, \psi)$ have parameters $(s, \mathcal{O}', \psi')$ such that $\mathcal{O} \subset \overline{\mathcal{O}'}$ and $\mathcal{O} \neq \mathcal{O}'$.

A parameter is called *tempered* if $s_0 = 0$. In this case the module $X(s, \mathcal{O}, \psi)$ is irreducible and corresponds to the Iwahori fixed vectors of an irreducible tempered representation of the group. The parameter is called a *Discrete Series* if in addition the orbit $\mathcal{O}$ of e does not meet any proper Levi component of $\check{\mathfrak{g}}$. Such modules correspond to the $\mathcal{I}-$fixed vectors of a *Discrete Series*of the p–adic group G. Now suppose that the pair (s, e) is contained in a Levi component $\check{\mathfrak{m}}$. Then we can form $X_M(s, \mathcal{O})$ and $X_G(s, \mathcal{O})$. The relation between them is

$$X_G = Ind_{\mathbb{H}_M}^{\mathbb{H}}[X_M] = \mathbb{H} \otimes_{\mathbb{H}_M} X_M. \tag{2.16.1}$$

More generally, write $A(s, e, M)$ and $A(s, e, G)$ for the corresponding component groups. Then

$$Ind_{\mathbb{H}_M}^{\mathbb{H}}[X_M(s, e, \phi)] = \sum [\psi|_{A(s,e,M)} : \phi] X_G(s, e, \psi). \tag{2.16.2}$$

In other words, $A(s, e)$ plays the role of an R–group.

We can use s_0 to construct the Levi component $\check{\mathfrak{m}}$. We then find a tempered representation $\mathcal{W}$ equal to $X_M(h, \mathcal{O}, \psi)$ tensored with a character ν corresponding to s_0. Then

$$X_G(s, \mathcal{O}, \psi) = Ind_{\mathbb{H}_M}^{\mathbb{H}}[\mathcal{W} \otimes \nu]. \tag{2.16.3}$$

Thus we recover the usual Langlands classification. We remark that $\mathbb{H}$ is defined in terms of a fixed system of positive roots. The data s, e can be conjugated so that $s_0 = \nu$ is dominant with respect to this system; we assume that this is the case. When we want to emphasize that we are using the usual Langlands classification, the standard module will be denoted by $X(M, \mathcal{W}, \nu)$.

The W–structure of the standard modules is also known. Let $\mathcal{B}_e$ be the variety of Borel subgroups that contain e. Then $H^*(\mathcal{B}_e)$ carries an action of W called the Springer action. It is usually normalized so that for the principal nilpotent, $H^*(\mathcal{B}_e)$ is the trivial module. It commutes with the action of the component group $A(e)$. Let $d(e) = dim\mathcal{B}_e$. Then $H^{d(e)}(\mathcal{B}_e)$ decomposes according to characters of $A(e)$. Each isotypic component is irreducible as a $W \times A(e)$–module and the ensuing $\phi \leftrightarrow \sigma_{\mathcal{O},\phi}$ is called the Springer correspondence.

Results of Borho–MacPherson imply that if $[\sigma\colon H^*(\mathcal{B}_e)^{\phi}] \neq 0$, then σ is of the form $\sigma_{\mathcal{O}',\phi'}$ where $\mathcal{O}'$ contains $\mathcal{O}$ in its closure. $\mathcal{O} \neq \mathcal{O}'$ unless $\phi = \phi'$ as well and this representation occurs with multiplicity 1. We call $\sigma_{\phi,\mathcal{O}}$ a *lowest K−type* of $H^*(\mathcal{B}_e)^{\phi}$.

Proposition (Kazhdan-Lusztig)**.** *There is an isomorphism of W–modules*

$$X(s,e) \cong H^*(\mathcal{B}_e) \otimes sgn.$$

Then $A(e)$ acts on the right hand side; the action of $A(s,e)$ on the left hand side is via the natural map $A(s,e) \to A(e)$. In particular we can talk about *lowest K−types* for $X(s,e,\psi)$. They occur with multiplicity 1 and, given our discussion above, $L(s,\mathcal{O},\psi)$ is the unique subquotient which contains the *lowest K−types* $\sigma_{\phi,\mathcal{O}} \otimes sgn$ for which the $\phi \in \widehat{A(e)}$ contains $\psi \in \widehat{A(s,e)}$ in its restriction. If $s = h/2$, then $A(s,e) = A(e)$, and $X(s,e,\psi)$ has a unique *lowest K−type* namely $\sigma_{\psi,\mathcal{O}} \otimes sgn$.

