# UNITARY SPHERICAL DUAL FOR SPLIT CLASSICAL GROUPS

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## 1. Introduction

The spherical unitary dual of a split groups plays a special role. For one thing, local factors of automorphic forms are unitary spherical for almost all places. They are also full principal series which maybe makes them a little less interesting. However if one just considers the local places by them selves,

**Complex Case.** The annihilator of a spherical irreducible representation in the enveloping algebra is the unique maximal primitive ideal at the given infinitesimal character.

**P-adic Case.** $IM$ of a spherical irreducible representation is the *largest* representation at this infinitesimal character (with Iwahori fixed vectors).

**Real Case.** Almost like the complex case if the group is split there is a parity condition (or block) when this is true. Otherwise not quite as nice. For $U(p,q)$, there are a lot of unitary spherical representations which are derived functor induced from unitary non-spherical modules.

**Lefschetz Principle.** Results should be true independent of the local field.

This is of course false in such generality. Still it is intriguing to what extent this might be true for the unitary dual. David Vogan asked me about a year ago if it might not be true at some level. He had of course serious reasons to expect it.

**Example.** Let $G = Sp(2, \mathbb{F})$ of rank 2. A parameter for a spherical irreducible module with *real* infinitesimal character is a pair $\underline{\nu} = (\nu_1, \nu_2)$ with $\nu_1 \geq \nu_2 \geq 0$. This gives a character of the split maximal Cartan subgroup, the representation is the spherical quotient $\overline{X}(\underline{\nu})$ of the principal series $X(\underline{\nu})$. Consider the case when $\overline{X}(\underline{\nu})$ is a subquotient of the induced from a spherical 1–dimensional module on the maximal parabolic subgroup $P_1$ with Levi component $M_1 = GL(2)$. The parameter is $(1/2+\nu, -1/2+\nu)$. The induced module is

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1. irreducible for all $\nu$ except $1/2, 3/2, 5/2, \ldots$ and unitary in the interval $[-3/2, 3/2]$ if $\mathbb{F} = \mathbb{C}$,
2. irreducible for all $\nu$ except $3/2, 5/2, \ldots$ and unitary in the interval $[-1/2, 1/2]$ if $\mathbb{F} = \mathbb{R}$, or p-adic.

For the case when $\overline{X}(\underline{\nu})$ is contained in the induced module from a 1–dimensional spherical module from the parabolic subgroup $P_2$ with Levi component $M_2 = GL(1) \times Sp(2)$, the parameter is $(\nu, 1)$. The induced module is

1. irreducible for all $\nu$ except $0, 1, \ldots$ and unitary in only for $\nu = 0$ if $\mathbb{F} = \mathbb{C}$,
2. irreducible for all $\nu$ except $0, 2, \ldots$ and unitary only for $\nu = 0$ if $\mathbb{F} = \mathbb{R}$, or p-adic.

So this example suggests that split groups should have a spherical dual independent of $\mathbb{F}$. We will make some comments about $\mathbb{C}$ later, complex groups are quasi-split not split.

For p-adic classical groups, the spherical dual is known [BM2]. For real groups the results are new. But it was more or less clear that one could write down the answer using the same ideas plus [B1] and [B2]. What got me interested was the possibility that the proofs and parameters of a lot of other unitary representatons are the same. Also the proofs in the p-adic case involve using $IM$ and going over to representations that are very closed to tempered. To make any headway with the general unitary dual for Iwahori-spherical modules one needs to deal with representations whose parameters are far from tempered, *e.g.* spherical.

The main theorem is the following:

**Theorem 1.0.1.** *A spherical irreducible representation is unitary if and only if it is unitarily induced from*

$$(unipotent) \otimes (gl - complementary\ series)$$

*or a complementary series from such a module.*

The actual parameters are hard to describe in detail, I can't draw pictures like for rank 1, 2 and 3. The existence of these *complementary series from complementary series* were observed by many people long time ago, *e.g.* Duflo, Cowling ... .

Of course one immediately makes a **conjecture** that theorem (1.0.1) holds in general.

Here are some examples to illustrate the difficulties in giving a combinatorial description or picture for the unitary dual.

**Example 1.** Consider $G = Sp(12)$ and $\overline{X}(1+\nu, 0+\nu, -1+\nu, 1-\nu, 0-\nu, -1-\nu)$. This is a subquotient of $Ind_P^G(\pi_\nu)$ where $P$ is a parabolic subgroup with Levi component $M = GL(6)$, and for $0 \leq \nu < 1/2$, $\pi_\nu$ is a complementary series from an induced from a parabolic subgroup with Levi component

$GL(3) \times GL(3)$. So $\overline{X}$ is unitary, but the induced module is **reducible** at $\nu = 0$.

**Example 2.** Consider $G = SO(7)$ and the parabolic subgroup $P$ with Levi component $M = So(3) \times GL(2)$. The representation $\overline{X}(\underline{\nu})$ with $\underline{\nu} = (1/2, 1/2 + \nu, -1/2 + \nu)$ is a complemnntary series for $0 \leq \nu < 1$.

**Example 3.** Consider $G = SO(5)$ and the parabolic subgroup $P$ with Levi component $M = So(3) \times GL(1)$. The representation $\overline{X}(\underline{\nu})$ with $\underline{\nu} = (1/2, \nu)$ is a complementary series for $0 \leq \nu < 1/2$.

