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UNITARY SPHERICAL SPECTRUM FOR SPLIT CLASSICAL GROUPS

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1. Introduction

{sec:1}

This paper gives a complete classification of the spherical unitary dual of the split groups $Sp(n)$ and $So(n)$ over the real and $p-$adic field. Most of the results were known earlier from [B1], [B2], [B3] and [BM3]. As is explained in these references, in the $p-$adic case the classification of the spherical unitary dual is equivalent to the classification of the unitary generic (in the sense of admitting Whittaker models) Iwahori-spherical modules. The new result is the proof of necessary conditions for unitarity in the real case. Following a suggestion of D. Vogan, I find a set of $K-$types which I call *relevant* which detect the nonunitarity. They have the property that they are in 1-1 correspondence with certain irreducible Weyl group representations (called relevant) so that the intertwining operators are *the same* in the real and $p-$adic case. The fact that these relevant $W-$types detect unitarity in the $p-$adic case is also new. Thus the same proof applies in both cases. Since the answer is independent of the field, this establishes a form of the Lefschetz principle.

Let G be a split symplectic or orthogonal group over a local field $\mathbb{F}$ which is either $\mathbb{R}$ or a $p-$adic field. Fix a maximal compact subgroup K. In the real case, there is only one conjugacy class. In the $p-$adic case, let $K = G(\mathcal{R})$ where $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$, with $\mathcal{R}$ the ring of integers and $\mathcal{P}$ the maximal prime ideal. Fix also a rational Borel subgroup $B = AN$. Then $G = KB$. An admissible representation (π, V) is called spherical if $V^K \neq (0)$.

The classification of irreducible admissible spherical modules is well known. For every irreducible spherical π, there is a character $\chi \in \widehat{A}$ such that $\chi|_{A\cap K} = triv$, and π is the unique spherical subquotient of $Ind_B^G[\chi \otimes \mathbb{1}]$. We will call a character χ whose restriction to $A \cap K$ is trivial, *unramified.* Write $X(\chi)$ for the induced module (principal series) and $L(\chi)$ for the irreducible spherical subquotient. Two such modules $L(\chi)$ and $L(\chi')$ are equivalent if and only if there is an element in the Weyl group W such that $w\chi = \chi'$. An $L(\chi)$ admits a nondegenerate hermitian form if and only if there is $w \in W$ such that $w\chi = -\overline{\chi}$.

The character χ is called *real* if it takes only positive real values. For real groups, χ is real if and only if $L(\chi)$ has real infinitesimal character ([K], chapter 16). As is proved there, any unitary representation of a real reductive group with nonreal infinitesimal character is unitarily induced from a unitary representation with real infinitesimal character on a proper Levi component. So for real groups it makes sense to consider only real infinitesimal character. In the $p-$adic case, χ is called real if the infinitesimal character is real in the sense of [BM2]. The results in [BM1] show that the problem of determining the unitary irreducible representations with Iwahori fixed vectors is equivalent to the same problem for the Iwahori-Hecke algebra. In [BM2], it is shown that the problem of classifying the unitary dual for the Hecke algebra reduces to determining the unitary dual with real infinitesimal character of some smaller Hecke algebra (not necessarily one for a proper Levi subgroup). So for $p-$adic groups as well it is sufficient to consider only real χ.

So we start by parametrizing real unramified characters of A. Since G is split, $A \cong (\mathbb{F}^{\times})^n$ where n is the rank. Define

{eq:1.1}
$$\mathfrak{a}^* = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{1.0.1}$$

where $X^*(A)$ is the lattice of characters of the algebraic torus A. Each element $\nu \in \mathfrak{a}^*$ defines an unramified character χ_ν of A, characterized by the formula

{eq:1.2}
$$\chi_\nu(\tau(f)) = |f|^{\langle \tau, \nu \rangle}, \qquad f \in \mathbb{F}^{\times}, \tag{1.0.2}$$

where τ is an element of the lattice of one parameter subgroups $X_*(A)$. Since the torus is split, each element of $X_*(A)$ can be regarded as a homomorphism of $\mathbb{F}^{\times}$ into A. The pairing in the exponent in (1.0.2) corresponds to the natural identification of $\mathfrak{a}^*$ with $\mathrm{Hom}[X_*(A), \mathbb{R}]$. The map $\nu \longrightarrow \chi_\nu$ from $\mathfrak{a}^*$ to real unramified characters of A is an isomorphism. We will often identify the two sets writing simply $\chi \in \mathfrak{a}^*$.

Let $\check{G}$ be the (complex) dual group, and let $\check{A}$ be the torus dual to A. Then $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to $\check{\mathfrak{a}}$, the Lie algebra of $\check{A}$. So we can regard χ as an element of $\check{\mathfrak{a}}$. We attach to each χ a nilpotent orbit $\check{\mathcal{O}}(\chi)$ as follows. By the Jacobson-Morozov theorem, there is a 1-1 correspondence between nilpotent orbits $\check{\mathcal{O}}$ and $\check{G}$-conjugacy classes of Lie triples $\{\check{e}, \check{h}, \check{f}\}$; the correspondence satisfies $\check{e} \in \check{\mathcal{O}}$. Choose the Lie triple such that $\check{h} \in \check{\mathfrak{a}}$. Then there are many $\check{\mathcal{O}}$ such that χ can be written as $w\chi = \check{h}/2 + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$, the centralizer in $\check{\mathfrak{g}}$ of the triple. For example this is always possible with $\check{\mathcal{O}} = (0)$. The results in [BM1] guarantee that for any χ there is a unique $\check{\mathcal{O}}(\chi)$ satisfying

(1) there exists $w \in W$ such that $w\chi = \frac{1}{2}\check{h} + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$,
(2) if χ satisfies property (1) for any other $\check{\mathcal{O}}'$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}(\chi)$.

Here is another characterization of the orbit $\check{\mathcal{O}}(\chi)$. Let

$$\check{\mathfrak{g}}_1 := \{\ x \in \check{\mathfrak{g}}\ :\ [\chi, x] = x\ \}, \qquad \check{\mathfrak{g}}_0 := \{x \in \check{\mathfrak{g}}\ :\ [\chi, x] = 0\ \}.$$

Then $\check{G}_0$, the Lie group corresponding to the Lie algebra $\check{\mathfrak{g}}_0$ has an open dense orbit in $\check{\mathfrak{g}}_1$. Its $\check{G}$ saturation in $\check{\mathfrak{g}}$ is $\check{\mathcal{O}}(\chi)$.

The pair $(\check{\mathcal{O}}(\chi),\nu)$ has further nice properties. For example assume that $\nu=0$ in (1) above. Then the representation $L(\chi)$ is one of the parameters that the Arthur conjectures predict to play a role in the residual spectrum. In particular, $L(\chi)$ should be unitary. In the $p-$adic case one can verify the unitarity directly as follows. In [BM1] it is shown how to calculate the Iwahori-Matsumoto dual of $L(\chi)$ in the Kazhdan-Lusztig classification of representations with Iwahori-fixed vector. It turns out that in the case $\nu=0$, it is a tempered module, and therefore unitary. Since the results in [BM1] show that the Iwahori-Matsumoto involution preserves unitarity, $L(\chi)$ is unitary as well. In the real case, a direct proof of the unitarity of $L(\chi)$ (still with $\nu=0$ as in (1) above) is given in [B3], and in section 9 of this paper.

In the classical Lie algebras, the centralizer $\mathfrak{z}(\check{e},\check{h},\check{f})$ is a product of symplectic and orthogonal Lie algebras. We will often abbreviate it as $\mathfrak{z}(\check{\mathcal{O}})$. The orbit $\check{\mathcal{O}}$ is called *distinguished* if $\mathfrak{z}(\check{\mathcal{O}})$ does not contain a nontrivial torus; equivalently, the orbit does not meet any proper Levi component. Let $\check{\mathfrak{m}}_{BC}$ be the centralizer of a Cartan subalgebra in $\mathfrak{z}(\check{\mathcal{O}})$. This is the Levi component of a parabolic subalgebra. The subalgebra $\check{\mathfrak{m}}_{BC}$ is the Levi subalgebra attached to $\check{\mathcal{O}}$ by the Bala-Carter classification of nilpotent orbits. The intersection of $\check{\mathcal{O}}$ with $\check{\mathfrak{m}}_{BC}$ is the other datum attached to $\check{\mathcal{O}}$, a distinguished orbit in $\check{\mathfrak{m}}_{BC}$. We will usually denote it $\check{\mathfrak{m}}_{BC}(\check{\mathcal{O}})$ if we need to emphasize the dependence on the nilpotent orbit. Let $M_{BC}\subset G$ be the Levi subgroup whose Lie algebra $\mathfrak{m}_{BC}$ has $\check{\mathfrak{m}}_{BC}$ as its dual.

The parameter χ gives rise to a spherical irreducible representation $L_{M_{BC}}(\chi)$ on M_{BC} as well as a $L(\chi)$. Then $L(\chi)$ is the unique spherical irreducible subquotient of

$$I_{M_{BC}}(\chi):=Ind_{M_{BC}}^{G}[L_{M_{BC}}(\chi)]. \tag{1.0.3}$$ {eq:1.3}

To motivate why we consider $M_{BC}(\check{\mathcal{O}})$, we need to recall some facts about the Kazhdan-Lusztig classification of representations with Iwahori fixed vectors in the p-adic case. Denote by τ the Iwahori-Matsumoto involution. Then the space of Iwahori fixed vectors of $\tau(L(\chi))$ is a $W-$representation (see 5.2), and contains the $W-$representation *sgn*. Irreducible representations with Iwahori-fixed vectors are parametrized by Kazhdan-Lusztig data; these are $\check{G}$ conjugacy classes of $(\check{e},\chi,\psi)$ where $\check{e}\in\check{\mathfrak{g}}$ is such that $[\chi,\check{e}]=\check{e}$, and ψ is an irreducible representation of the component group $A(\chi,\check{e})$. To each such parameter there is associated a standard module $X(\check{e},\chi,\psi)$ which contains a unique irreducible submodule $L(\check{e},\chi,\psi)$. All other factors have parameters $(\check{e}',\chi',\psi')$ such that

$$\check{\mathcal{O}}(\check{e})\subset\overline{\check{\mathcal{O}}(\check{e}')},\qquad \check{\mathcal{O}}(\check{e})\neq\check{\mathcal{O}}(\check{e}').$$

As explained in section 4 and 8 in [BM1], $X(\check{e}',\chi',\psi')$ contains *sgn* if and only if $\psi'=triv$. Thus if we assume $\check{\mathcal{O}}$ satisfies (1) and (2) with respect to χ, it follows that $X(\check{e},\chi,triv)=L(\check{e},\chi,triv)$. We would like it to equal $I_{M_{BC}}$ but this is not true. In general (for an M which contains M_{BC}),

$L(\check{e},\chi,triv) = Ind_M^G[X_M(\check{e},\chi,triv)]$ if and only if the component $A_M(\check{e},\chi)$ equals the component group $A(\check{e},\chi)$. We will enlarge $M_{BC}(\check{\mathcal{O}})$ to an M_{KL} so that $A_{M_{KL}}(\check{e},\chi) = A(\check{e},\chi)$. Note that if $\check{\mathfrak{m}} \subset \check{\mathfrak{m}}'$, then $A_M(\check{e},\chi) \subset A_{M'}(\check{e},\chi)$. Then

{eq:1.4}
$$Ind_{M_{KL}}^G[X_{M_{KL}}(\check{e},\chi,triv)] = X(\check{e},\chi,triv) = L(\check{e},\chi,triv) \qquad (1.0.4)$$

and

{eq:1.5}
$$L(\chi) = I_{M_{KL}}(\chi) := Ind_{M_{KL}}^G[L_{M_{KL}}(\chi)] \qquad (1.0.5)$$

follows by applying τ. We remark that M_{KL} depends on χ as well as $\check{e}$. It will be described explicitly in section 2. A more general discussion about how canonical $\check{\mathfrak{m}}_{KL}$ is, appears in [BC2].

In the real case, we use the same Levi components as in the $p-$adic case. Then equality (1.0.5) does not hold for any proper Levi component. A result essential for the paper is that equality *does* hold at the level of multiplicities of the *relevant* $K-$types (section 4.2).

We will use the data $(\check{\mathcal{O}},\nu)$ to parametrize the unitary dual. Fix an $\check{\mathcal{O}}$. A representation $L(\chi)$ will be called a *complementary series attached to* $\check{\mathcal{O}}$, if it is unitary, and $\check{\mathcal{O}}(\chi) = \check{\mathcal{O}}$. To describe it, we need to give the set of ν such that $L(\chi)$ with $\chi = \check{h}/2 + \nu$ is unitary. Viewed as an element of $\mathfrak{z}(\check{\mathcal{O}})$, the element ν gives rise to a spherical parameter $((0),\nu)$ where (0) denotes the trivial nilpotent orbit. The main result in section 3.2 says that the ν giving rise to the complementary series for $\check{\mathcal{O}}$ coincide with the ones giving rise to the complementary series for (0) on $\mathfrak{z}(\check{\mathcal{O}})$. This is suggestive of Langlands functoriality.

It is natural to conjecture that such a result will hold for all split groups. Recent work of D. Ciubotaru for F_4, and by D. Ciubotaru and myself for the other exceptional cases, show that this is generally true, but there are exceptions.

I give a more detailed outline of the paper. Section 2 reviews notation from earlier papers. Section 3 gives a statement of the main results. A representation is called *spherical unipotent* if its parameter is of the form $\check{h}/2$ for the neutral element of a Lie triple associated to a nilpotent orbit $\check{\mathcal{O}}$. The unitarity of the spherical unipotent representations is dealt with in section 8. For the $p-$adic case I simply cite [BM3]. The real case (sketched in [B2]) is redone in section 9.5. The proofs are simpler than the original ones because I take advantage of the fact that wave front sets, asymptotic supports and associated varieties "coincide" due to [SV]. Section 10.1 proves an irreducibility result in the real case which is clear in the $p-$adic case from the work of Kazhdan-Lusztig. This is needed for determining the complementary series (definition 3.1 in section 3.1).

Sections 4 and 5 deal with the nonunitarity. The decomposition $\chi = \check{h}/2 + \nu$ is introduced in section 3. It is more common to parametrize the

χ by representatives in $\check{\mathfrak{a}}$ which are dominant with respect to some positive root system. We use Bourbaki's standard realization of the positive system. It is quite messy to determine the data $(\check{\mathcal{O}},\nu)$ from a dominant parameter, because of the nature of the nilpotent orbits and the Weyl group. Sections 2.3 and 2.8 give a combinatorial description of $(\check{\mathcal{O}},\nu)$ starting from a dominant χ.

In the classical cases, the orbit $\check{\mathcal{O}}$ is given in terms of partitions. To such a partition we associate the Levi component

$$\check{\mathfrak{m}}_{BC} := gl(a_1)\times\cdots\times gl(a_k)\times\check{\mathfrak{g}}_0(n_0)$$

given by the Bala-Carter classification. (The $\check{\mathfrak{g}}_0$ in this formula is *not related* to the one just after conditions (1) and (2)). The intersection of $\check{\mathcal{O}}$ with $\check{\mathfrak{m}}_{BC}$ is an orbit of the form

$$(a_1)\times\cdots\times(a_r)\times\check{\mathcal{O}}_0$$

where $\check{\mathcal{O}}_0$ is a distinguished nilpotent orbit, and (a_i) is the principal nilpotent orbit on $gl(a_i)$. This is the distinguished orbit associated to $\check{\mathcal{O}}$ by Bala-Carter. Then χ gives rise to irreducible spherical modules $L_M(\chi)$, $L(\chi)$ and $I_M(\chi)$ as in (1.0.3) and (1.0.5). The module $L(\chi)$ is the irreducible spherical subquotient of $I_M(\chi)$. As already mentioned, $I_{M_{KL}}(\chi)=L(\chi)$ in the $p-$adic case, but not the real case. In all cases, the multiplicities of the *relevant* $K-$*types* in $L(\chi)$, $I_M(\chi)$ coincide. These are representations of the Weyl group in the $p-$adic case, representations of the maximal compact subgroup in the real case. Their definition is in section 4.2; they are a small finite set of representations which provide necessary conditions for unitarity which are also sufficient. The relationship between the real and $p-$adic case is investigated in chapter 4. In particular the issue is addressed of how the relevant $K-$types allow us to deal with the $p-$adic case only. A more general class of $K-$types for split real groups (named *petite* K-types), on which the intertwining operator is equal to the p-adic operator, is defined in [B6], and the proofs are more conceptual. Sections 4.4, and 4.5 are included for completeness. The interested reader can consult [B6] and [BC1] for results where these kinds of $K-$types and $W-$types are useful.

The determination of the nonunitary parameters proceeds by induction on the rank of $\check{\mathfrak{g}}$ and by the inclusion relations of the closure of the orbit $\check{\mathcal{O}}$. Section 5 completes the induction step; it shows that conditions (B) in section 3.1 is necessary. The last part of the induction step is actually done in section 3.1.

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2. Description of the spherical parameters {sec}2

2.1. **Explicit Langlands parameters.** We consider spherical irreducible representations of the split connected classical groups of rank n of type $B,\ C,\ D$, precisely, $G = So(2n+1),\ G = Sp(2n)$ and $G = So(2n)$. These groups will be denoted by $G(n)$ when there is no danger of confusion (n is the rank). Levi components will be written as

{eq:1.1levi} $$M = GL(k_1) \times \cdots \times GL(k_r) \times G_0(n_0), \tag{2.1.1}$$

where $G_0(n_0)$ is the factor of the same type as G. The corresponding complex Lie algebras are denoted $\mathfrak{g}(n)$ and $\mathfrak{m} = gl(k_1) \times \cdots \times gl(k_r) \times \mathfrak{g}_0(n_0)$.

As already explained in the introduction, we deal with real unramified characters only. In the case of classical groups, such a character can be represented by a vector of size the rank of the group. Two such vectors parametrize the same irreducible spherical module if they are conjugate via the Weyl group which acts by permutations and sign changes for type B, C and by permutations and an even number of sign changes in type D. For a given χ, let $L(\chi)$ be the corresponding irreducible spherical module. We will occasionally refer to χ as the infinitesimal character.

For any nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ we attach a parameter $\chi_{\check{\mathcal{O}}} \in \mathfrak{a}^*$ as follows. Recall from the introduction that $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to $\check{\mathfrak{a}}$. Let $\{\check{e}, \check{h}, \check{f}\}$ be representatives for the Lie triple associated to a nilpotent orbit $\check{\mathcal{O}}$. Then $\chi_{\check{\mathcal{O}}} := \check{h}/2$.