2.17. **Example.** Suppose $\check{G} = Sp(2n,\mathbb{C})$. Then nilpotent orbits in $\check{\mathfrak{g}}$ are parametrized by partitions of $(x_0,\dots,x_k)$ of $2n$ (with $x_i \le x_{i+1}$) such that every odd part occurs an even number of times. The centralizer of the corresponding Lie triple is of the form

$$\prod O(r_{2a}) \times \prod Sp(s_{2b+1})$$

where r_{2a} is the number of x_i equal to $2a$ and s_{2b+1} is the number of x_i equal to $2b+1$. The component group is then $\prod \mathbb{Z}_2$. The number of $\mathbb{Z}_2$'s equals the number of (distinct) even parts occuring in the partition.

A nilpotent orbit meets a proper Levi component if and only if $x_i = x_{i+1}$ for some i. More precisely, if say $x_i = x_{i+1} = a$, then the nilpotent orbit meets the maximal Levi component $GL(a) \times Sp(2n-2a)$. In general we will write $\check{M} = GL(a) \times G(n-a)$ for such Levi components. The intersection contains the nilpotent corresponding to the partition $(a) \times (x_0,\dots,\widehat{x_i},\widehat{x_{i+1}},\dots,x_k)$. Here (a) denotes the principal nilpotent in $GL(a)$. Thus we can write the nilpotent orbits corresponding to discrete series as

$$(2x_0,\dots,2x_k) \qquad \text{with} \qquad x_i < x_{i+1}.$$

The corresponding standard modules behave as described in 1.4. Not all characters of the component group give rise to standard modules, (or equivalently occur in the Springer correspondence). The ones that do, and the corresponding lowest $K-$types, are described in [L3].

References

[B] A. Borel *Admissible representations of a semisimple group over a local field with fixed vectors under an Iwahori subgroup*, Invent. Math., vol 35, 1976, pages 233–259

[BM1] D. Barbasch, A. Moy *A unitarity criterion for p–adic groups*, Invent. Math., vol 98, 1989, pages 19–38

[BM2] ——— *Reduction to real infinitesimal character in affine Hecke algebras*, Journal of the AMS, vol 6, Number 3, 1993, pages 611–635

[BM3] ——— *Unitary spherical spectrum for p− adic classical groups*, Acta Appl. Math., vol 44, 1996, pages 1–37

[C] W. Casselman *A new nonunitarity argument for p–adic representations* J. Fac. Sci. Tokyo Univ. vol sect IA Math. 28, no. 3, 1981, pages 907–928

[Ca] R. Carter, Finite groups of Lie type, Wiley-Interscience, New York, 1985

[Car] P. Cartier *Representations of p–adic groups: a survey* pages 111–157, Automorphic Forms, Representations, and L–functions, American Mathematical Society, Providence, Rhode Island, vol 31 part 1, 1979

[KL] D. Kazhdan, G. Lusztig Proof of the Deligne–Langlands conjecture for Hecke algebras, Inv Math. , vol 87, 1987 pages 153–215,

[L1] G. Lusztig, Characters of reductive groups over a finite field, Annals of Math. Studies, Princeton University Press, vol 107

[L2] ——— *Affine Hecke algebras and their graded version* Jour. AMS, vol 2, 1989 pages 599–635

[L3] ——— *Intersection cohomology complexes on a reductive group*, Invent. Math., vol 75, 1984 pages 205–272

[L4] ——— *Cuspidal local systems and graded algebras I*, Publ. Math de l'IHES, vol 67, 1988, pages 145–202

[L5] ——— *Cuspidal local systems and graded algebras II*, Proceedings of conference in Banff, 1994

[T] M. Tadic *Classification of unitary representations in irreducible representations of general linear group (nonarchimedean case)* , vol 19, 1986, pages 335–382