## 2. Uniform Proofs

Here are some of the ingredients that make the proofs uniform (within reason!)

**Intertwining operators.** This is the same for all fields. One checks separately for rank one groups. Other than that, if $B = AN \subset G$ is a Borel subgroup and $\nu$ represents a character, there is an operator

$$A_w : X(\nu) \longrightarrow X(w\nu) \quad \text{for each } w \in W$$

normalized to be the identity on the trivial K-type. If $w = s_1 \cdot \dots \cdot s_k$ is a reduced decomposition, then $A_w = A_1 \cdot \dots \cdot A_k$, where the $A_i$ are induced from the same kind of operators on rank 1 Levi components. For rank one groups, everything is known about the situation.

The definition of $\overline{X}(\underline{\nu})$ and its unitarity can be expressed in terms of these operators. There is a serious difference. In the p-adic case, $A_i$ are isomorphisms whenever $(\nu, \check{\alpha}) \neq 1$, in the real case whenever $(\nu, \check{\alpha}) \notin \mathbb{N}^+$. To remedy this, in the real case one should consider only certain K-types which I will call *spherical.* The condition is roughly that *t*wice a root is not a weight. But in fact a much smaller set is needed for the unitarity questions. Let

$$M := \mathcal{Z}_K(A),$$

the centralizer of $A$ in $K$. Frobenius reciprocity says that if $\sigma, V_\sigma)$ is a representation of $K$, then

$$[\sigma : X(\nu)] = [\sigma|_M : triv] = dim V_\sigma^M.$$

Then $V_\sigma^M$ is a representation of $W := N(A)/\mathcal{Z}(A)$, and in principle we want to do the unitarity calculation on $V_\sigma^M$ in fact we want to involve just the $W$-structure if we can. For the p-adic case this is in fact all there is if we reduce the problem to the equivalent one for the Iwahori-Hecke algebra.

To match the real case to the p-adic case, we introduce *spherical K-types.* This is a set of K-types such that $V_\sigma^M$ is irreducible and the unitarity can be detected on them.

$$
\begin{array}{cccr}
 & \widehat{W} & \widehat{K} & (2.1)\\
B,D & (n-k)\times(k) & (1,\dots,1,0,\dots,0)\otimes(1,\dots,1,0,\dots,0) & (2.2)\\
C & (n-k)\times(k) & (2,\dots,2,0\dots,0) & (2.3)\\
B,D & (k,n-k)\times(0) & (2,\dots,2,0\dots,0)\otimes(0,\dots,0) & (2.4)\\
C & (k,n-k)\times(0) & (1,\dots,1,0,\dots,0,-1,\dots,-1) & (2.5)
\end{array}
$$

**Theorem 2.0.2.** *$\overline{X}(\nu)$ is unitary if and only if the form is definite on the spherical K-types.*

What one might like is a stronger version that in fact the intertwining operators are "the same" in the sense described above. I think this is formal possible to do. My calculation just keeps track of the signatures which is a lot easier.

## 3. Unitarity

Given the result above, we need to show that spherical unipotent representations are unitary. They are parametrized by maps

$$\psi : SL(2) \longrightarrow \check{G}$$

where $\check{G}$ is the dual (complex) group. These are called Arthur parameters; the semisimple element $\psi \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$ is the $\underline{\nu}$ from before. If $Im\psi$ is contained in a proper Levi component of $\check{G}$, the representation is contained in a unitarily induced module from the same kind of parameter. So it is enough to treat the *distinguished* case, when the image does not meet any proper Levi component.

**Real Case.** Argument is in [B1] and [B2].

**P-adic case.** Argument is in [BM1]. The main point is that $IM$ preserves unitarity and $IM(spherical)$ is tempered therefore unitary.

There is not much chance to duplicate the p-adic argument in the real case.

But one can do it the other way around. In both cases a representation $\pi$ has an *asymptotic support* which is a combination

$$AS(\pi) = \sum m(\pi, \mathcal{O})\widehat{\mu}_{\mathcal{O}}$$

where the $\mathcal{O}$ are nilpotent orbits in the Lie algebra (with the same dimension called Gelfand-Kirillov dimension) and the $m(\pi, \mathcal{O})$ are integers. In the real case these are results of [BV1] and [SV1], in the p-adic case [MW]. The unitarity is a formal proof, provided we know that there is only one $m(\pi, \mathcal{O})$ and it is 1.

**Another example.** Yesterday J.-S. Li talked about the nontrivial spherical module which is furthest from the temperd spectrum. For split classical groups this would correspond to the Arthur parameter given by the subregular nilpotent orbit.

$$\begin{array}{rrr}
\check{G} & \check{\mathcal{O}} & Parameter\ in\ G \\
So(2n+1) & (1,1,2n-1) & (n-1,\dots,1,0) \\
Sp(2n) & (2,2n-2) & (n-3/2,\dots,1/2,1/2) \\
So(2n) & (3,2n-3) & (n-2,\dots,1,0,1) \\
GL(n) & (1,n-1) & (\frac{n-2}{2},\dots,-\frac{n-2}{2},0)
\end{array}$$

The case of $So(2n+1)$ looks like it should give rise to a complementary series, because the nilpotent is contained in $GL(1) \times So(2n-1)$; it doesn't.

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