Conversely, to each χ we will attach a nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ and Levi components $M_{BC},\ M_{KL} := GL(k_1) \times \cdots \times GL(k_r) \times G_0(n_0)$. In addition we will specify an even nilpotent orbit $\check{\mathcal{O}}_0 \subset \check{\mathfrak{g}}_0(n_0)$ with unramified character $\chi_0 := \chi_{\check{\mathcal{O}}_0}$ on $\mathfrak{g}_0(n_0)$, and unramified characters χ_i on the $GL(k_i)$. These data have the property that $L(\chi)$ is the spherical subquotient of

{1.1.0} $$Ind_{M_{KL}}^{G}[\bigotimes_i L(\chi_i) \otimes L(\chi_0)]. \tag{2.1.2}$$

{1.2a}

2.2. We introduce the following notation (a variant of the one used by Zelevinski [ZE]).

{d:1.1}

Definition. *A* string *is a sequence*

$$(a, a+1, \dots, b-1, b)$$

of numbers increasing by 1 from a to b. A set of strings is called nested *if for any two strings either the coordinates* do not *differ by integers, or if they do, then their coordinates, say* $(a_1, \dots, b_1)$ *and* $(a_2, \dots, b_2)$, *satisfy*

{eq:d1} $$a_1 \le a_2 \le b_2 \le b_1 \qquad or \qquad a_2 \le a_1 \le b_1 \le b_2, \tag{2.2.1}$$

or

{eq:d2} $$b_1 + 1 < a_2 \qquad or \qquad b_2 + 1 < a_1. \quad \square \tag{2.2.2}$$

A set of strings is called strongly nested *if the coordinates of any two strings either* do not *differ by integers or else satisfy (2.2.1).*

Each string represents a 1-dimensional spherical representation of a $GL(n_i)$ with $n_i = b_i - a_i + 1$. The matchup is

{eq:1.1.1a} $$(a,\dots,b) \longleftrightarrow \big|\det\big|^{\frac{a+b}{2}}, \quad \text{of} \quad GL(b-a+1). \tag{2.2.3}$$

In the case of $G = GL(n)$, we record the following result. For the $p-$adic case, it originates in the work of Zelevinski, and Bernstein-Zelevinski ([ZE] and references therein). To each set of strings $(a_1,\dots,b_1;\dots;a_k,\dots,b_k)$ we can attach a Levi component $M_{BC} := \prod_{1\le i\le k} GL(n_i)$, and an induced module

$$I(\chi) := Ind_{M_{BC}}^{GL(n)}[\bigotimes L(\chi_i)] \tag{2.2.4}$$ {eq:1.1.1b}

where χ_i are obtained from the strings as in (2.2.3).

In general, if the set of strings is not nested, then the corresponding induced module is not irreducible. The coordinates of χ in $\mathfrak{a}^* \simeq \mathbb{R}^n$ determine a set of nested strings as follows. Extract the longest sequence starting with the smallest element in χ that can form a string. Continue to extract sequences from the remainder until there are no elements left. This set of strings is, up to the order of the strings, the unique set of nested strings one can form out of the entries of χ.

{t:2.1}

Theorem. *Suppose* $\mathbb{F}$ *is p-adic. Let* $(a_1,\dots,b_1;\dots a_r,\dots,b_r)$ *be a set of nested strings, and* $M := GL(b_1 - a_1 + 1)\times\dots\times GL(b-r-a_r+1)$*. Then*

$$L(\chi) = Ind_M^{GL(n)}\Big[\ \big|\det\big|^{\frac{a_1+b_1}{2}}\cdot\dots\cdot\big|\det\big|^{\frac{a_r+b_r}{2}}\ \Big].$$

In the language of section 2.1, $M_{BC} = M_{KL} = M$, where M is the one defined in the theorem. The nilpotent orbit $\check{\mathcal{O}}$ corresponds to the partition of n with entries $(b_i - a_i + 1)$; it is the orbit $\check{\mathcal{O}}(\chi)$ satisfying (1) and (2) in the introduction, with respect to $\chi = (a_1,\dots,b_1;\dots;a_r,\dots,b_r)$.

For the real case (still $GL(n)$), the induced module in theorem 2.2 fails to be irreducible. However equality holds on the level of multiplicities of relevant K-types. These K-types will be defined in section 4.7.

We will generalize this procedure to the other classical groups. As before, the induced modules that we construct fail to be irreducible in the real case, but equality of multiplicity of *relevant* $K-$types in the two sides of (1.0.5) holds.

{sec:2.3}

2.3. Nilpotent orbits. In this section we attach a set of parameters to each nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$. Let $\{\check{e},\check{h},\check{f}\}$ be a Lie triple so that $\check{e}\in\check{\mathcal{O}}$, and let $\mathfrak{z}(\check{\mathcal{O}})$ be its centralizer. In order for χ to be a parameter attached to $\check{\mathcal{O}}$ we require that

$$\chi = \check{h}/2 + \nu, \qquad \nu \in \mathfrak{z}(\check{\mathcal{O}}), \ \ \text{semisimple}, \tag{2.3.1}$$ {eq:2.3.1}

but also that if

$$\chi = \check{h}'/2 + \nu', \qquad \nu' \in \mathfrak{z}(\check{\mathcal{O}}'), \quad \text{semisimple} \tag{2.3.2}$$ {eq:2.3.2}

for another nilpotent orbit $\check{\mathcal{O}}' \subset \check{\mathfrak{g}}$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$. In [BM1], it is shown that the orbit of χ, uniquely determines $\check{\mathcal{O}}$ and the conjugacy class of $\nu \in \mathfrak{z}(\check{\mathcal{O}})$. We describe the pairs $(\check{\mathcal{O}}, \nu)$ explicitly in the classical cases.

Nilpotent orbits are parametrized by partitions

{eq:2.3.3} $$(\underbrace{1,\dots,1}_{r_1}, \underbrace{2,\dots,2}_{r_2}, \dots, \underbrace{j,\dots,j}_{r_j}, \dots), \tag{2.3.3}$$

satisfying the following constraints.

- **A_{n-1}:** $gl(n)$, partitions of n.
- **B_n:** $so(2n+1)$, partitions of $2n+1$ such that every even part occurs an even number of times.
- **C_n:** $sp(2n)$, partitions of $2n$ such that every odd part occurs an even number of times.
- **D_n:** $so(2n)$, partitions of $2n$ such that every even part occurs an even number of times. In the case when every part of the partition is even, there are two conjugacy classes of nilpotent orbits with the same Jordan blocks, labelled (I) and (II). The two orbits are conjugate under the action of $O(2n)$.

The Bala-Carter classification is particularly well suited for describing the parameter spaces attached to the $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$. An orbit is called *distinguished* if it does not meet any proper Levi component. In type A, the only distinguished orbit is the principal nilpotent orbit, where the partition has only one part. In the other cases, the distinguished orbits are the ones where each part of the partition occurs at most once. In particular, these are *even nilpotent orbits*, *i.e.* $\operatorname{ad}\check{h}$ has even eigenvalues only. Let $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ be an arbitrary nilpotent orbit. We need to put it into as small as possible Levi component $\check{\mathfrak{m}}$. In type A, if the partition is $(a_1, \dots, a_k)$, the Levi component is $\check{\mathfrak{m}}_{BC} = gl(a_1) \times \cdots \times gl(a_k)$. In the other classical types, the orbit $\check{\mathcal{O}}$ meets a proper Levi component if and only if one of the $r_j > 1$. So separate as many pairs (a, a) from the partition as possible, and rewrite it as

{eq:2.3.4} $$((a_1, a_1), \dots, (a_k, a_k); d_1, \dots, d_l), \tag{2.3.4}$$

with $d_i < d_{i+1}$. The Levi component $\check{\mathfrak{m}}_{BC}$ attached to this nilpotent by Bala-Carter is

{eq:2.3.5} $$\check{\mathfrak{m}}_{BC} = gl(a_1) \times \cdots \times gl(a_k) \times \check{\mathfrak{g}}_0(n_0) \quad n_0 := n - \sum a_i, \tag{2.3.5}$$

The distinguished nilpotent orbit is the one with partition (d_i) on $\check{\mathfrak{g}}(n_0)$, principal nilpotent on each $gl(a_j)$. The χ of the form $\check{h}/2 + \nu$ are the ones with ν an element of the center of $\check{\mathfrak{m}}_{BC}$. The explicit form is

{eq:2.3.6} $$(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i, \dots; \check{h}_0/2), \tag{2.3.6}$$

where $\check{h}_0$ is the middle element of a triple corresponding to (d_i). We will write out (d_i) and $\check{h}_0/2$ in sections 2.4-2.7.

We will consider more general cases where we write the partition of $\check{\mathcal{O}}$ in the form (2.3.4) so that the d_i are not necessarily distinct, but (d_i) forms an even nilpotent orbit in $\check{\mathfrak{g}}_0(n_0)$. This will be the situation for $\check{\mathfrak{m}}_{KL}$.

The parameter χ determines an irreducible spherical module $L(\chi)$ for G as well as an $L_M(\chi)$ for $M = M_{BC}$ or M_{KL} of the form

$$L_1(\chi_1) \otimes \cdots \otimes L_k(\chi_k) \otimes L_0(\chi_0), \tag{2.3.7}$$ {eq:2.3.7}

where the $L_i(\chi_i)$ $i = 1, \dots, k$ are one dimensional. We will consider the relation between the induced module

$$I_M(\chi) := Ind_M^G[L_M(\chi)], \tag{2.3.8}$$ {eq:2.3.8}

and $L(\chi)$.

{sec:2.3a}

2.4. G of Type A. We write the $\check{h}/2$ for a nilpotent $\check{\mathcal{O}}$ corresponding to $(a_1, \dots, a_k)$ with $a_i \le a_{i+1}$ as

$$(\dots; -\frac{a_i - 1}{2}, \dots, \frac{a_i - 1}{2}; \dots).$$

The parameters of the form $\chi = \check{h}/2 + \nu$ are then

$$(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots). \tag{2.4.1}$$ {2.3a.1}

Conversely, given a parameter as a concatenation of strings

$$\chi = (\dots; A_i, \dots, B_i; \dots), \tag{2.4.2}$$ {eq:2.3a.2}

it is of the form $\check{h}/2 + \nu$ where $\check{h}$ is the neutral element for the nilpotent orbit with partition $(A_i + B_i + 1)$ (the parts need not be in any particular order) and $\nu_i = \frac{A_i - B_i}{2}$. We recall the following well known result about closures of nilpotent orbits.

{l:2.3a}

Lemma. *Assume $\check{\mathcal{O}}$ and $\check{\mathcal{O}}'$ correspond to the (increasing) partitions $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ respectively, where some of the a_i or b_j may be zero in order to have the same number k. The following are equivalent*

(1) $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$.
(2) $\sum_{i \ge s} a_i \ge \sum_{i \ge s} b_i$ *for all* $k \ge s \ge 1$.

{p:2.3a}

Proposition. *A parameter χ as in (2.4.1) is attached to $\check{\mathcal{O}}$ in the sense of satisfying (2.3.1) and (2.3.2) if and only if it is nested.*

Proof. Assume the strings are not nested. There must be two strings

$$(A, \dots, B), \qquad (C, \dots, D) \tag{2.4.3}$$ {eq:2.3a.3}

such that $A - C \in \mathbb{Z}$, and $A < C \le B < D$, or $C = B + 1$. Then by conjugating χ by the Weyl group to a χ', we can rearrange the coordinates of the two strings in (2.4.3) so that the strings

$$(A, \dots, D), \quad (C, \dots B), \qquad \text{or} \quad (A, \dots, D). \tag{2.4.4}$$ {eq:2.3a.4}

appear. Then by the lemma, $\chi' = \check{h}'/2 + \nu'$ for a strictly larger nilpotent $\check{\mathcal{O}}'$.

Conversely, assume $\chi = \check{h}/2 + \nu$, so it is written as strings, and they are nested. The nilpotent orbit for which the neutral element is $\check{h}/2$ has partition given by the lengths of the strings, say $(a_1, \dots a_k)$ in increasing order. If χ is nested, then a_k is the length of the longest string of entries we can extract from the coordinates of χ, a_{k-1} the longest string we can extract from the remaining coordinates and so on. Then (2) of lemma 2.4 precludes the possibility that some conjugate χ' equals $\check{h}'/2 + \nu'$ for a strictly larger nilpotent orbit. □

In type A, $\check{\mathfrak{m}}_{KL} = \check{\mathfrak{m}}_{BC}$.

{sec:2.3b}

2.5. G of Type B. Rearrange the parts of the partition of $\check{\mathcal{O}} \subset sp(2n, \mathbb{C})$, in the form (2.3.4),

{eq:2.3b.1}
$$((a_1, a_1), \dots, (a_k, a_k); 2x_0, \dots, 2x_{2m}) \tag{2.5.1}$$

The d_i have been relabeled as $2x_i$ and a $2x_0 = 0$ is added if necessary, to insure that there is an odd number. The x_i are integers, because all the odd parts of the partition of $\check{\mathcal{O}}$ occur an even number of times, and were therefore extracted as (a_i, a_i). The χ of the form $\check{h}/2 + \nu$ are

{eq:2.3b.2}
$$\begin{aligned}(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots;\\ \underbrace{1/2, \dots, 1/2}_{n_{1/2}}, \dots, \underbrace{x_{2m} - 1/2, \dots, x_{2m} - 1/2}_{n_{x_{2m}-1/2}}).\end{aligned} \tag{2.5.2}$$

where

{eq:2.3b.3}
$$n_{l-1/2} = \#\{x_i \geq l\}. \tag{2.5.3}$$

Lemma 2.4 holds for this type verbatim. So the following proposition holds.

{p:2.3b}

Proposition. *A parameter $\chi = \check{h}/2 + \nu$ cannot be conjugated to one of the form $\check{h}'/2 + \nu'$ for any larger nilpotent $\check{\mathcal{O}}'$ if and only if*

(1) *the set of strings satisfying $\frac{a_i-1}{2} + \nu_i - \frac{a_j-1}{2} - \nu_j \in \mathbb{Z}$ are nested.*
(2) *the strings satisfying $\frac{a_i-1}{2} + \nu_i \in 1/2 + \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1/2 < -\frac{a_i-1}{2} + \nu_i$ or there is j such that*

{eq:2.3b.4}
$$x_j + 1/2 < -\frac{a_i - 1}{2} + \nu_i \leq \frac{a_i - 1}{2} + \nu_i < x_{j+1} + 1/2. \tag{2.5.4}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is obtained from $\check{\mathfrak{m}}_{BC}$ as follows. Consider the strings for which a_i is even, and $\nu_i = 0$. If a_i is not equal to any $2x_j$, then remove one pair (a_i, a_i), and add two $2x_j = a_i$ to the last part of (2.5.1). For example, if the nilpotent orbit $\check{\mathcal{O}}$ is

{eq:2.3b.5}
$$(2, 2, 2, 3, 3, 4, 4), \tag{2.5.5}$$

then the parameters of the form $\check{h}/2+\nu$ are

{eq:2.3b.6}
$$\begin{aligned}(&-1/2+\nu_1, 1/2+\nu_1; -1+\nu_2, \nu_2, 1+\nu_2;\\ &-3/2+\nu_3, -1/2+\nu_3, 1/2+\nu_3, 3/2+\nu_3; 1/2)\end{aligned} \tag{2.5.6}$$

The Levi component is $\check{\mathfrak{m}}_{BC}=gl(2)\times gl(3)\times gl(4)\times\check{\mathfrak{g}}(1)$. If $\nu_3\neq 0$, then $\check{\mathfrak{m}}_{BC}=\check{\mathfrak{m}}_{KL}$. But if $\nu_3=0$, then $\check{\mathfrak{m}}_{KL}=gl(2)\times gl(3)\times\check{\mathfrak{g}}(5)$. The parameter is rewritten

$$\begin{aligned}&\check{\mathcal{O}}\longleftrightarrow((2,2)(3,3);2,4,4) &(2.5.7)\\ &\chi\longleftrightarrow(-1/2+\nu_1,1/2+\nu_1;-1+\nu_2,\nu_2,1+\nu_2;1/2,1/2,1/2,3/2,3/2).\end{aligned}$$ {eq:2.3b.7}

The explanation is as follows. For a partition (2.3.3),

$$\mathfrak{z}(\check{\mathcal{O}})=sp(r_1)\times so(r_2)\times sp(r_3)\times\dots \tag{2.5.8}$$ {eq:2.3b.8}

and the centralizer in $\check{G}$ is a product of $Sp(r_{2j+1})$ and $O(r_{2j})$, *i.e.* Sp for the odd parts, O for the even parts. Thus the component group $A(\check{h},\check{e})$, which by [BV2] also equals $A(\check{e})$, is a product of $\mathbb{Z}_2$, one for each $r_{2j}\neq 0$. Then $A(\chi,\check{e})=A(\nu,\check{h},\check{e})$. In general $A_{M_{BC}}(\chi,\check{e})=A_{M_{BC}}(\check{e})$ embeds canonically into $A(\chi,\check{e})$, but the two are not necesarily equal. In this case they are unless one of the $\nu_i=0$ for an even a_i with the additional property that there is no $2x_j=a_i$.

We can rewrite each of the remaining strings

$$\left(-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i\right) \tag{2.5.9}$$ {eq:2.3b.9}

as

$$\chi_i:=(f_i+\tau_i, f_i+1+\tau_i,\dots,F_i+\tau_i), \tag{2.5.10}$$ {eq:2.3b.10}

satisfying

$$\begin{aligned}&f_i\in\mathbb{Z}+1/2,\quad 0\le\tau_i\le 1/2,\quad F_i=f_i+a_i,. &(2.5.11)\\ &|f_i+\tau_i|\ge|F_i+\tau_i| \text{ if } \tau_i=1/2\end{aligned}$$ {eq:2.3b.11}

This is done as follows. We can immediately get an expression like (2.5.10) with $0\le\tau_i<1$, by defining f_i to be the largest element in $\mathbb{Z}+1/2$ less than or equal to $-\frac{a_i-1}{2}+\nu_i$. If $\tau_i\le 1/2$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i-1$, and τ_i by $1-\tau_i$. The presentation of the strings subject to (2.5.11) is unique except when $\tau_i=1/2$. In this case the argument just given provides the presentation $(f_i+1/2,\dots,F_i+1/2)$, but also provides the presentation

$$(-F_i-1+1/2,\dots,-f_i-1+1/2). \tag{2.5.12}$$ {eq:2.3b.12}

We choose between (2.5.10) and (2.5.12) the one whose leftmost term is larger in absolute value. That is, we require $|f_i+\tau_i|\ge|F_i+\tau_i|$ whenever $\tau_i=1/2$.

{sec:2.3c}

2.6. G of Type C. Rearrange the parts of the partition of $\check{\mathcal{O}} \subset so(2n+1,\mathbb{C})$, in the form (2.3.4),

$$((a_1, a_1), \dots, (a_k, a_k); 2x_0 + 1, \dots, 2x_{2m} + 1); \tag{2.6.1}$$ {eq:2.3c.1}

The d_i have been relabeled as $2x_i + 1$. In this case it is automatic that there is an odd number of nonzero x_i. The x_i are integers, because all the even parts of the partition of $\check{\mathcal{O}}$ occur an even number of times, and were threrefore extracted as (a_i, a_i). The χ of the form $\check{h}/2 + \nu$ are

{eq:2.3c.2} $$(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots; \underbrace{0, \dots, 0}_{n_0}, \dots, \underbrace{x_{2m}, \dots, x_{2m}}_{n_{x_{2m}}}). \tag{2.6.2}$$

where

{eq:2.3c.3} $$n_l = \begin{cases} m & \text{if } l = 0, \\ \#\{x_i \geq l\} & \text{if } l \neq 0. \end{cases} \tag{2.6.3}$$

Lemma 2.4 holds for this type verbatim. So the following proposition holds.

{p:2.3c}

Proposition. *A parameter $\chi = \check{h}/2 + \nu$ cannot be conjugated to one of the form $\check{h}'/2 + \nu'$ for any larger nilpotent $\check{\mathcal{O}}'$ if and only if*

(1) *the set of strings satisfying $\frac{a_i-1}{2} + \nu_i - \frac{a_j-1}{2} - \nu_j \in \mathbb{Z}$ are nested.*
(2) *the strings satisfying $\frac{a_i-1}{2} + \nu_i \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1 < -\frac{a_i-1}{2} + \nu_i$ or there is j such that*

{eq:2.3c.4} $$x_j + 1 < -\frac{a_i - 1}{2} + \nu_i \leq \frac{a_i - 1}{2} + \nu_i < x_{j+1} + 1. \tag{2.6.4}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is obtained from $\check{\mathfrak{m}}_{BC}$ as follows. Consider the strings for which a_i is odd and $\nu_i = 0$. If a_i is not equal to any $2x_j + 1$, then remove one pair (a_i, a_i), and add two $2x_j + 1 = a_i$ to the last part of (2.6.1). For example, if the nilpotent orbit is

{eq:2.3c.5} $$(1, 1, 1, 3, 3, 4, 4) = ((1, 1), (3, 3), (4, 4); 1), \tag{2.6.5}$$

then the parameters of the form $\check{h}/2 + \nu$ are

{eq:2.3c.6} $$\begin{aligned} &(\nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2; \\ &\quad -3/2 + \nu_3, -1/2 + \nu_3, 1/2 + \nu_3, 3/2 + \nu_3) \end{aligned} \tag{2.6.6}$$

The Levi component is $\check{\mathfrak{m}}_{BC} = gl(1) \times gl(3) \times gl(4)$. If $\nu_2 \neq 0$, then $\check{\mathfrak{m}}_{BC} = \check{\mathfrak{m}}_{KL}$. But if $\nu_2 = 0$, then $\check{\mathfrak{m}}_{KL} = gl(1) \times gl(4) \times \check{\mathfrak{g}}(3)$. The parameter is rewritten

{eq:2.3c.7} $$\begin{aligned} &\check{\mathcal{O}} \longleftrightarrow ((1, 1), (4, 4); 1, 3, 3) \\ &\chi \longleftrightarrow (\nu_1; -3/2 + \nu_3, -1/2 + \nu_3, 1/2 + \nu_3, 3/2 + \nu_3; 0, 1, 1). \end{aligned} \tag{2.6.7}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is unchanged if $\nu_1 = 0$.

The explanation is as follows. For a partition (2.3.3),

{eq:2.3c.8} $$\mathfrak{z}(\check{\mathcal{O}}) = so(r_1) \times sp(r_2) \times so(r_3) \times \dots \tag{2.6.8}$$

and the centralizer in $\check{G}$ is a product of $O(r_{2j+1})$ and $Sp(r_{2j})$, *i.e.* O for the odd parts, Sp for the even parts. Thus the component group is a product of $\mathbb{Z}_2$, one for each $r_{2j+1} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$, and so $A_{M_{BC}}(\chi, \check{e}) = A(\chi, \check{e})$ unless one of the $\nu_i = 0$ for an odd a_i with the additional property that there is no $2x_j + 1 = a_i$.

We can rewrite each of the remaining strings

$$(-\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i) \tag{2.6.9}$$ {eq:2.3c.9}

as

$$\chi_i := (f_i + \tau_i, f_i + 1 + \tau_i, \dots, F_i + \tau_i), \tag{2.6.10}$$ {eq:2.3c.10}

satisfying

$$f_i \in \mathbb{Z}, \quad 0 \leq \tau_i \leq 1/2, \quad F_i = f_i + a_i \tag{2.6.11}$$ {eq:2.3c.11}

$$|f_i + \tau_i| \geq |F_i + \tau_i| \text{ if } \tau_i = 1/2.$$

This is done as follows. We can immediately get an expression like (2.6.10) with $0 \leq \tau_i < 1$, by defining f_i to be the largest element in $\mathbb{Z}$ less than or equal to $-\frac{a_i-1}{2} + \nu_i$. If $\tau_i \leq 1/2$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i - 1$, and τ_i by $1 - \tau_i$. The presentation of the strings subject to (2.6.11) is unique except when $\tau_i = 1/2$. In this case the argument just given also provides the presentation

$$(-F_i - 1 + 1/2, \dots, -f_i - 1 + 1/2). \tag{2.6.12}$$ {eq:2.3c.12}

We choose between (2.6.10) and (2.6.12) the one whose leftmost term is larger in absolute value. That is, we require $|f_i + \tau_i| \geq |F_i + \tau_i|$ whenever $\tau_i = 1/2$.

{sec:2.3d}

2.7. G of Type D. Rearrange the parts of the partition of $\check{\mathcal{O}} \subset so(2n, \mathbb{C})$, in the form (2.3.4),

$$((a_1, a_1), \dots, (a_k, a_k); 2x_0 + 1, \dots, 2x_{2m-1} + 1) \tag{2.7.1}$$ {eq:2.3d.1}

The d_i have been relabeled as $2x_i + 1$. In this case it is automatic that there is an even number of nonzero $2x_i + 1$. The x_i are integers, because all the even parts of the partition of $\check{\mathcal{O}}$ occur an even number of times, and were therefore extracted as (a_i, a_i). The χ of the form $\check{h}/2 + \nu$ are

$$(\dots; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots; \underbrace{0, \dots, 0}_{n_0}, \dots, \underbrace{x_{2m-1}, \dots, x_{2m-1}}_{n_{x_{2m-1}}}). \tag{2.7.2}$$ {eq:2.3d.2}

where

$$n_l = \begin{cases} m & \text{if } l = 0, \\ \#\{x_i \geq l\} & \text{if } l \neq 0. \end{cases} \tag{2.7.3}$$ {eq:2.3d.3}

Lemma 2.4 holds for this type verbatim. So the following proposition holds.

{p:2.3d}

Proposition. *A parameter* $\chi = \check{h}/2 + \nu$ *cannot be conjugated to one of the form* $\check{h}'/2 + \nu'$ *for any larger nilpotent* $\check{\mathcal{O}}'$ *if and only if*

(1) *the set of strings satisfying* $\frac{a_i - 1}{2} + \nu_i - \frac{a_j - 1}{2} - \nu_j \in \mathbb{Z}$ *are nested.*
(2) *the strings satisfying* $\frac{a_i-1}{2} + \nu_i \in \mathbb{Z}$ *satisfy the additional condition that either* $x_{2m-1} + 1 < -\frac{a_i-1}{2} + \nu_i$ *or there is* j *such that*

{eq:2.3d.4}
$$x_j + 1 < -\frac{a_i - 1}{2} + \nu_i \leq \frac{a_i - 1}{2} + \nu_i < x_{j+1} + 1. \tag{2.7.4}$$

The Levi component $\check{\mathfrak{m}}_{KL}$ is obtained from $\check{\mathfrak{m}}_{BC}$ as follows. Consider the strings for which a_i is odd and $\nu_i = 0$. If a_i is not equal to any $2x_j + 1$, then remove one pair (a_i, a_i), and add two $2x_j + 1 = a_i$ to the last part of (2.7.1). For example, if the nilpotent orbit is

{eq:2.3d.5}
$$(1, 1, 3, 3, 4, 4), \tag{2.7.5}$$

then the parameters of the form $\check{h}/2 + \nu$ are

{eq:2.3d.6}
$$\begin{aligned}&(\nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2;\\ &-3/2 + \nu_3, -1/2 + \nu_3, 1/2 + \nu_3, 3/2 + \nu_3)\end{aligned} \tag{2.7.6}$$

The Levi component is $\check{\mathfrak{m}}_{BC} = gl(1) \times gl(3) \times gl(4)$. If $\nu_2 \neq 0$ and $\nu_1 \neq 0$, then $\check{\mathfrak{m}}_{BC} = \check{\mathfrak{m}}_{KL}$. If $\nu_2 = 0$ and $\nu_1 \neq 0$, then $\check{\mathfrak{m}}_{KL} = \check{\mathfrak{g}}(3) \times gl(1) \times gl(4)$. If $\nu_2 \neq 0$ and $\nu_1 = 0$, then $\check{\mathfrak{m}}_{KL} = gl(3) \times gl(4) \times \check{\mathfrak{g}}(1)$. If $\nu_1 = \nu_2 = 0$, then $\check{\mathfrak{m}}_{KL} = gl(4) \times \check{\mathfrak{g}}(4)$. The parameter is rewritten

{eq:2.3d.7}
$$\begin{aligned}&\check{\mathcal{O}} \longleftrightarrow ((1, 1), (4, 4); 3, 3)\\ &\chi \longleftrightarrow (\nu_1; -3/2 + \nu_3, -1/2 + \nu_3; 1/2 + \nu_3, 3/2 + \nu_3; 0, 1, 1).\end{aligned} \tag{2.7.7}$$

The explanation is as follows. For a partition (2.3.3),

{eq:2.3d.8}
$$\mathfrak{z}(\check{\mathcal{O}}) = so(r_1) \times sp(r_2) \times so(r_3) \times \dots \tag{2.7.8}$$

and the centralizer in $\check{G}$ is a product of $O(r_{2j+1})$ and $Sp(r_{2j})$, *i.e.* O for the odd parts, Sp for the even parts. Thus the component group is a product of $\mathbb{Z}_2$, one for each $r_{2j+1} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$, and so $A_{M_{BC}}(\chi, \check{e}) = A(\chi, \check{e})$ unless one of the $\nu_i = 0$ for an odd a_i with the additional property that there is no $2x_j + 1 = a_i$.

We can rewrite each of the remaining strings

{eq:2.3d.9}
$$(-\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i) \tag{2.7.9}$$

as

{eq:2.3d.10}
$$\chi_i := (f_i + \tau_i, f_i + 1 + \tau_i, \dots, F_i + \tau_i), \tag{2.7.10}$$

{eq:2.3d.11}
$$\begin{aligned}&\text{satisfying} \quad f_i \in \mathbb{Z}, \quad 0 \leq \tau_i \leq 1/2, \quad F_i = f_i + a_i\\ &|f_i + \tau_i| \geq |F_i + \tau_i| \text{ if } \tau_i = 1/2.\end{aligned} \tag{2.7.11}$$

This is done as in types B and C, but see the remarks which have to do with the fact that $-Id$ is not in the Weyl group. We can immediately get an expression like (2.7.10) with $0 \leq \tau_i < 1$, by defining f_i to be the largest element in $\mathbb{Z}$ less than or equal to $-\frac{a_i-1}{2}+\nu_i$. If $\tau_i \le 1/2$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i-1$, and τ_i by $1-\tau_i$. The presentation of the strings subject to (2.7.11) is unique except when $\tau_i = 1/2$. In this case the argument just given also provides the presentation

$$(-F_i-1+1/2,\dots,-f_i-1+1/2). \tag{2.7.12}$$ {eq:2.3d.12}

We choose between (2.7.10) and (2.7.12) the one whose leftmost term is larger in absolute value. That is, we require $|f_i+\tau_i| \ge |F_i+\tau_i|$ whenever $\tau_i = 1/2$.

Remarks

(1) A (real) spherical parameter χ is hermitian if and only if there is $w \in W(D_n)$ such that $w\chi = -\chi$. This is the case if the parameter has a coordinate equal to zero, or if none of the coordinates are 0, but then n must be even.

(2) Assume the nilpotent orbit $\check{\mathcal{O}}$ is very even, *i.e.* all the parts of the partition are even (and therefore occur an even number of times). The nilpotent orbits labelled (I) and (II) are characterized by the fact that $\check{\mathfrak{m}}_{BC}$ is of the form

$$(I) \longleftrightarrow gl(a_1)\times\dots\times gl(a_{k-1})\times gl(a_k),$$
$$(II) \longleftrightarrow gl(a_1)\times\dots\times gl(a_{k-1})\times gl(a_k)'.$$

The last gl factors differ by which extremal root of the fork at the end of the diagram for D_n is in the Levi component. The string for k is

$$(I) \longleftrightarrow (-\frac{a_k-1}{2}+\nu_k,\dots,\frac{a_k-1}{2}+\nu_k),$$
$$(II) \longleftrightarrow (-\frac{a_k-1}{2}+\nu_k,\dots\frac{a_k-3}{2}+\nu_k,-\frac{a_k-1}{2}-\nu_k).$$

We can put the parameter in the form (2.7.10) and (2.7.11), because all strings are even length. In any case (I) and (II) are conjugate by the outer automorphism, and for unitarity it is enough to consider the case of (I).

The assignment of a nilpotent orbit (I) or (II) to a parameter is unambiguous. If a χ has a coordinate equal to 0, it might be written as $h_I/2+\nu_I$ or $h_{II}/2+\nu_{II}$. But then it can also be written as $h'/2+\nu'$ for a larger nilpotent orbit. For example, in type D_2, the two cases are $(2,2)_I$ and $(2,2)_{II}$, and we can write

$$(I) \longleftrightarrow (1/2,-1/2)+(\nu,\nu),$$
$$(II) \longleftrightarrow (1/2,1/2)+(\nu,-\nu).$$

The two forms are not conjugate unless the parameter contains a 0. But then it has to be $(1,0)$ and this corresponds to $(1,3)$, the larger principal nilpotent orbit.

(3) Because we can only change an even number of signs using the Weyl group, we might not be able to change all the signs of a string. We can always do this if the parameter contains a coordinate equal to 0, or if the length of the string is even. If there is an odd length string, and none of the coordinates of χ are 0, changing all of the signs of the string cannot be achieved unless some other coordinate changes sign as well. However if $\chi = \check{h}/2 + \nu$ cannot be made to satisfy (2.7.10) and (2.7.11), then χ', the parameter obtained from χ by applying the outer automorphism, can. Since $L(\chi)$ and $L(\chi')$ are either both unitary of both nonunitary, it is enough to consider just the cases that can be made to satisfy (2.7.10) and (2.7.11). For example, the parameters

$$(-1/3, 2/3, 5/3; -7/4, -3/4, 1/4),$$
$$(-5/3, -2/3, 1/3; -7/4, -3/4, 1/4)$$

in type D_6 are of this kind. Both parameters are in a form satisfying (2.7.10) but only the second one satisfies (2.7.11). The first one cannot be conjugated by $W(D_6)$ to one satisfying (2.7.11).

{sec:2.4}

2.8. Relation between infinitesimal characters and strings. In the previous sections we described for each nilpotent orbit $\check{\mathcal{O}}$ the parameters of the form $\check{h}/2 + \nu$ with $\nu \in \mathfrak{z}(\check{\mathcal{O}})$ semisimple, along with condition (2.3.2). In this section we give algorithms to find the data $(\check{\mathcal{O}}, \nu)$ satisfying (2.3.1) and (2.3.2), and the various Levi components from a $\chi \in \check{\mathfrak{a}}$. The formulation was suggested by S. Sahi. Given a $\chi \in \check{\mathfrak{a}}$, we need to specify,

(a): strings, same as sequences of coordinates with increment 1,
(b): a partition, same as a nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$,
(c): the centralizer of a Lie triple corresponding to $\mathfrak{z}(\check{\mathcal{O}})$,
(d): coordinates of the parameter ν, coming from the decompositon $\chi = \check{h}/2 + \nu$,

Furthermore, we give algorithms for

(e): two Levi components $\check{\mathfrak{m}}_{BC}$ and $\check{\mathfrak{m}}_{KL}$,
(f): another two Levi components $\check{\mathfrak{m}}_e$ and $\check{\mathfrak{m}}_o$,
(g): one dimensional characters χ_e and χ_o of the Levi components $\check{\mathfrak{m}}_e$ and $\check{\mathfrak{m}}_o$.

Parts (f) and (g) are described in detail in section 5.3. These Levi components are used to compute multiplicities of relevant $K-$types in $L(\chi)$.

Algorithms for (a) and (b).

Step 0.
G of type C. Double the number of 0's and add one more.
G of type D. Double the number of 0's. If there are no coordinates equal to 0 and the rank is odd, the parameter is not hermitian. If the rank is even, only an even number of sign changes are allowed in the subsequent steps.

Step 1.
G of type C,D. Extract maximal strings of the form $(0, 1, \dots)$. Each contributes a part in the partition of size *2(length of string)-1* to $\check{\mathcal{O}}$.
G of type B. Extract maximal strings of type $(1/2, 3/2, \dots)$. Each contributes a part of size *2(length)* to $\check{\mathcal{O}}$.

Step 2.
For all types, extract maximal strings from the remaining entries after Step 1, changing signs if necessary. Each string contributes two parts of size *(length of string)* to $\check{\mathcal{O}}$. In type D, if the rank is odd and no coordinate of the original χ is 0, the parameter is not hermitian. If there are no 0's and the rank of type D is even, only an even number of sign changes is allowed. In this case, the last string might be $(\dots, b, -b-1)$. If so, and all strings are of even size, $\check{\mathcal{O}}$ is very even, and is labelled II. If all strings are of the form $(\dots, b, b+1)$, then the very even orbit is labelled I.

Algorithms for (c).

G of type C,D. $\mathfrak{z}(\check{\mathcal{O}}) = so(m_1) \times sp(m_2) \times so(m_3) \times \dots$, where m_i are the number of parts of $\check{\mathcal{O}}$ equal to i.
G of type B. $\mathfrak{z}(\check{\mathcal{O}}) = sp(m_1) \times so(m_2) \dots sp(m_3) \times \dots$ where again m_i is the number of parts of $\check{\mathcal{O}}$ equal to i.

Algorithms for (d).

The parameter ν is a vector of size equal to $\operatorname{rk} G$. For each $\mathfrak{z}_i$, of $\mathfrak{z}(\mathcal{O})$, add $\operatorname{rk} \mathfrak{z}_i$ coordinates each equal to the average of the string cooresponding to the size i part of $\check{\mathcal{O}}$. For each factor $\mathfrak{z}_i$, of $\mathfrak{z}(\check{\mathcal{O}})$, the coordinates are the averages of the corresponding strings. The remaining coordinates of ν are all zero.

Algorithm for (e).

The Levi subgroups are determined by specifying the GL factors. There is at most one other factor $G_0(n_0)$ of the same type as the group.

For $\check{\mathfrak{m}}_{BC}$, each pair of parts (k, k) yields a $GL(k)$. If the corresponding string comes from Step 2, then the character on $GL(k)$ is given by $|\det(*)|^{\text{average of string}}$. Otherwise it is the trivial character. The parts of the remaining partition have multiplicity 1 corresponding to a distinguished orbit in $\check{\mathfrak{g}}_0(n_0)$.

For $\check{\mathfrak{m}}_{KL}$, apply the same procedure as for $\check{\mathfrak{m}}_{BC}$, except for pairs coming from Step 1. If originally there was an odd number of parts, then there is no change. If there was an even number, leave behind one pair. The parts in the remaining partition have multiplicity 1 or 2 corresponding to an even orbit in $\check{\mathfrak{g}}_0(n_0)$.

Algorithms for (f) and (g).

Both $\check{\mathfrak{m}}_e$ and $\check{\mathfrak{m}}_o$ acquire a $GL(k)$ factor for each pair of parts (k,k) in Step 2, with character given by the average of the corresponding string as before.

For the parts coming from Step 1, write them in decreasing order $a_r \geq \dots \geq a_1 > 0$.

For $\check{\mathfrak{m}}_e$*,* there are additional GL factors

G of type B: $(a_1+a_2)/2, (a_3+a_4)/2, \dots, (a_{r-2}+a_{r-1})/2$ if r is odd, $(a_1)/2, (a_2+a_3)/2, \dots, (a_{r-2}+a_{r-1})/2$ when r is even. The characters are given by the averages of the strings $(-(a_{r-1}-1)/2, \dots, (a_{r-2}-1)/2)$ and so on, and $(-(a_1-1)/2, \dots, -1/2), \dots$ when r is even. Recall that the a_i are all even because $\check{\mathcal{O}}$ is a nilpotent orbit in type C.

G of type C: $(a_1+a_2)/2, (a_3+a_4)/2, \dots,$ with characters given by the averages of the strings $(-(a_1-1)/2, \dots, (a_2-1)/2)$, and so on. In this case $\check{\mathcal{O}}$ is type B, so there are an odd number of odd parts.

G of type D: $(a_1+a_2)/2, (a_3+a_4)/2, \dots,$ with characters obtained by the same procedure as in type C. In this case $\check{\mathcal{O}}$ has an even number of odd parts.

For $\check{\mathfrak{m}}_o$*,* there are additional GL factors

G of type B: $(a_2+a_3)/2, (a_4+a_5)/2, \dots, (a_{r-1}+a_r)/2$ leaving a_1 out if r is odd, $(a_1+a_2)/2, (a_3+a_4)/2, \dots, (a_{r-1}+a_r)/2$ if r is even. The characters are given by the averages of the strings $(-(a_r-1)/2, \dots, (a_{r-1}-1)/2) \dots$ and so on.

G of type C: $(a_2+a_3)/2, (a_4+a_5)/2, \dots,$ and characters given by the averages of the strings $(-(a_3-1)/2, \dots, (a_2-1)/2), \dots.$ In this case $\check{\mathcal{O}}$ has an odd number of odd sized parts.

G of type D: $(a_1+a_2)/2, \dots, (a_{r-3}+a_{r-2})/2, ((a_{r-1}-1)/2)$ with characters obtained by the averages of the strings $(-(a_2-1)/2, \dots, (a_1-1)/2), \dots, ((-a_{r-1}-1)/2, \dots, -1)$. In this case $\check{\mathcal{O}}$ has an even number of odd sized parts.

{sec:2.6}

2.9. Let $\chi = \check{h}/2 + \nu$ be associated to the orbit $\check{\mathcal{O}}$. Recall from 2.3

{eq:2.6.1}
$$I_M(\chi) := Ind_M^G[L_M(\chi)], \tag{2.9.1}$$

where $L_M(\chi)$ is the irreducible spherical module of M with parameter χ. Write the nilpotent orbit in (2.3.4) with the $(d_1, \dots, d_l)$ as in sections 2.5-2.7 depending on the Lie algebra type. Then $\check{\mathfrak{m}}_{BC} = gl(a_1)\times\dots\times gl(a_k)\times\check{\mathfrak{g}}_0(n_0)$ is as in (2.3.5). Thus χ determines a spherical irreducible module

{eq:2.6.2}
$$L_{M_{BC}}(\chi) = L_1(\chi_1)\otimes\dots\otimes L_k(\chi_k)\otimes L_0(\chi_0), \tag{2.9.2}$$

with $\chi_i = (-\frac{a_i-1}{2}+\nu_i, \dots, \frac{a_i-1}{2}+\nu_i)$, while $\chi_0 = \check{h}_0/2$ for the nilpotent (d_i).

Let $\check{\mathfrak{m}}_{KL}$ be the Levi component attached to $\chi = \check{h}/2 + \nu$ in sections 2.5-2.7. As for $\check{\mathfrak{m}}_{BC}$ we have a parameter $L_{M_{KL}}(\chi)$. In this case $\check{\mathcal{O}} = ((a_1', a_1'), \dots, (a_r', a_r'); d_1', \dots d_l')$ as described in 2.5-2.7. Then (a W-conjugate of) χ can be written as in (2.5.2)-(2.7.2)), and

$$\begin{aligned} \check{\mathfrak{m}}_{KL} &= gl(a_1') \times \dots \times gl(a_r') \times \check{\mathfrak{g}}_0(n_0'), \\ L_{M_{KL}}(\chi) &= L_1(\chi_1') \otimes \dots \otimes L_r(\chi_r') \otimes L_0(\chi_0'). \end{aligned} \tag{2.9.3}$$ {eq:2.6.3}

{t:2.6}

Theorem. *In the p-adic case*

$$I_{M_{KL}}(\chi) = L(\chi).$$

Proof. This is in [BM1], $\check{\mathfrak{m}}_{KL}$ was defined in such a way that this result holds. □

{c:2.6}

Corollary. *The module $I_{M_{BC}}(\chi)$ equals $L(\chi)$ in the $p-$adic case if all the $\nu_i \neq 0$.*

3. The Main Result {Sect}3

3.1. Recall that $\check{G}$ is the (complex) dual group, and $\check{A} \subset \check{G}$ the maximal torus dual to A. Assuming as we may that the parameter is real, a spherical irreducible representation corresponds to an orbit of a hyperbolic element $\chi \in \check{\mathfrak{a}}$, the Lie algebra of $\check{A}$. In section 2 we attached a nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$ with partition $(\underbrace{a_1, \dots a_1}_{r_1}, \dots, \underbrace{a_k, \dots, a_k}_{r_k})$ to such a parameter. Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple attached to $\check{\mathcal{O}}$. Let $\chi := \check{h}/2 + \nu$ satisfy (2.3.1)-(2.3.2).

{d:2.1}

Definition. *A representation $L(\chi)$ is said to be in the complementary series for $\check{\mathcal{O}}$, if the parameter χ is attached to $\check{\mathcal{O}}$ in the sense of satisfying (2.3.1) and (2.3.2), and is unitary.*

We will describe the complementary series explicitly in coordinates.

The centralizer $Z_{\check{G}}(\check{e}, \check{h}, \check{f})$ has Lie algebra $\mathfrak{z}(\check{\mathcal{O}})$ which is a product of $sp(r_l, \mathbb{C})$ or $so(r_l, \mathbb{C})$, $1 \le l \le k$, according to the rule

Ǧ of type B, D: $sp(r_l)$ for a_l even, $so(r_l)$ for a_l odd,

Ǧ of type C: $sp(r_l)$ for a_l odd, $so(r_l)$ for a_l even.

The parameter ν determines a spherical irreducible module $L_{\check{\mathcal{O}}}(\nu)$ for the split group whose dual is $Z_{\check{G}}(\check{e}, \check{h}, \check{f})^0$. It is attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$.

{thm:3.1}

Theorem. *The complementary series attached to $\check{\mathcal{O}}$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$. For the trivial orbit* (0) *in each of the classical cases, the complementary series are*

G of type B:

$$0 \le \nu_1 \le \dots \le \nu_k < 1/2.$$

G of type C, D:

$$0 \le \nu_1 \le \dots \le \nu_k \le 1/2 < \nu_{k+1} < \dots < \nu_{k+l} < 1$$

so that $\nu_i + \nu_j \le 1$. *There are*

(1) *an even number of* ν_i *such that* $1-\nu_{k+1} < \nu_i \le 1/2$,
(2) *for every* $1 \le j \le l$, *there is an odd number of* ν_i *such that* $1-\nu_{k+j+1} < \nu_i < 1-\nu_{k+j}$.
(3) *In type D of odd rank,* $\nu_1 = 0$ *or else the parameter is not hermitian.*

Remarks.

(1) The complementary series for $\check{\mathcal{O}} = (0)$ consists of representations which are both spherical and generic in the sense that they have Whittaker models.
(2) The condition that $\nu_i + \nu_j \neq 1$ implies that in types C,D there is at most one $\nu_k = 1/2$.
(3) In the case of $\check{\mathcal{O}} \neq (0)$, $\chi = \check{h}/2 + \nu$, and each of the coordinates ν_i for the parameter on $\mathfrak{z}(\check{\mathcal{O}})$ comes from a string, *i.e.* each ν_i comes from $(-\frac{a_i-1}{2}+\nu_i, \dots, \frac{a_i-1}{2}+\nu_i)$. The parameter does not satisfy (2.7.11). For (2.7.11) to hold, it suffices to change ν_{k+j} for types C, D to $1-\nu_{k+j}$. More precisely, for $1/2 < \nu_{k+j} < 1$ the connection with the strings in the form (2.7.10) and (2.7.11) is as follows. Write $(-\frac{a_{k+j}-1}{2}+\nu_{k+j}, \dots, \frac{a_{k+j}-1}{2}+\nu_{k+j})$ as $(-\frac{a_{k+j}-3}{2} + (\nu_{k+j}-1), \dots, \frac{a_{k+j}+1}{2} + (\nu_{k+j}-1))$ and then conjugate each entry to its negative to form $(-\frac{a_{k+j}-3}{2}+\nu'_{k+j}, \dots, \frac{a_{k+j}+1}{2}+\nu'_{k+j})$, with $0 < \nu'_{k+j} = 1-\nu_{k+j} < 1/2$.

An Algorithm. For types C and D we give an algorithm, due to S. Sahi, to decide whether a parameter in types C and D is unitary. This algorithm is for the complememntary series for $\check{\mathcal{O}} = (0)$. For arbitrary $\check{\mathcal{O}}$ it applies to the parameter for $\mathfrak{z}(\check{\mathcal{O}})$ obtained as in remark (3) above.

Order the parameter in dominant form,

{eq:3.1.1}
$$\begin{aligned} &0 \le \nu_1 \le \dots \le \nu_n, \text{ for type C,} \\ &0 \le |\nu_1| \le \dots \le \nu_n, \text{ for type D.} \end{aligned} \tag{3.1.1}$$

The first condition is that $\nu_n < 1$, and in addition that if the type is D_n with n odd, then $\nu_1 = 0$. Next replace each coordinate $1/2 < \nu_i$ by $1-\nu_i$. Reorder the new coordinates in increasing order as in (3.1.1). Let $F(\nu)$ be the set of new positions of the $1-\nu_i$. If any position is ambiguous, the parameter is not unitary, or is attached to a different nilpotent orbit. This corresponds to either a $\nu_i+\nu_j = 1$, or a $1/2 < \nu_i = \nu_j$. Finally, $L(\nu)$ is unitary if and only if $F(\nu)$ consists of odd numbers only.

{sec:3.2}
3.2. We prove the unitarity of the parameters in the theorem for $\check{\mathcal{O}} = (0)$ for types B,C, and D. First we record some facts.

Let $G := GL(2a)$ and

{eq:3.2.1}
$$\chi := (-\frac{a-1}{2} - \nu, \dots, \frac{a-1}{2} - \nu; -\frac{a-1}{2} + \nu, \dots, \frac{a-1}{2} + \nu). \quad (3.2.1)$$

Let $M := GL(a) \times GL(a) \subset GL(2a)$. Then the two strings of χ determine an irreducible spherical (1-dimensional) representation $L_M(\chi)$ on M. Recall $I_M(\chi) := Ind_M^G[L_M(\chi)]$.

{l1:3.2}
Lemma (1)**.** *The representation $I_M(\chi)$ is unitary irreducible for $0 \le \nu < 1/2$. The irreducible spherical module $L(\chi)$ is not unitary for $\nu > \frac{1}{2}$, $2\nu \notin \mathbb{Z}$.*

Proof. This is well known and goes back to [Stein] (see also [T] and [V1]). □

We also recall the following well known result due to Kostant in the real case, Casselman in the $p-$adic case.

{l2:3.2}
Lemma (2)**.** *If none of the $\langle \chi, \alpha \rangle$ for $\alpha \in \Delta(\check{\mathfrak{g}}, \check{\mathfrak{a}})$ is a nonzero integer, then $X(\chi)$ is irreducible. In particular, if $\chi = 0$, then*

$$L(\chi) = X(\chi) = Ind_A^G[\chi],$$

and it is unitary.

Let $\check{\mathfrak{m}} \subset \check{\mathfrak{g}}$ be a Levi component, and $\xi_t \in \mathfrak{z}(\check{\mathfrak{m}})$, where $\mathfrak{z}(\check{\mathfrak{m}})$ is the center of $\check{\mathfrak{m}}$, depending continuously on $t \in [a, b]$.

{l3:3.2}
Lemma (3)**.** *Assume that*

$$I_M(\chi_t) := Ind_M^G[L_M(\chi_0) \otimes \xi_t]$$

is irreducible for $a \le t \le b$, and $L_M(\chi_0) \otimes \xi_t$ is hermitian. Then $I_M(\chi_t)$ (equal to $L(\chi_t)$) is unitary if and only if $L_M(\chi_0)$ is unitary.

This is well known, and amounts to the fact that (normalized) induction preserves unitarity. I don't know the original reference.

When the conditions of lemma 3 are satisfied, we say that $I_M(\chi_t)$ is a *continuous deformation* of $L(\chi_0)$.

We now start the proof of the unitarity.

Type B. In this case there are no roots $\alpha \in \Delta(\check{\mathfrak{g}}, \check{\mathfrak{a}})$ such that $\langle \chi, \alpha \rangle$ is a nonzero integer. Thus

$$L(\chi) = Ind_A^G[\chi]$$

as well. When deforming χ to 0 continuously, the induced module stays irreducible. Since $Ind_A^G[0]$ is unitary, so is $L(\chi)$.

Type C,D. There is no root such that $\langle\chi,\alpha\rangle$ is a nonzero integer, so $L(\chi) = Ind_A^G[\chi]$. If there are no $\nu_{k+i} > 1/2$ the argument for type B carries over word for word. When there are $\nu_{k+i} > 1/2$ we have to be more careful with the deformation. We will do an induction on the rank. Suppose that $\nu_{j-1} = \nu_j$ for some j. Necessarily, $\nu_j < 1/2$. Conjugate χ by the Weyl group so that

$$\chi = (\nu_1, \dots, \nu_i, \dots \widehat{\nu_{j-1}}, \widehat{\nu_j}, \dots, ; \nu_{j-1}; \nu_j) := (\chi_0; \nu_{j-1}; \nu_j). \qquad (3.2.2)$$ {eq:3.2.2}

Let $\check{\mathfrak{m}} := \check{\mathfrak{g}}(n-2) \times gl(2)$, and denote by M the corresponding Levi component. Then by induction in stages,

{eq:3.2.3} $$L(\chi) = Ind_M^G[L_M(\chi)], \qquad (3.2.3)$$

where $L_M(\chi) = L_0(\chi_0) \otimes L_1(\nu_{j-1}, \nu_j)$. By lemma (1) of 3.2, $L_1(\nu_{j-1}, \nu_j)$ is unitary. Thus $L(\chi)$ is unitary if and only if $L_0(\chi_0)$ is unitary. If χ satisfies the assumptions of the theorem, then so does χ_0. By the induction hypothesis, $L_0(\chi_0)$ is unitary, and therefore so is $L(\chi)$. Thus we may assume that

{eq:3.2.4} $$0 \le \nu_1 < \dots < \nu_k \le 1/2 < \nu_{k+1} < \dots < \nu_{k+l}. \qquad (3.2.4)$$

If $\nu_k < 1 - \nu_{k+1}$, then the assumptions imply $1 - \nu_{k+2} < \nu_k$. Consider the parameter

{eq:3.2.5} $$\chi_t := (\dots, \nu_k, \nu_{k+1} - t, \dots). \qquad (3.2.5)$$

Then

{eq:3.2.6} $$L(\chi_t) = Ind_A^G[\chi_t], \qquad \text{for } 0 \le t \le \nu_{k+1} - \nu_k, \qquad (3.2.6)$$

because no $\langle\chi_t, \alpha\rangle$ is a nonzero integer. At $t = \nu_{k+1} - \nu_k$, the parameter is in the case just considered earlier. By induction we are done.

If on the other hand $1 - \nu_{k+1} < \nu_k$, the assumptions on the parameter are such that necessarily $1 - \nu_{k+1} < \nu_{k-1} < \nu_k$. Then repeat the argument with

{eq:3.2.7} $$\chi_t := (\dots, \nu_{k-1}, \nu_k - t, \dots), \qquad 0 \le t \le \nu_k - \nu_{k-1}. \qquad (3.2.7)$$

This completes the proof of the unitarity of the parameters in theorem 3.1 when $\check{\mathcal{O}} = (0)$.

{sec:3.3}

3.3. We prove the unitarity of the parameters in theorem 3.1 in the general case when $\check{\mathcal{O}} \ne (0)$.

The proof is essentially the same as for $\check{\mathcal{O}} = (0)$, but special care is needed to justify the irreducibility of the modules. Recall the notation of the partition of $\check{\mathcal{O}}$ (2.3.3).

The factors of $\mathfrak{z}(\check{\mathcal{O}})$ isomorphic to $sp(r_j)$, contribute $r_j/2$ factors of the form $gl(a_i)$ to $\check{\mathfrak{m}}_{KL}$. The factors of type $so(r_j)$ with r_j odd, contribute a d_i (notation (2.3.4)) to the expression (2.3.4) of the partition of $\check{\mathcal{O}}$, and $\frac{r_j-1}{2}$ $gl(a_i)$. The factors $so(r_j)$ of type D (r_j even) are more complicated. Write the strings coming from this factor as in (2.3.6),

$$(-\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i)$$

with the ν_i satisfying the assumptions of theorem 3.1. If r_j is not divisible by 4, then there must be a $\nu_1 = 0$, (otherwise the corresponding spherical parameter is not hermitian), and $\check{\mathfrak{m}}_{BC} \neq \check{\mathfrak{m}}_{KL}$. Similarly when r_j is divisible by 4 and $\nu_1 = 0$, $\check{\mathfrak{m}}_{BC} \neq \check{\mathfrak{m}}_{KL}$. In all situations, we consider

$$I_{M_{KL}}(\chi) \qquad (3.3.1)$$ {eq:3.3.1}

as for $\check{\mathcal{O}} = (0)$. We aim to show that this module stays irreducible under the deformations used for $\check{\mathcal{O}} = (0)$, separately for the ν_i for the same partition size or equivalently simple factor of $\mathfrak{z}(\check{\mathcal{O}})$. It is enough to prove that under these deformations the strings stay strongly nested at all the values of the parameters. Then proposition 3.3 applies.

Using the conventions of section 2, the strings are of the form

$$(-A-1+\nu_1, \dots, A-1+\nu_1) \qquad (3.3.2)$$ {eq:3.3.1a}

$$(-B+\nu_2, \dots, B-1+\nu_2) \qquad (3.3.3)$$ {eq:3.3.1b}

$$(-C+\nu_3, \dots, C+\nu_3). \qquad (3.3.4)$$ {eq:3.3.1c}

The ν_i satisfy $0 \le \nu_i \le 1/2$. When $\nu_1 = 1/2$, the string is $(-A-1/2, \dots, A-1/2)$ so it conforms to (2.5.11), (2.6.11), (2.7.11). Similarly when $\nu_2 = 1/2$, the string is $(-B+1/2, \dots, B-1/2)$. But when $\nu_3 = 1/2$, the string is $(-C+1/2, \dots, C+1/2)$ and it must be replaced by $(-C-1/2, \dots, C-1/2)$ to conform to (2.5.11), (2.6.11), (2.7.11). The string in (3.3.2) gives a $\nu_{1,\mathfrak{z}(\check{\mathcal{O}})} = 1-\nu_1$ (as explained in remark (3) of section 3.1). The other ones ν_2 and ν_3 give $\nu_{j,\mathfrak{z}(\check{\mathcal{O}})} = \nu_2$ or ν_3 respectively. Suppose first that there is only one size of string. This means that the corresponding entries $\nu_{j,\mathfrak{z}(\check{\mathcal{O}})}$ belong to the same simple factor of $\mathfrak{z}(\check{\mathcal{O}})$. Then the strings are either all of the form (3.3.2) and (3.3.4) with $A = C$ or all of the form (3.3.3). Consider the first case. If there is a string (3.3.2) then $\nu_{1,\mathfrak{z}(\check{\mathcal{O}})} = 1-\nu_1 \ge 1/2$, and $\nu_{1,\mathfrak{z}(\check{\mathcal{O}})}$ is deformed downward. By the assumptions $\nu_{1,\mathfrak{z}(\check{\mathcal{O}})}$ does not equal any $\nu_{j,\mathfrak{z}(\check{\mathcal{O}})}$ nor does $\nu_{1,\mathfrak{z}(\check{\mathcal{O}})} + \nu_{j,\mathfrak{z}(\check{\mathcal{O}})} = 1$ for any j. The module stays irreducible. When $\nu_{1,\mathfrak{z}(\check{\mathcal{O}})}$ crosses $1/2$, the string becomes $(-A+\nu_1', \dots, A+\nu_1')$, and ν_1' is deformed downward from $1/2$ to either some ν_3 or to 0. Again $\nu_1' + \nu_{j,\mathfrak{z}(\check{\mathcal{O}})} \neq 1$ for any j, so no reducibility occurs. The irreduciblity in the case when ν_1' reaches 0 is dealt with by section 10.

Remains to check that in these deformations no reducibility occurs because the string interacts with one of a different length, in other words, when a $\nu_{i,\mathfrak{z}(\check{\mathcal{O}})}$ for one size string or equivalently factor of $\mathfrak{z}(\check{\mathcal{O}})$ becomes equal to a $\nu_{j,\mathfrak{z}(\check{\mathcal{O}})}$ from a distinct factors of $\mathfrak{z}(\check{\mathcal{O}})$. We explain the case when the deformation involves a string of type (3.3.2), the others are similar and easier. Consider the deformation of ν_1 from 0 to 1/2. If the module is to become reducible ν_1 must reach a ν_3 so that there is a string of the form (3.3.4) satisfying

$$-A-1 < -C, \qquad A-1 < C. \qquad (3.3.5)$$ {eq:3.3.1d}

This is the condition that the strings are not nested for some value of the parameter. This implies that $A-1<C<A+1$ must hold. Thus $A=C$, and the two strings correspond to the same size partition or simple factor in $\mathfrak{z}(\check{\mathcal{O}})$.

In the p-adic case, the irreducibility of (3.3.1) in the case of strongly nested strings follows from the results of Kazhdan-Lusztig. In the case of real groups, the same irreducibility results hold, but are harder to prove. Given χ, consider the root system

{eq:3.3.5}
$$\check{\Delta}_\chi := \{\check{\alpha} \in \check{\Delta} : \langle \chi, \check{\alpha}\rangle \in \mathbb{Z}\}. \tag{3.3.6}$$

Let G_χ be the connected split real group whose dual root system is $\check{\Delta}_\chi$. Then χ determines an irreducible spherical representation $L_{G_\chi}(\chi)$. The Kazhdan-Lusztig conjectures for nonintegral infinitesimal character provide a way to prove any statement about the character of $L(\chi)$ by proving it for $L_{G_\chi}(\chi)$. This is beyond the scope of this paper (or my competence), I refer to [ABV], chapters 16 and 17 for an explanation.

Since G_χ is not simple, it is sufficient to prove the needed irreducibility result for each simple factor. This root system is a product of classical systems as follows. For each $0 \le \tau \le 1/2$ let A_τ be the set of coordinates of χ congruent to τ modulo $\mathbb{Z}$. Each A_τ contributes to $\check{\Delta}_\chi$ as follows.

G of type B: Every $0<\tau<1/2$ contributes a type A of size equal to the number of coordinates in A_τ. Every $\tau = 0, 1/2$ contributes a type C of rank equal to the number of coordinates in A_τ.

G of type C: Every $0<\tau<1/2$ contributes a type A as for type B. Every $\tau=0$ contributes a type B, while $\tau=1/2$ contributes type D.

G of type D: Every $0<\tau<1/2$ contributes a type A. Every $\tau = 0, 1/2$ contributes a type D.

The irreducibility results for $I_{M_{KL}}(\chi_t)$ needed to carry out the proof are contained in the following proposition.

{p:3.3}

Proposition. *Let*

$$\chi := (\dots; -\frac{a_i-1}{2}+\nu_i, \dots, \frac{a_i-1}{2}+\nu_i; \dots)$$

be given in terms of strings, and let $\check{\mathfrak{m}} = gl(a_1)\times\dots\times gl(a_k)$ *be the corresponding Levi component. Assume that* χ *is integral (i.e.* $\langle\chi,\check{\alpha}\rangle\in\mathbb{Z}$*). In addition assume that the coordinates of* χ *are*

- *in* $\mathbb{Z}$, *in type B,*
- *in* $1/2+\mathbb{Z}$ *for type D.*

If the strings are strongly nested, then

$$I_M(\chi) = Ind_M^G[L_M(\chi)].$$

The proof of the proposition will be given in section 10.

Remark. For the ν_j attached to factors of type D in $\mathfrak{z}(\check{\mathcal{O}})$, it is important in the argument that we do not deform to (0). The next example illustrates why.

Assume $\check{\mathcal{O}} = (2,2,2,2) \subset sp(4)$. The parameters of the form $\check{h}/2+\nu$ are

$$(-1/2+\nu_1, 1/2+\nu_1; -1/2+\nu_2, 1/2+\nu_2), \qquad (3.3.7)$$ {eq:3.3.2}

and, because parameters are up to $W-$conjugacy, we may restrict attention to the region $0 \le \nu_1 \le \nu_2$. In this case $\mathfrak{z}(\check{\mathcal{O}}) = so(4)$, and the unitarity region is $0 \le \pm\nu_1+\nu_2 < 1$. Furthermore $\check{\mathfrak{m}}_{BC} = gl(2)\times gl(2)$, but $\check{\mathfrak{m}}_{KL} = \check{\mathfrak{m}}_{BC}$ only if $0 < \nu_1$. When $\nu_1 = 0$, $\check{\mathfrak{m}}_{KL} = sp(2)\times gl(2)$, the nilpotent orbit is rewritten $(2,2;(2,2))$, and $\check{h}_0/2 = (1/2,1/2)$. For $\nu_1 = 0$, the induced representations

$$I_{M_{KL}}(\chi_{\nu_2}) := Ind_{Sp(2)\times GL(2)}^{Sp(4)}[L_0(1/2,1/2)\otimes L_1(-1/2+\nu_2, 1/2+\nu_2)] \quad (3.3.8)$$ {eq:3.3.3}

are induced irreducible in the range $0 \le \nu_2 < 1$. For $0 < \nu_1$ the representation

$$Ind_{GL(4)}^{Sp(4)}[L((-1/2+\nu_1+t, 1/2+\nu_1+t); (-1/2-\nu_1-t, 1/2-\nu_1-t))] \quad (3.3.9)$$ {eq:3.3.4}

is induced irreducible for $0 \le t \le 1/2-\nu_1$.

The main point of the example is that $Ind_{GL(2)}^{Sp(2)}[L(-1/2+t, 1/2+t)]$ is **reducible** at $t = 0$. So we cannot conclude that $L(\chi)$ is unitary for a (ν_1,ν_2) with $0 < \nu_1$ from the unitarity of $L(\chi)$ for a parameter with $\nu_1 = 0$. Instead we conclude that the representation is unitary in the region $0 \le \pm\nu_1+\nu_2 < 1$ because it is a deformation of the irreducible module for $\nu_1 = \nu_2$ which is unitarily induced irreducible from a Stein complementary series on $GL(4)$. □

4. Relevant $K-$types

{sec:4}

4.1. In the real case we will call a $K-$type (μ, V) *quasi-spherical* if it occurs in the spherical principal series. In section 4, we will use the notation $M = K\cap B$. By Frobenius reciprocity (μ, V) is quasi-spherical if and only if $V^{K\cap B} \neq 0$. Because the Weyl group $W(G,A)$ may be realized as $N_K(A)/Z_K(A)$, this Weyl group acts naturally on this space.

The representations of $W(A_{n-1}) = S_n$ are parametrized by partitions $(a) := (a_1,\dots,a_k)$, $a_i \le a_{i+1}$, of n, and we write $\sigma((a))$ for the corresponding representation. The representations of $W(B_n) \cong W(C_n)$ are parametrized as in [L1] by pairs of partitions, and we write as

$$\sigma((a_1,\dots,a_r),(b_1,\dots,b_s)),$$
$$a_i \le a_{i+1}, \quad b_j \le b_{j+1}, \quad \sum a_i + \sum b_j = n. \qquad (4.1.1)$$ {eq:4.0.4}

Precisely the representation parametrized by (4.1.1) is as follows. Let $k = \sum a_i$, $l = \sum b_j$. Recall that $W \cong S_n \ltimes \mathbb{Z}_2^n$. Let χ be the character of $\mathbb{Z}_2^n$ which is trivial on the first k $\mathbb{Z}_2$'s, sign on the last l. Its centralizer in S_n is $S_k\times S_l$. Let $\sigma((a))$ and $\sigma((b))$ be the representations of S_k, S_l corresponding to the partitions (a) and (b). Then let $\sigma((a),(b),\chi)$ be the unique representation of

$(S_k\times S_l)\ltimes Z_2^n$ which is a multiple of χ when restricted to Z_2^n, and $\sigma(a)\otimes\sigma(b)$ when restricted to $S_k\times S_l$. The representation in (4.1.1), is

{eq:4.0.5}
$$\sigma((a),(b)) = Ind^{W}_{(S_k\times S_l)\ltimes \mathbb{Z}_2^n}\,[\sigma(a,b,\chi)]. \tag{4.1.2}$$

If $(a)\neq(b)$, the representations $\sigma((a),b)$ and $\sigma((b),(a))$ restrict to the same irreducible representation of $W(D_n)$, which we denote again by the same symbol. When $a=b$, the restriction is a sum of two inequivalent representations which we denote $\sigma((a),(a))_{I,\ II}$. Let $W_{(a),I}:=S_{a_1}\times\dots\times S_{a_r}$ and $W_{(a),II}:=S_{a_1}\times\dots\times S'_{a_r}$, be the Weyl groups corresponding to the Levi components considered in Remark (2) in section 2.7. Then $\sigma((a),(a))_I$ is characterized by the fact that its restriction to $W_{(a),I}$ contains the trivial representation. Similarly $\sigma((a),(a))_{II}$ is the one that contains the trivial representation of $W_{(a),II}$.

{sec:4.2}

4.2. Symplectic Groups. The group is $Sp(n)$ and the maximal compact subgroup is $U(n)$. The highest weight of a K-type will be written as $\mu(a_1,\dots,a_n)$ with $a_i\ge a_{i+1}$ and $a_i\in\mathbb{Z}$, or

{eq:4.2.1}
$$\mu(a_1^{r_1},\dots,a_k^{r_k}) := (\underbrace{a_1,\dots,a_1}_{r_1},\dots,\underbrace{a_k,\dots,a_k}_{r_k}). \tag{4.2.1}$$

when we want to emphasize the repetitions. We will repeatedly use the following restriction formula

{l:4.2}

Lemma. *The restriction of $\mu(a_1,\dots,a_n)$ to $U(n-1)\times U(1)$ is*

$$\sum \mu(b_1,\dots,b_{n-1})\otimes\mu(b_n),$$

where the sum ranges over all possible $a_1\ge b_1\ge a_2\ge\dots\ge b_{n-1}\ge a_n$, and $b_n=\sum_{1\le i\le n}a_i-\sum_{1\le j\le n-1}b_j$.

{def:4.2}

Definition. *The representations $\mu_e(n-r,r):=\mu(2^r,0^{n-r})$ and $\mu_o(k,n-k):=\mu(1^k,0^{n-2k},-1^k)$ are called **relevant**.*

{p:4.2}

Proposition. *The relevant $K-$types are quasispherical. The representation of $W(C_n)$ on V^M is*

$$\mu_e(n-r,r)\longleftrightarrow\sigma[(n-r),(r)],$$
$$\mu_o(k,n-k)\longleftrightarrow\sigma[(k,n-k),(0)],$$

The K-types $\mu(0^{n-r},(-2)^r)$, dual to $\mu_e(n-r,r)$ are also quasispherical, and could be used in the same way as $\mu_e(n-r,r)$.

Proof. We do an induction on n. When $n=1$, the only relevant representations of $U(1)$ are $\mu_e(1,0)=\mu(0)$ and $\mu_e(0,1)=(2)$, which correspond to the trivial and the sign representations of $W(C_1)=\mathbb{Z}/2\mathbb{Z}$, respectively. Consider the case $n=2$. There are four relevant representations of $U(2)$ with highest weights $(2,0)$, $(1,-1)$, $(2,2)$ and $(0,0)$. The first representation is the symmetric square of the standard representation, the second one is the adjoint representation and the fourth one is the trivial representation. The normalizer of A in K can be identified with the diagonal subgroup $(\pm1, \pm1)$ inside $U(1) \times U(1) \subset U(2)$. The Weyl group is generated by the elements

$$\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.2.2}$$ {eq:4.2.3}

The restriction to $U(1) \times U(1)$ of the four representations of $U(2)$ is

$$\begin{aligned} (2,0) &\longrightarrow (2)\otimes(0) + (1)\otimes(1) + (0)\otimes(2), \\ (1,-1) &\longrightarrow (1)\otimes(-1) + (0)\otimes(0) + (-1)\otimes(1), \\ (2,2) &\longrightarrow (2)\otimes(2), \\ (0,0) &\longrightarrow (0)\otimes(0). \end{aligned} \tag{4.2.3}$$ {eq:4.2.4}

The space V^M is the sum of all the weight spaces $(p)\otimes(q)$ with both p and q even. For the last one, the representation of W on V^M is $\sigma[(2),(0)]$. The third one is 1-dimensional so V^M is 1-dimensional; the Weyl group representation is $\sigma((0),(2))$. The second one has V^M 1-dimensional and the Weyl group representation is $\sigma((11),(0))$. For the first one, V^M is 2-dimensional and the Weyl group representation is $\sigma((1),(1))$. These facts can be read off from explicit realizations of the representations.

Assume that the claim is proved for $n-1$. Choose a parabolic subgroup so that its Levi component is $M' = Sp(n-1) \times GL(1)$ and M is contained in it. Let $H = U(n-1) \times U(1)$ be such that $M \subset M' \cap K \subset H$.

Suppose that μ is relevant. The cases when $k = 0$ or $r = 0$ are 1-dimensional and are straightforward. So we only consider $k,\ r > 0$. The K-type $\mu(2^r, 0^{n-r})$ restricts to the sum of

$$\mu(2^r, 0^{n-r-1}) \otimes \mu(0) \tag{4.2.4}$$ {eq:4.2.5}

$$\mu(2^{r-1}, 1, 0^{n-r-1}) \otimes \mu(1) \tag{4.2.5}$$ {eq:4.2.6}

$$\mu(2^{r-1}, 0^{n-r}) \otimes \mu(2). \tag{4.2.6}$$ {eq:4.2.7}

Of the representations appearing, only $\mu(2^r, 0^{n-r-1}) \otimes \mu(0)$ and $\mu(2^{r-1}, 1, 0^{n-r-1}) \otimes \mu(2)$ are quasispherical. So the restriction of V^M to $W(C_{n-1}) \times W(C_1)$ is the sum of

$$\sigma[(n-r-1),(r)] \otimes \sigma[(1),(0)] \tag{4.2.7}$$ {eq:4.2.8}

$$\sigma[(n-r),(r-1)] \otimes \sigma[(0),(1)] \tag{4.2.8}$$ {eq:4.2.9}

The only irreducible representations of $W(C_n)$ containing (4.2.7) in their restrictions to $W(C_{n-1})$ are

$$\sigma[(1,n-r-1),(r)] \tag{4.2.9}$$ {eq:4.2.10}

$$\sigma[(n-r),(r)]. \tag{4.2.10}$$ {eq:4.2.11}

But the restriction of $\sigma[(1,n-r-1),(r)]$ to $W(C_{n-1}) \times W(C_1)$ contains $\sigma[(1,n-r-1),(r-1)] \otimes \sigma[(0),(1)]$, and this does not appear in (4.2.7)-(4.2.8). Thus the representation of $W(C_n)$ on V^M for (4.2.9) must be (4.2.5), and the claim is proved in this case.

Consider the case $\mu(1^k, 0^l, -1^k)$ for $k > 0,\ 2k + l = n$. The restriction of this K-type to $U(n-1) \times U(1)$ is the sum of

$$\mu(1^k, 0^l, -1^{k-1}) \otimes \mu(-1) \qquad (4.2.11)$$ {eq:4.2.12}

$$\mu(1^{k-1}, 0^l, -1^k) \otimes \mu(1) \qquad (4.2.12)$$ {eq:4.2.13}

$$\mu(1^{k-1}, 0^{l+1}, -1^{k-1}) \otimes \mu(0) \qquad (4.2.13)$$ {eq:4.2.14}

$$\mu(1^k, 0^{l-1}, -1^k) \otimes \mu(0) \qquad (4.2.14)$$ {eq:4.2.15}

Of the representations appearing, only (4.2.13) and (4.2.14) are quasispherical. So the restriction of V^M to $W(C_{n-1}) \times W(C_1)$ is the sum of

{eq:4.2.16} $$\sigma[(k-1, k+l), (0)] \otimes \sigma[(1), (0)], \qquad (4.2.15)$$

{eq:4.2.17} $$\sigma[(k, k+l-1), (0)] \otimes \sigma[(1), (0)]. \qquad (4.2.16)$$

The representation (4.2.16) can only occur in the restriction to $W(C_{n-1}) \times W(C_1)$ of $\sigma[(1, k, k+l-1), (0)]$ or $\sigma[(k, k+l), (0)]$. If $k > 1$, the first one contains $\sigma[(1, k-1, k+l-1), (0)]$ in its restriction, which is not in the sum of (4.2.15) and (4.2.16). If $k = 1$ then (4.2.15) can only occur in the restriction of $\sigma[(0, l+2), (0)]$, or $\sigma[(1, l+1), (0)]$. But V^M cannot consist of $\sigma[(0, l+2), (0)]$ alone, because (4.2.15) does not occur in its restriction. If it consists of both $\sigma[(0, l+2), (0)]$ and $\sigma[(1, l), (0)]$, then the restriction is too large. The claim is proved in this case. $\square$

{sec:4.3}

4.3. Orthogonal groups. Because we are dealing with the spherical case, we can use the groups $O(a, b)$, $SO(a, b)$, or the connected component of the identity, $SO_e(a, b)$. The corresponding K's are $O(a) \times O(b)$, $S(O(a) \times O(b))$, and $SO(a) \times SO(b)$, respectively. We will use $O(a, b)$ for the calculation of relevant K-types. For $SO(a)$, an irreducible representation will be identified by its highest weight in coordinates, $\mu(x_1, \dots, x_{[a/2]})$, or if there are repetitions, $\mu(x_1^{n_1}, \dots, x_k^{n_k})$. For $O(a)$ we use the parametrization of Weyl, [Weyl]. Embed $O(a) \subset U(a)$ in the standard way. Then we denote by $\mu(x_1, \dots, x_k, 0^{[a/2]-k}; \epsilon)$ the irreducible $O(a)-$component generated by the highest weight of the representation

$$\mu(x_1, \dots, x_k, 1^{(1-\epsilon)(a/2-k)}, 0^{a-k-(1-\epsilon)(a/2-k)})$$

of $U(a)$. In these formulas, $\epsilon = \pm$, is often written as $+$ for 1, and $-$ for -1.

{sec:4.4}

4.4. We describe the **relevant** K-types for the orthogonal groups $O(a, a)$.

{def:4.4}

Definition (even orthogonal groups)**.** *The **relevant** $K-$types for $O(a, a)$*

{eq:4.4.1} $$\mu_e([a/2] - r, r) := \mu(0^{[a/2]}; +) \otimes \mu(2^r, 0^l; +) \qquad (4.4.1)$$

{eq:4.4.2} $$\mu_o(r, [a/2] - r) := \mu(1^r, 0^l; +) \otimes \mu(1^r, 0^l; +). \qquad (4.4.2)$$

where $r + l = [a/2]$.

{p:4.4}

Proposition. *The relevant $K-$types are quasispherical. The representation of $W(D_a)$ of $O(a,a)$ on V^M is*

{eq:4.4.3}
$$\sigma[(r,a-r),(0)] \longleftrightarrow \mu(0^{[a/2]};+)\otimes\mu(2^r,0^l;+), \tag{4.4.3}$$

{eq:4.4.4}
$$\sigma[(a-k),(k)], \longleftrightarrow \mu(1^k,0^l;+)\otimes\mu(1^k,0^l;+), \tag{4.4.4}$$

When $l=0$, and a is even,

$$\sigma[(a/2,a/2),(0)] \longleftrightarrow \mu(0^{a/2})\otimes\mu(2^{a/2-1},\pm 2), \tag{4.4.5}$$ {eq:4.4.5}

$$\sigma[(a/2),(a/2)]_{I,II} \longleftrightarrow \mu(1^{a/2-1},\pm 1)\otimes\mu(1^{a/2-1},\pm 1). \tag{4.4.6}$$

We will prove this together with the corresponding proposition for $O(a+1,a)$ in section 4.6.

{sec:4.5}

4.5. We describe the relevant K-types for $O(a+1,a)$

{def:4.5}

Definition (odd orthogonal groups)**.** *The **relevant** K-types for $O(a+1,a)$ are*

$$\mu_e(a-r,r) := \mu(0^{[(a+1)/2]};+)\otimes\mu(2^r,0^l;+) \tag{4.5.1}$$ {eq:4.5.1}

$$\mu_o(a-k,k) := \mu(1^k,0^l;+)\otimes\mu(1^k,0^s;+) \tag{4.5.2}$$ {eq:4.5.2}

$$\mu_o(k,a-k) := \mu(1^{k+1},0^l;+)\otimes\mu(1^k,0^s;+) \tag{4.5.3}$$ {eq:4.5.3}

where $r+l=[a/2]$ in (4.5.1), $k+l=[(a+1)/2]$, $k+s=[a/2]$ in (4.5.2), and $k+1+l=[(a+1)/2]$, $k+s=[a/2]$ in (4.5.3).

{p:4.5}

Proposition. *The representations of $W(B_a)$ on V^M for the relevant K-types are*

$$\sigma[(r,a-r),(0)] \longleftrightarrow \mu(0^{[(a+1)/2]};+)\otimes\mu(2^r,0^l;+) \tag{4.5.4}$$ {eq:4.5.4}

$$\sigma[(a-k),(k)] \longleftrightarrow \mu(1^k,0^{[(a+1)/2]-k};+)\otimes\mu(1^k,0^{[a/2]-l};+), \tag{4.5.5}$$ {eq:4.5.5}

$$\sigma[(k),(a-k)] \longleftrightarrow \mu(1^{k+1},0^{[(a+1)/2]-k-1};+)\otimes\mu(1^k,0^{[a/2]-k};+). \tag{4.5.6}$$ {eq:4.5.6}

When a is even,

$$\sigma[(a/2),(a/2)] \longleftrightarrow \mu(1^{a/2})\otimes\mu(1^{a/2-1},\pm 1). \tag{4.5.7}$$ {eq:4.5.7}

When a is odd,

$$\sigma[(\frac{a-1}{2}),(\frac{a-1}{2})] \longleftrightarrow \mu(1^{(a-1)/2},\pm 1)\otimes\mu(1^{(a-1)/2}). \tag{4.5.8}$$ {eq:4.5.8}

The proof will be in section 4.6.

4.6. Proof of propositions 4.4 and 4.5. We use the standard realization of the orthogonal groups $O(a+1,a)$ and $O(a,a)$. Let {sec:4.6}

$$\widetilde{M} := \{(\eta_0, \eta_1, \dots, \eta_a, \epsilon_1, \dots, \epsilon_a) \ : \ \eta_i,\ \epsilon_j = \pm 1, \prod \eta_i = \prod \epsilon_j = 1\}, \quad (4.6.1)$$ {eq:4.6.1}

viewed as the subgroup of $O(a+1) \times O(a)$ with the $\eta_i,\ \epsilon_j$ on the diagonal. With the appropriate choice of $\mathfrak{a} \cong \mathbb{R}^a$, $\widetilde{M} \subset N_K(\mathfrak{a})$, and the action is

$$(\eta_i, \epsilon_j) \cdot (\dots, x_k, \dots) = (\dots, \eta_k \epsilon_k x_k, \dots). \quad (4.6.2)$$ {eq:4.6.2}

Then $M := K \cap B$ is the subgroup of $\widetilde{M}$ determined by the relations $\eta_j = \epsilon_j,\ j = 1, \dots, a$. Similarly for $O(a) \times O(a)$ but there is no η_0.

We do the case $O(a+1,a)$, $O(a,a)$ is similar. The representations $\mu_o(a-k,k)$ and $\mu_o(k, a-k)$ can be realized as $\bigwedge^k \mathbb{C}^{a+1} \otimes \bigwedge^k \mathbb{C}^a$, respectively $\bigwedge^{k+1} \mathbb{C}^{a+1} \otimes \bigwedge^k(\mathbb{C}^a)$. Let e_i be a basis of $\mathbb{C}^{a+1}$ and f_j a basis of $\mathbb{C}^a$. The space V^M is the span of the vectors $e_{i_1} \wedge \dots \wedge e_{i_k} \otimes f_{i_1} \wedge \dots \wedge f_{i_k}$, and $e_0 \wedge e_{i_1} \wedge \dots \wedge e_{i_k} \otimes f_{i_1} \wedge \dots \wedge f_{i_k}$. The elements of W corresponding to short root reflections all have representatives of the form $\eta_0 = -1, \eta_j = -1$, the rest zero. The action of $S_a \subset W$ on the space V^M is by permuting the e_i, f_j diagonally. Claims (4.4.4-4.4.5) and (4.5.5-4.5.6) follow from these considerations, we omit further details.

For cases (4.4.3) and (4.5.4) we do an induction on r. We do the case $O(a,a)$ only. The claim is clear for $r = 0$. Since the first factor of $\mu_e([a/2]-r, r)$ is the trivial representation, we only concern ourselves with the second factor. Consider $\bigwedge^r \mathbb{C}^a \otimes \bigwedge^r \mathbb{C}^a$. The space of $M-$fixed vectors has dimension $\binom{a}{r}$, and a basis is

{eq:4.6.3} $$e_{i_1} \wedge \dots \wedge e_{i_r} \otimes e_{i_1} \wedge \dots \wedge e_{i_r} \quad (4.6.3)$$

As a module of S_a, this is

{eq:4.6.4} $$Ind_{S_r \times S_{a-r}}^{S_a}[triv \otimes triv] = \sum_{1 \le j \le r} (j, a-j) \quad (4.6.4)$$

On the other hand, the tensor product $\bigwedge^r \mathbb{C}^r \otimes \bigwedge^r \mathbb{C}^a$ consists of representations with highest weight $\mu(2^\alpha, 1^\beta, 0^\gamma)$. From the explicit description of $\bigwedge^k \mathbb{C}^a$, and the action of M, we can infer that V^M for $\beta \neq 0$ is (0). This is because the representation occurs in $\bigwedge^{\alpha+\beta} \mathbb{C}^a \otimes \bigwedge^\alpha \mathbb{C}^a$, which has no $M-$fixed vectors. But $\mu(2^j, 0^l)$ for $j \le r$ occurs (for example by the P-R-V conjecture). By the induction hypothesis, $(j, a-j)$ occurs in $\mu(2^j, 0^l)$, for $j < r$, and so only $(r, a-r)$ is unaccounted for. Thus V^M for $\mu_e([a/2]-r, r)$ be $(r, a-r)$. The claim now follows from the fact that the action of the short root reflections is trivial, and the description of the irreducible representations of $W(B_a)$.

{sec:4.7}

4.7. General linear groups. The maximal compact subgroup of $GL(a,\mathbb{R})$ is $O(a)$, the Weyl group is $W(A_{a-1}) = S_a$ and $M \cong \underbrace{O(1)\times\cdots\times O(1)}_{a}$. We list the case of the connected component $GL(a,\mathbb{R})^+$ (matrices with positive determinant) instead, because its maximal compact group is $K = SO(a)$ which is connected, and irreducible representations are parametrized by their highest weights.

{def:4.7}

Definition. *The* ***relevant*** *$K-$types are the ones with highest weights*

$$\mu(2^k, 0^l).$$

The corresponding Weyl group representations on V^M are $\sigma[(k, a-k)]$.

We omit the details, the proof is essentially the discussion about the representation of S_a on $\bigwedge\mathbb{C}^a \otimes \bigwedge\mathbb{C}^a$ for the orthogonal groups.

{sec:5.8}

4.8. Relevant $W-$types.

Definition. *Let W be the Weyl group of type B,C,D. The following $W-$types will be called* ***relevant****.* {d:5.8}

$$\sigma_e(n-r, r) := \sigma[(n-r),(r)], \qquad \sigma_o(k, n-k) := \sigma[(k, n-k),(0)] \quad (4.8.1)$$ {eq:4.8.1}

In type D for n even, and $r = n/2$ there are two $W-$ types, $\sigma_e[(n/2),(n/2)]_{I,II} := \sigma[(n/2),(n/2)]_{I,II}$. If the root system is not simple, the relevant $W-$types are tensor products of relevant $W-$types on each factor.

5. Intertwining Operators

{sec:5} {sec:5.1}

5.1. Recall that $X(\nu)$ denotes the spherical principal series. Let $w \in W$. Then there is an intertwining operator

$$I(w,\nu) : X(\nu) \longrightarrow X(w\nu). \qquad (5.1.1)$$ {eq:5.1.1}

If (μ, V) is a $K-$type, then $I(w,\nu)$ induces a map

$$I_V(w,\nu) : \mathrm{Hom}_K[V, X(\nu)] \longrightarrow \mathrm{Hom}_K[V, X(w\nu)]. \qquad (5.1.2)$$ {eq:5.1.2}

By Frobenius reciprocity, we get a map

$$R_V(w,\nu) : (V^*)^{K\cap B} \longrightarrow (V^*)^{K\cap B}. \qquad (5.1.3)$$ {eq:5.1.3}

In case (μ, V) is trivial the spaces are 1-dimensional and $R_V(w,\nu)$ is a scalar. We normalize $I(w,\nu)$ so that this scalar is 1. The $R_V(w,\nu)$ are meromorphic functions in ν, and the $I(w,\nu)$ have the following additional properties.

(1) If $w = w_1 \cdot w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$, then $I(w,\nu) = I(w_1, w_2\nu) \circ I(w_2,\nu)$. In particular if $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced decomposition, then $I(w,\nu)$ factors into a product of intertwining operators I_j, one for each s_{α_j}. These operators are

{eq:5.1.4} $$I_j \ : \ X(s_{\alpha_{j+1}} \dots s_{\alpha_k} \cdot \nu) \longrightarrow X(s_{\alpha_j} \dots s_{\alpha_k} \cdot \nu) \tag{5.1.4}$$

(2) Let $P = MN$ be a standard parabolic subgroup (so $A \subset M$) and $w \in W(M,A)$. The intertwining operator

$$I(w,\nu) \ : \ X(\nu) = Ind_P^G[X_M(\nu)] \longrightarrow X(w\nu) = Ind_P^G[X_M(w\nu)]$$

is of the form $I(w,\nu) = Ind_M^G[I_M(w,\nu)]$.

(3) If $Re\langle \nu, \alpha \rangle \geq 0$ for all positive roots α, then $R_V(w_0,\nu)$ has no poles, and the image of $I(w_0,\nu)$ ($w_0 \in W$ is the long element) is $L(\nu)$.

(4) If $-\overline{\nu}$ is in the same Weyl group orbit as ν, let w be the shortest element so that $w\nu = -\overline{\nu}$. Then $L(\nu)$ is hermitian with inner product

$$\langle v_1, v_2 \rangle := \langle v_1, I(w,\nu) v_2 \rangle.$$

Let α be a simple root and $P_\alpha = M_\alpha N$ be the standard parabolic subgroup so that the Lie algebra of M_α is isomorphic to the $sl(2,\mathbb{R})$ generated by the root vectors $E_{\pm\alpha}$. We assume that $\theta E_\alpha = -E_{-\alpha}$, where θ is the Cartan involution corresponding to K. Let $D_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ and $s_\alpha = e^{\sqrt{-1}\pi D_\alpha/2}$. Here by s_α, we actually mean the representative in $N_K(A)$ of the Weyl group reflection. Then $s_\alpha^2 = m_\alpha$ is in $K \cap B \cap M_\alpha$. Since the square of any element in $K \cap B$ is in the center, and $K \cap B$ normalizes the the root vectors, $\mathrm{Ad}\, m(D_\alpha) = \pm D_\alpha$. Grade $V^* = \oplus V_i^*$ according to the absolute values of the eigenvalues of D_α (which are integers). Then $K \cap B$ preserves this grading and

$$(V^*)^{K\cap B} = \bigoplus_{i \ even} (V_i^*)^{K\cap B}.$$

The map $\psi_\alpha : sl(2,\mathbb{R}) \longrightarrow \mathfrak{g}$ determined by

$$\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_\alpha, \qquad \psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{-\alpha}$$

determines a map

{eq:5.1.5} $$\Psi_\alpha \ : \ SL(2,\mathbb{R}) \longrightarrow G \tag{5.1.5}$$

with image G_α, a connected group with Lie algebra isomorphic to $sl(2,\mathbb{R})$.

{p:5.1} **Proposition.** *On* $(V_{2m}^*)^{K\cap B}$,

$$R_V(s_\alpha,\nu) = \begin{cases} Id & \text{if } m = 0, \\ \prod_{0\leq j<m} \frac{2j+1-<\nu,\check{\alpha}>}{2j+1+<\nu,\check{\alpha}>}\, Id & \text{if } m \neq 0. \end{cases}$$

In particular, $I(w,\nu)$ *is an isomorphism unless* $\langle \nu, \check{\alpha} \rangle \in 2\mathbb{Z}+1$.

Proof. The formula is well known for $SL(2,\mathbb{R})$. The second assertion follows from this and the listed properties of intertwining operators. ☐

{c:5.1}

Corollary. *For relevant $K-$types the formula is*

$$R_V(s_\alpha,\nu)=\begin{cases} Id & \text{on the +1 eigenspace of } s_\alpha,\\ \frac{1-<\nu,\check\alpha>}{1+<\nu,\check\alpha>}\ Id & \text{on the -1 eigenspace of } s_\alpha.\end{cases}$$

When restricted to $(V^*)^{K\cap B}$, the long intertwining operator is the product of the $R_V(s_\alpha,*)$ corresponding to the reduced decomposition of w_0 and depends only on the Weyl group structure of $(V^*)^{K\cap B}$.

Proof. Relevant $K-$types are distinguished by the property that the even eigenvalues of D_α are $0,\pm2$ only. The element s_α acts by 1 on the zero eigenspace of D_α and by -1 on the ±2 eigenspace. The claim follows from this. □

{sec:5.2}

5.2. We now show that the formulas in the previous section coincide with corresponding ones in the $p-$adic case. In the split p-adic case, spherical representations are a subset of representations with $\mathcal{I}$-fixed vectors, where $\mathcal{I}$ is an Iwahori subgroup. As explained in [B], the category of representations with $\mathcal{I}$ fixed vectors is equivalent to the category of finite dimensional representations of the Iwahori-Hecke algebra $\mathcal{H}:=\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$. The equivalence is

$$\mathcal{V}\longrightarrow\mathcal{V}^{\mathcal{I}}. \tag{5.2.1}$$ {eq:5.2.1}

The papers [BM1] and [BM2] show that the problem of the determination of the unitary dual of representations with $\mathcal{I}$ fixed vectors, is equivalent to the problem of the determination of the unitary irreducible representations of $\mathcal{H}$ with real infinitesimal character. In fact it is the affine graded Hecke algebras we will need to consider, and they are as follows.

Let $\mathbb{A}:=S(\mathfrak{a})$, and define the affine graded Hecke algebra to be $\mathbb{H}:=\mathbb{C}[W]\otimes\mathbb{A}$ as a vector space, and usual algebra structure for $\mathbb{C}[W]$ and $\mathbb{A}$.

The generators of $\mathbb{C}[W]$ are denoted by t_α corresponding to the simple reflections s_α, while the generators of $\mathbb{A}$ are $\omega\in\mathfrak{a}$. Impose the additional relation

$$\omega t_\alpha=s_\alpha(\omega)t_\alpha+<\omega,\alpha>,\qquad \omega\in\mathfrak{a}, \tag{5.2.2}$$ {eq:5.2.2}

where t_α is the element in $\mathbb{C}[W]$ corresponding to the simple root α. If $X(\chi)$ is the standard (principal series) module determined by χ, then

$$X(\chi)^{\mathcal{I}}=\mathbb{H}\otimes_{\mathbb{A}}\mathbb{C}_\chi. \tag{5.2.3}$$ {eq:5.2.3}

The intertwining operator $I(w,\chi)$ is a product of operators I_{α_i} according to a reduced decomposition of $w=s_{\alpha_1}\cdot\dots\cdot s_{\alpha_k}$. If α is a simple root,

$$r_\alpha:=(t_\alpha\alpha-1)\frac{1}{\alpha-1},\qquad I_\alpha\ :\ x\otimes\mathbb{1}_\chi\mapsto xr_\alpha\otimes\mathbb{1}_{s_\alpha\chi}. \tag{5.2.4}$$ {eq:5.2.4}

The $I(w,\nu)$ have the same properties as in the real case. The r_α are multiplied on the right, so we can replace α with $-\langle\nu,\alpha\rangle$ in the formulas. Furthermore,

$$\mathbb{C}[W]=\sum_{\sigma\in\widehat{W}} V_\sigma\otimes V_\sigma^*.$$

Since r_α acts as multiplication on the right, it gives rise to an operator

$$r_\sigma(s_\alpha,\nu):V_\sigma^*\longrightarrow V_\sigma^*.$$

{t:5.2}

Theorem. *The $R_V(s_\alpha,\nu)$ for the real case on relevant $K-$types coincide with the $r_\sigma(s_\alpha,\nu)$ on the $V_\sigma^*\cong (V^*)^{K\cap B}$*

Proof. These operators act the same way:

{eq:5.2.5}

$$r_\sigma(s_\alpha,\nu)=\begin{cases} Id & \text{on the } +1 \text{ eigenspace of } s_\alpha,\\ \frac{1-\langle\nu,\alpha\rangle}{1+\langle\nu,\alpha\rangle}Id & \text{on the } -1 \text{ eigenspace of } s_\alpha.\end{cases} \tag{5.2.5}$$

The assertion is now clear from corollary (5.1) and formula (5.2.2). We emphasize that the Hecke algebra for a p-adic group G is defined using the dual root system of the complex group $\check{G}$ so that there is no discrepancy between α and $\check{\alpha}$ in the formulas. ☐

{sec:5.3}

5.3. The main point of section 5.2 is that for the real case, and a relevant K-type (V,μ), the intertwining operator calculations coincide with the intertwining operator calculations for the affine graded Hecke algebra on the space $V^{K\cap B}$. Thus we will deal with the Hecke algebra calculations exclusively, but the conclusions hold for both the real and p-adic case. Recall from section 2.3 that to each χ we have associated a nilpotent orbit $\check{\mathcal{O}}$, and Levi components $\check{\mathfrak{m}}_{BC}$ and $\check{\mathfrak{m}}_{KL}$. These are special instances of the following situation. Assume that $\check{\mathcal{O}}$ is written as in (2.3.4) (*i.e.* $((a_1,a_1),\dots,(a_k,a_k);(d_i))$ with

- $\check{\mathfrak{g}}$ **of type B:** (d_i) all odd; they are relabelled $(2x_0+1,\dots,2x_{2m}+1)$,
- $\check{\mathfrak{g}}$ **of type C:** (d_i) all even; they are relabelled $(2x_0,\dots,2x_{2m})$,
- $\check{\mathfrak{g}}$ **of type D:** (d_i) all odd; they are relabelled $(2x_0+1,\dots,2x_{2m-1}+1)$.

Similar to (2.3.5), let

{eq:5.3.1}

$$\check{\mathfrak{m}}:=gl(a_1)\times\dots\times gl(a_k)\times\check{\mathfrak{g}}(n_0),\quad n_0=n-\sum a_i. \tag{5.3.1}$$

We consider parameters of the form $\chi=\check{h}/2+\nu$.

Write χ_0 for the parameter $\check{h}/2$, and $\chi_i := (-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i)$. We focus on χ_0 as a parameter on $\check{\mathfrak{g}}(n_0)$. We attach two Levi components

{eq:5.3.2}

$$
\begin{array}{ll}
\check{\mathfrak{g}}_e : & \\
B & gl(x_{2m-1}+x_{2m-2}+1)\times\cdots\times gl(x_1+x_0+1)\times\check{\mathfrak{g}}(x_{2m})\\
C & gl(x_{2m-1}+x_{2m-2})\times\cdots\times gl(x_1+x_0)\times\check{\mathfrak{g}}(x_{2m})\\
D & gl(x_{2m-1}+x_{2m-2}+1)\times\cdots\times gl(x_1+x_0+1)\\
\check{\mathfrak{g}}_o : & \\
B & gl(x_{2m}+x_{2m-1}+1)\times\cdots\times gl(x_2+x_1+1)\times\check{\mathfrak{g}}(x_0)\\
C & gl(x_{2m}+x_{2m-1})\times\cdots\times gl(x_2+x_1)\times\check{\mathfrak{g}}(x_0)\\
D & gl(x_{2m-3}+x_{2m-4}+1)\times\cdots\times gl(x_{2m-2})\times\check{\mathfrak{g}}(x_{2m-1}+1).
\end{array}
\tag{5.3.2}
$$

There are 1-dimensional representations $L(\chi_e)$ and $L(\chi_o)$ such that the spherical irreducible representation $L(\chi_0)=\overline{X}(\chi_0)$ with infinitesimal character χ_0 is the spherical irreducible subquotient of $X_e := Ind_{P_e}^G(L(\chi_e))$ and $X_o := Ind_{P_o}^G(L(\chi_o))$ respectively

The parameters χ_e and χ_o are written in terms of strings as follows:

X_e**:**

$$
\begin{array}{l}
B:\ \dots(-x_{2i-1},\dots,x_{2i-2})\dots(-x_{2m},\dots,-1)\\
C:\ \dots(-x_{2i-1}+1/2,\dots,x_{2i-2}-1/2)\dots(-x_{2m}+1/2,\dots,-1/2)\\
D:\ \dots(-x_{2i-1},\dots,x_{2i-2})\dots
\end{array}
\tag{5.3.3}
$$

{eq:5.3.3}

X_o**:**

$$
\begin{array}{l}
B:\ \dots(-x_{2i},\dots,x_{2i-1})\dots(-x_0,\dots,-1)\\
C:\ \dots(-x_{2i}+1/2,\dots,x_{2i-1}-1/2)\dots(-x_0+1/2,\dots,-1/2)\\
D:\ \dots(-x_{2i},\dots,x_{2i-1})\dots(-x_{2m-2},\dots,-1)(-x_{2m-1}+1,\dots,0)
\end{array}
\tag{5.3.4}
$$

{th:5.3}

Theorem. *For the Hecke algebra, p-adic groups,*

$$
\begin{aligned}
&[\sigma[(n-r),(r)]\ :\ X_e]=[\sigma[(n-r),(r)]\ :\ L(\chi_0)],\\
&[\sigma[(k,n-k),(0)]\ :\ X_o]=[\sigma[(k,n-k),(0)]\ :\ L(\chi_0)]
\end{aligned}
$$

hold.

The proof is in section 6.8.

For a general parameter $\chi=\chi_0+\nu$, the strings defined in section 2 and the above construction define parabolic subgroups with Levi components $gl(a_1)\times\cdots\times gl(a_k)\times\check{\mathfrak{g}}_e$ and $gl(a_1)\times\cdots\times gl(a_r)\times\check{\mathfrak{g}}_o$, and corresponding $L_e(\chi)$ and $L_o(\chi)$. We denote these induced modules by X_e and X_o as well.

Corollary. *The relations*

$$
\begin{aligned}
&[\sigma[(n-r),(r)]\ :\ X_e]=[\sigma[(n-r),(r)]\ :\ L(\chi)],\\
&[\sigma[(k,n-k),(0)]\ :\ X_o]=[\sigma[(k,n-k),(0)]\ :\ L(\chi)]
\end{aligned}
$$

hold in general. For real groups, in the notation of sections 4.2-4.4,

$$[\mu_e(r, n-r)\ :\ X_e] = [\mu_e(r, n-r)\ :\ L(\chi)],$$
$$[\mu_o(k, n-k)\ :\ X_o] = [\mu_o(k, n-k)\ :\ L(\chi)].$$

Proof. The results in section 5.2 show that the intertwining operators on $\sigma_e(k, n-k)$ for the $p-$adic group equal the intertwining operators for $\mu_e(k, n-k)$ for the real group, and similarly for σ_o and μ_o. Thus the multiplicities of the σ_e/σ_o in $L(\chi)$ for the $p-$adic case equal the multiplicities of the corresponding μ_e/μ_o in $L(\chi)$ in the real case. We do the $p-$adic case first. Recall theorem 2.9 which states that $I_{M_{KL}}(\chi) = L(\chi)$. The Levi subgroup M_{KL} is a product of GL factors, which we will denote M_A, with a factor $G(n_0)$. So for $W-$type multiplicities we can replace $I_{M_{KL}}(\chi)$ by $Ind_{M_A \times G(n_0)}[\otimes triv \otimes L(\chi_0)]$. We explain the case of $\sigma_e = \sigma_e(k, n-k)$, that of σ_o being identical. By Frobenius reciprocity,

{eq:5.3.4}

$$\operatorname{Hom}_W[\sigma_e : L(\chi)] = \operatorname{Hom}_W[\sigma_e : I_{M_{KL}}(\chi)] \tag{5.3.5}$$
$$= \operatorname{Hom}_{W(M_A)\times W(G(n_0))}[\sigma_e : triv \otimes L(\chi_0)]. \tag{5.3.6}$$

Using the formulas for restrictions of representations for Weyl groups of classical types, it follows that

{eq:5.3.5}

$$\dim \operatorname{Hom}_W[\sigma_e : L(\chi)] = \sum_{k'} \dim \operatorname{Hom}_{W(G(n_0))}[\sigma_e(k', n_0 - k') : L(\chi_0)]$$
$$= \sum_{k'} \dim \operatorname{Hom}_{W(G(n_0))}[\sigma_e(k', n_0 - k') : X_{e,G(n_0)}], \tag{5.3.7}$$

where the last step is theorem 5.3. Since $X_e = Ind^G_{M_A \times G(n_0)}[\otimes L(\chi_i) \otimes X_{e,G(n_0)}(\chi_0)]$, again by Frobenius reciprocity, one can show that this is also equal to $\dim \operatorname{Hom}_W[\sigma_e : X_e]$. This proves the claim for the p-adic case.

In the real case the proof is complete once we observe that in all the steps for the $p-$adic case, the multiplicity of $triv \otimes \mu_e(k', n_0 - k')$ in the restriction of $\mu_e(k, n-k)$ matches the multiplicity of $triv \otimes \sigma_e(k', n_0 - k')$ in the restriction of $\sigma_e(k, n-k)$. Similarly for μ_o and σ_o.

□

{sec:6} {sec:6.1}

6. Hecke algebra calculations

6.1. The proof of the results in 5.3 is by a computation of intertwining operators on the relevant $K-$types. It only depends on the $W-$type of $V^{K\cap B}$, so we work in the setting of the Hecke algebra. The fact that we can deal exclusively with $W-$types, is a big advantage. In particular we do not have to worry about disconnectedness of Levi components. We will write $GL(k)$ for the Hecke algebra of type A and $G(n)$ for the types B, C or D as the case may be. This is so as to emphasize that the results are about groups, real or $p-$adic.

The intertwining operators will be decomposed into products of simpler operators induced from operators coming from maximal Levi subgroups. We introduce these first.

Suppose M is a Levi component of the form

$$GL(a_1) \times \cdots \times GL(a_l) \times G(n_0). \tag{6.1.1}$$ {eq:6.1.1}

Let χ_i be characters for $GL(a_i)$. We simplify the notation somewhat by writing

$$\chi_i \longleftrightarrow (\nu_i) := (-\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i). \tag{6.1.2}$$ {eq:6.1.2}

The parameter is antidominant, and so $L(\chi_i)$ occurs as a submodule of the principal series $X((\nu_i))$. The module is spherical 1-dimensional, and the action of $\mathfrak{a}$ is

$$\chi_i(\omega) = \langle \omega, (\frac{a_i-1}{2} + \nu_i, \dots, -\frac{a_i-1}{2} + \nu_i) \rangle, \qquad \omega \in \mathfrak{a}, \tag{6.1.3}$$ {eq:6.1.3}

while W acts trivially. The trivial representation of $G(n_0)$ corresponds to the string $(-n_0 + \epsilon, \dots, -1 + \epsilon)$ where

$$\epsilon := \begin{cases} 0 & \mathbb{H} \text{ of type B,} \\ 1/2, & \mathbb{H} \text{ of type C,} \\ 1, & \mathbb{H} \text{ of type D.} \end{cases} \tag{6.1.4}$$ {eq:6.1.4}

We abbreviate this as (ν_0). Again $L(\chi_0)$ is the trivial representation, and because χ_0 is antidominant, it appears as a submodule of the principal series $X(\chi_0)$. We abbreviate

$$X_M(\dots(\nu_i)\dots) := Ind_{\prod GL(a_i)\times G(n_0)}^{G}[\otimes\chi_i \otimes triv]. \tag{6.1.5}$$ {eq:6.1.5}

The module $X_M(\dots(\nu_i)\dots)$ is a submodule of the standard module $X(\chi)$ with parameter corresponding to the strings

$$\chi := (\dots, -\frac{a_i-1}{2} + \nu_i, \dots, \frac{a_i-1}{2} + \nu_i, \dots, -n_0 + \epsilon, \dots, -1 + \epsilon). \tag{6.1.6}$$ {eq:6.1.6}

In the setting of the Hecke algebra, the induced modules (6.1.5) is really $X_M(\dots(\nu_i)\dots) = \mathbb{H} \otimes_{\mathbb{H}_M} [\bigotimes \chi_i \otimes triv]$.

Let $w_{i,i+1} \in W$ be the shortest Weyl group element which interchanges the strings (ν_i) and (ν_{i+1}) in ν, and fixes all other coordinates. The intertwining operator $I_{w_{i,i+1}} : X(\nu) \longrightarrow X(w_{i,i+1}\nu)$ restricts to an intertwining operator

$$\begin{aligned} &I_{M,i,i+1}(\dots(\nu_i)(\nu_{i+1})\dots) : \\ &\qquad X_M(\dots(\nu_i)(\nu_{i+1})\dots) \longrightarrow X_{w_{i,i+1}M}(\dots(\nu_{i+1})(\nu_i)\dots). \end{aligned} \tag{6.1.7}$$ {eq:6.1.7}

This operator is induced from the same kind for $GL(a_i+a_{i+1})$ where $M_{i,i+1} = GL(a_i) \times GL(a_{i+1}) \subset GL(a_i + a_{i+1})$ is the Levi component of a maximal parabolic subgroup.

Let $w_l \in W$ be the shortest element which changes (ν_l) to $(-\nu_l)$, and fixes all other coordinates. It induces an intertwining operator

$$I_{M,l}(\dots(\nu_l)(\nu_0)) : X_M(\dots(\nu_l),(\nu_0)) \longrightarrow X_{w_lM}(\dots(-\nu_l),(\nu_0)). \qquad (6.1.8)$$ {eq:6.1.8}

In this case, $w_lM = M$, so we will not always include it in the notation. In type D, if $n_0 = 0$, the last entry of the resulting string might have to stay $-\frac{a_l-1}{2} + \nu_l$ instead of $\frac{a_l-1}{2} - \nu_l$. This operator is induced from the same kind on $G(a_l + n_0)$ with $M_l = GL(a_l) \times G(n_0) \subset G(a_l + n_0)$ the Levi component of a maximal parabolic subgroup.

Lemma. *The operators $I_{M,i,i+1}$ and $I_{M,l}$ are meromorphic in ν_i in both the real and p-adic case.*

(1) *$I_{M,i,i+1}$ has poles only if $\frac{a_i-1}{2} + \nu_i - \frac{a_{i+1}-1}{2} - \nu_{i+1} \in \mathbb{Z}$. If so, a pole only occurs if*

$$-\frac{a_i - 1}{2} + \nu_i < -\frac{a_{i+1} - 1}{2} - \nu_{i+1}, \qquad \frac{a_i - 1}{2} + \nu_i < \frac{a_{i+1} - 1}{2} + \nu_{i+1}.$$

(2) *$I_{M,l}$ has a pole only if $\frac{a_l-1}{2} + \nu_l \equiv \epsilon (mod\ \mathbb{Z})$. In that case, a pole only occurs if*

$$-\frac{a_l - 1}{2} + \nu_l < 0.$$

Proof. We prove the assertion for $I_{M,i,i+1}$, the other one is similar. The fact that the integrality condition is necessary is clear. For the second condition, it is sufficient to consider the case $M = GL(a_1) \times GL(a_2) \subset GL(a_1 + a_2)$. If the strings are strongly nested, then the operator cannot have any pole because X_M is irreducible. The remaining case is to show there is no pole in the case when $-\frac{a_2-1}{2} + \nu_2 \le -\frac{a_1-1}{2} + \nu_1$, and $\frac{a_1-1}{2} + \nu_1 > \frac{a_2-1}{2} + \nu_2$. Let

{eq:6.1.9}
$$\begin{aligned} &M' := GL(\frac{a_1 + a_2}{2} + \nu_2 + \nu_1) \times GL(\frac{a_1 - a_2}{2} + \nu_1 - \nu_2) \times GL(a_2),\\ &(\nu_1') = (-\frac{a_1 - 1}{2} + \nu_1, \dots, \frac{a_2 - 1}{2} + \nu_2)\\ &(\nu_2') = (\frac{a_2 - 1}{2} + 1 + \nu_2, \dots, \frac{a_1 - 1}{2} + \nu_1)\\ &(\nu_3') = (\nu_2) = (-\frac{a_2 - 1}{2} + \nu_2, \dots, \frac{a_2 - 1}{2} + \nu_2). \end{aligned} \qquad (6.1.9)$$

Then $X_M((\nu_1)(\nu_2)) \subset X_{M'}((\nu_1')(\nu_2')(\nu_3'))$, and $I_{M,1,2}$ is the restriction of $I_{w_{2,3}M',1,2},((\nu_1')(\nu_3')(\nu_2') \circ I_{M',2,3}((\nu_1')(\nu_2')(\nu_3'))$ to X_M. Because the strings $(\nu_1')(\nu_3')$ are strongly nested, $I_{w_{2,3}M',1,2}$ has no pole, and $I_{M',2,3}$ has no pole because it is a restriction of operators coming from $SL(2)$'s which do not have poles. The claim follows. □

Let σ be a $W-$type. We are interested in computing $r_\sigma(w, \dots(\nu_i)\dots)$, where w changes all the ν_i for $1 \le i$ to $-\nu_i$. The operator can be factored into a product of $r_\sigma(w_{i,i+1}, *)$ of the type (6.1.7) and $r_\sigma(w_l, *)$ of the type (6.1.8).

These operators are more tractable. Here's a more precise explanation. Let M be the Levi component

{eq:6.1.11} $$GL(a_1)\times\dots\times GL(a_i+a_{i+1})\times\dots \text{ in case (6.1.7)} \tag{6.1.10}$$

{eq:6.1.12} $$GL(a_1)\times\dots\times G(a_l+n_0) \text{ in case (6.1.8)} \tag{6.1.11}$$

Since X_M is induced from the trivial $W(M)$ module,

$$\mathrm{Hom}_W[\sigma, X_M((\nu_i))] = Hom_{W(M)}[\sigma|_{W(M)}\ :\ triv\otimes X_{M_{i,i+1}}((\nu_i),(\nu_{i+1}))\otimes triv] \\ \text{in case (6.1.7)} \tag{6.1.12}$$ {eq:6.1.13}

$$\mathrm{Hom}_W[\sigma, X((\nu_i))] = Hom_{W(M)}[\sigma|_{W(M)}\ :\ triv\otimes X_{M_l}((\nu_l),(\nu_0))] \\ \text{in case (6.1.8)} \tag{6.1.13}$$ {eq:6.1.14}

where $M_{i,i+1} = GL(a_i)\times GL(a_{i+1})$ is a maximal Levi component of $GL(a_i+a_{i+1})$ and $M_l = GL(a_l)\times G(n_0)$ is a maximal Levi component of $G(a_l+n_0)$. To compute the $r_\sigma(w_{i,i+1},*)$ and $r_\sigma(w_l,*)$, it is enough to compute the corresponding r_{σ_j} for the σ_j ocuring in the restriction $\sigma\mid_{W(M)}$ in the cases $GL(a_i)\times GL(a_{i+1})\subset GL(a_i+a_{i+1})$ and $GL(a_l)\times G(n_0)\subset G(a_l+n_0)$. The restrictions of relevant $W-$types to Levi components consist of relevant $W-$types of the same kind, *i.e.* $\sigma[(n-r),(r)]$ restricts to a sum of representations of the kind σ_e, and $\sigma[(k,n-k),(0)]$ restricts to a sum of σ_o. Typically the multiplicities of the factors are 1.

We also note that

$$X_M\mid_W = \sum_{\sigma\in\widehat{W}} V_\sigma\otimes (V_\sigma^*)^{W(M)}. \tag{6.1.14}$$ {eq:6.1.15}

So the $r_\sigma(w,*)$ map $(V_\sigma^*)^{W(M)}$ to $(V_\sigma^*)^{W(wM)}$.

In the next sections we will compute the cases of Levi components of maximal parabolic subgroups.

{6.2}

6.2. $GL(a)\times GL(b)\subset GL(a+b)$**.** This is the case of $I_{i,i+1}$ with $i<l$. Let $n=a+b$ and $G=GL(n)$ and $M=GL(a)\times GL(b)$. The module $X_M((\nu_1),(\nu_2))$ induced from the characters corresponding to

$$(-\frac{a-1}{2}+\nu_1,\dots,\frac{a-1}{2}+\nu_1),(-\frac{b-1}{2}+\nu_2,\dots,\frac{b-1}{2}+\nu_2) \tag{6.2.1}$$ {eq:6.2.1}

has the following S_{a+b} structure. Let $m:=\min(a,b)$ and write $\sigma(k,a+b-k)$ for the module corresponding to the partition $(k,a+b-k)$, $0\le k\le m$. Then

$$X_M((\nu_1),(\nu_2))\mid_W = \bigoplus_{0\le k\le m}\sigma(k,a+b-k). \tag{6.2.2}$$ {eq:6.2.2}

{l:6.2}

Lemma. *For* $1\le k\le m$*, the intertwining operator* $I_{M,1,2}((\nu_1)(\nu_2))$ *restricted to* σ *gives*

$$r_{\sigma(k,a+b-k)}(a,b,\nu_1,\nu_2) = \prod_{0\le j\le k-1}\frac{(\nu_1-\frac{a-1}{2})-(\frac{b-1}{2}+\nu_2+1)+j}{(\nu_1+\frac{a-1}{2})-(-\frac{b-1}{2}+\nu_2-1)-j}.$$

Proof. The proof is an induction on a, b and k. We omit most details but give the general idea. Assume $0 < k < m$, the case $k = m$ is simpler. Embed $X_M((\nu_1),(\nu_2))$ into $X_{M'}((\nu'),(\nu''),(\nu_2))$, where $M' = GL(a-1)\times GL(1)\times GL(b)$, corresponding to the strings

{eq:6.2.3}
$$(-\frac{a-1}{2}+\nu_1,\dots,\frac{a-3}{2}+\nu_1),(\frac{a-1}{2}+\nu_1),(-\frac{b-1}{2}+\nu_2,\dots,\frac{b-1}{2}+\nu_2). \tag{6.2.3}$$

The intertwining operator $I_{M,1,2}(\nu_1,\nu_2)$ is the restriction of

{eq:6.2.4}
$$I_{M',1,2}(\nu',\nu_2,\nu'')\circ I_{M',2,3}(\nu';\nu'',\nu_2) \tag{6.2.4}$$

to $X_M((\nu_1),(\nu_2))\subset X_{M'}((\nu'),(\nu''),(\nu_2))$. By an induction on n we can assume that these operators are known. The $W-$type $\sigma(k,n-k)$ occurs with multiplicity 1 in $X_M((\nu_1),(\nu_2))$ and with multiplicity 2 in $X_{M'}((\nu'),(\nu''),(\nu_2))$. The restrictions are

$$\sigma(k,n-k)\mid_{W(M')} = triv\otimes\sigma(k-1,b+1-k)+triv\otimes\sigma(k,b-k) \tag{6.2.5}$$

{eq:6.2.5}
$$\text{for } I_{M',1,2} \tag{6.2.6}$$

{eq:6.2.6}
$$\sigma(k,n-k)\mid_{W(M')} = \sigma(1,b)+\sigma(0,b+1) \text{ for } I_{M',2,3} \tag{6.2.7}$$

The representation $\sigma(k,n-k)$ has a realization as harmonic polynomials in $S(\mathfrak{a})$ spanned by

{eq:6.2.7}
$$\prod_{1\le\ell\le k}(\epsilon_{i_\ell}-\epsilon_{j_\ell}) \tag{6.2.8}$$

where $(i_1,j_1),\dots,(i_k,j_k)$ are k pairs of integers satisfying $1\le i_\ell,\ j_\ell\le n$, and $i_\ell\ne j_\ell$. We apply the intertwining operator to the $S_a\times S_b-$fixed vector

{eq:6.2.8}
$$e := \sum_{x\in S_a\times S_b} x\cdot[(\epsilon_1-\epsilon_{a+1})\times\dots\times(\epsilon_k-\epsilon_{a+k})]. \tag{6.2.9}$$

The intertwining operator $I_{M',2,3}$, has a simple form on the vectors

{eq:6.2.9}
$$e_1 := \sum_{x\in S_{a-1}\times S_{b+1}} x\cdot[(\epsilon_1-\epsilon_{a+1})\times\dots\times(\epsilon_k-\epsilon_{a+k})],\ \text{in } \sigma(0,b+1) \tag{6.2.10}$$

{eq:6.2.10}
$$e_2 := \sum_{x\in S_{a-1}\times S_1\times S_b} x\cdot[(\epsilon_1-\epsilon_{a+1})\times\dots\times(\epsilon_{k-1}-\epsilon_{a+k-1})(\epsilon_a-\epsilon_{a+k})],\ \text{in } \sigma(1,b) \tag{6.2.11}$$

which appear in (6.2.7). They are mapped into scalar multiples (given by the lemma) of the vectors $e_1',\ e_2'$ which are invariant under $S_{a-1}\times S_b\times S_1$, and transform according to $triv\otimes\sigma(0,b+1)$ and $triv\otimes\sigma(1,b)$. We choose

{eq:6.2.11}
$$\begin{aligned} e_1' &= e_1,\\ e_2' &:= \sum_{x\in S_{a-1}\times S_b\times S_1} x\cdot[(\epsilon_1-\epsilon_a)\times\dots\times(\epsilon_{k-1}-\epsilon_{a+k-2})(\epsilon_n-\epsilon_{a+k-1})] \end{aligned} \tag{6.2.12}$$

The intertwining operator $I_{M',1,2}$ has a simple form on the vectors invariant under $S_{a-1}\times S_b\times S_1$ transforming according to $\sigma(k,n-k-1)$ and $\sigma(k-1,n-k)$. We can choose multiples of

$$f_1 := \sum_{x\in S_{a-1}\times S_b\times S_1} x\cdot[(\epsilon_1-\epsilon_a)\times\dots\times(\epsilon_{k-1}-\epsilon_{a+k-2})(\epsilon_k-\epsilon_{a+k-1})], \quad \text{in } \sigma(k-1,n-k) \tag{6.2.13}$$ {eq:6.2.12}

$$f_2 := \sum_{x\in S_{a-1}\times S_b\times S_1} x\cdot[(\epsilon_1-\epsilon_a)\times\dots\times(\epsilon_{k-1}-\epsilon_{a+k-2})\cdot \cdot(e_k+\dots+\epsilon_{a-1}+\epsilon_a+\epsilon_{a+k}+\dots+\epsilon_{n-1}-(n-2k+1)\epsilon_n)] \quad \text{in } \sigma(k,n-k-1) \tag{6.2.14}$$ {eq:6.2.13}

The fact that f_1 transforms according to $\sigma(k,n-1)$ follows from (6.2.8). The fact that f_2 transforms according to $\sigma(k-1,n)$ is slightly more complicated. The product $\prod(\epsilon_1-\epsilon_a)\times\dots\times(\epsilon_{k-1}-\epsilon_{a+k-2})$ transforms according to $\sigma(k-1,k-1)$ under S_{2k-2}. The vector $(e_k+\dots+\epsilon_{a-1}+\epsilon_a+\epsilon_{a+k}+\dots+\epsilon_{n-1}-(n-2k+1)\epsilon_n)$ is invariant under the S_{n-2k-1} acting on the coordinates $\epsilon_k,\dots\epsilon_a,\epsilon_{a+k},\dots,\epsilon_{n-1}$. Since $\sigma(k,n-k-1)$ does not have such invariant vectors, the product inside the sum in (6.2.14) must transform according to $\sigma(k-1,n-k)$. The average under x in (6.2.14) is nonzero. The operator $I_{M',2,3}$ maps f_1 and f_2 into multiples (using the induction hypothesis) of the vectors $f_1',\ f_2'$ which are the $S_b\times S_{a-1}\times S_1$ invariant vectors transforming according to $\sigma(k,n-1)$ and $\sigma(k-1,n-k)$. The composition $I_{M',1,2}\circ I_{M',2,3}$ maps e into a multiple of

$$e' := \sum_{\sigma\in S_b\times S_a}\sigma\cdot[(\epsilon_1-\epsilon_{b+1})\times\dots\times(\epsilon_k-\epsilon_{b+k})]. \tag{6.2.15}$$ {eq:6.2.14}

The multiple is computable by using the induction hypothesis and the expression of

e in terms of $e_1,\ e_2$,
$e_1',\ e_2'$ in terms of $f_1,\ f_2$, and
e' in terms of $f_1',\ f_2'$.

For example for the case $k = 1$, we get the following formulas.

$$\begin{aligned} e &= b(\epsilon_1 + \dots + \epsilon_a) - a(\epsilon_{a+1} + \dots + \epsilon_n),\\ e_1 &= (b+1)(\epsilon_1 + \dots + \epsilon_{a-1}) - (a-1)(\epsilon_a + \dots + \epsilon_n),\\ e_2 &= b\epsilon_a - (\epsilon_{a+1} + \dots + \epsilon_n),\\ f_1 &= b(\epsilon_1 + \dots + \epsilon_{a-1}) - (a-1)(\epsilon_a + \dots + \epsilon_{n-1}),\\ f_2 &= (\epsilon_1 + \dots + \epsilon_{a-1}) + (\epsilon_a + \dots + \epsilon_{n-1}) - (n-1)\epsilon_n,\\ e' &= -a(\epsilon_1 + \dots + \epsilon_b) - b(\epsilon_{b+1} + \dots + \epsilon_n),\\ e_1' &= (b+1)(\epsilon_1 + \dots + \epsilon_{a-1}) - (a-1)(\epsilon_a + \dots + \epsilon_n),\\ e_2' &= -(\epsilon_a + \dots + \epsilon_{n-1}) + b(\epsilon_n),\\ f_1' &= -(a+1)(\epsilon_1 + \dots + \epsilon_b) + b(\epsilon_{b+1} + \dots + \epsilon_{n-1}),\\ f_2' &= (\epsilon_1 + \dots + \epsilon_b) + (\epsilon_{b+1} + \dots + \epsilon_{n-1}) - (n-1)\epsilon_n. \end{aligned} \tag{6.2.16}$$ {eq:6.2.14a}

Then

{eq:6.2.15}
$$\begin{aligned} e &= \frac{a-1}{b+1}e_1 - \frac{n}{b+1}e_2,\\ e_1' &= \frac{n}{n-1}f_1 + \frac{a-1}{n-1}f_2,\\ e_2' &= \frac{1}{n-1}f_1 - \frac{b}{n-1}f_2,\\ e' &= \frac{n}{n-1}f_1' - \frac{b}{n-1}f_2'. \end{aligned} \tag{6.2.17}$$

□

{sec:6.3}

6.3. $GL(k) \times G(n) \subset G(n+k)$. In the next sections we prove theorem 5.3 in the case of a parabolic subgroup with Levi component $GL(k) \times G(n)$ for the induced module

{eq:6.3.1}
$$X_M((\nu_1)(\nu_0)) = Ind_M^G[L(\chi_1) \otimes L(\chi_0)]. \tag{6.3.1}$$

The notation is as in section 6.1.

The strings are

{eq:6.3.2}
$$(-\frac{k-1}{2} + \nu, \dots, \frac{k-1}{2} + \nu)(-n+1+\epsilon, \dots, -1+\epsilon). \tag{6.3.2}$$

Recall that $\epsilon = 0$ when the Hecke algebra is type B, $\epsilon = 1/2$ for type C, and $\epsilon = 1$ for type D, and

{eq:6.3.3}
$$r_\sigma(\nu) : (V_\sigma^*)^{W(M)} \longrightarrow (V_\sigma^*)^{W(M)}. \tag{6.3.3}$$

We will compute $r_\sigma(w_1, (\nu)(\nu_0))$ by induction on k. In this case the relevant $W-$types have multiplicity ≤ 1 so r_σ is a scalar.

{sec:6.4}

6.4. We start with the special case $k = 1$ when the maximal parabolic subgroup P has Levi component $M = GL(1) \times G(n) \subset G(n+1)$. In type D we assume $n \geq 1$. Then

{eq:6.4.1}
$$X_M \mid_W = \sigma[(n+1),(0)] \;+\; \sigma[(1,n),(0)] \;+\; \sigma[(n),(1)], \tag{6.4.1}$$

and all the $W-$types occuring are relevant. In types B,C the operator $r_\sigma(\nu)$ is the restriction to $(V_\sigma^*)^{W(M)}$ of the product

{eq:6.4.2}
$$r_{1,2} \circ \cdots \circ r_{n,n+1} \circ r_{n+1} \circ r_{n,n+1} \circ \cdots \circ r_{1,2} \tag{6.4.2}$$

as an operator on V_σ. Here $r_{i,j}$ is the $r_\sigma(w,*)$ corresponding to the root $\epsilon_i - \epsilon_j$ and r_{n+1} is the r_σ corresponding to ϵ_{n+1} or $2\epsilon_{n+1}$ in types B and C. In type D, the operator is

$$r_{1,2} \circ \cdots \circ r_{n,n+1} \circ \widetilde{r_{n,n+1}} \circ \cdots \circ r_{1,2} \tag{6.4.3}$$ {eq:6.4.2d}

where $r_{i,i+1}$ are as before, and $\widetilde{r_{n,n+1}}$ corresponds to $\epsilon_n + \epsilon_{n+1}$. Since the multiplicities are 1, this is a scalar.

{p:6.4}

Proposition. *The scalar* $r_\sigma(w_1,((\nu)(\nu_0)))$ *is*

$$\begin{array}{lll} & \sigma_e(1,n) = \sigma[(n),(1)] & \sigma_o(1,n) = \sigma[(1,n),(0)] \\ B & \frac{n+1-\nu}{n+1+\nu} & -\frac{n+1-\nu}{n+1+\nu} \\ C & \frac{1/2+n-\nu}{1/2+n+\nu} & \frac{1/2+n-\nu}{1/2+n+\nu} \cdot \frac{1/2-\nu}{1/2+\nu} \\ D & \frac{n-\nu}{n+\nu} & \frac{n-\nu}{n+\nu}\frac{1-\nu}{1+\nu} \end{array} \tag{6.4.4}$$ {eq:6.4.3}

Proof. We do an induction on n.

The reflection representation $\sigma[(n),(1)]$ has dimension $n+1$ and the usual basis $\{\epsilon_i\}$. The $W(M)-$fixed vector is ϵ_1. The representation $\sigma[(1,n),(0)]$ has a basis $\epsilon_i^2 - \epsilon_j^2$ with the symmetric square action. The $W(M)-$fixed vector is $\epsilon_1^2 - \frac{1}{n}(\epsilon_2^2 + \cdots + \epsilon_{n+1}^2)$.

The case $n = 0$ for type C is clear; the intertwining operator is 1 on $\mu_o(1,0) = triv$ and $\frac{1/2-\nu}{1/2+\nu}$ on $\mu_e(0,1) = sgn$. We omit the details for type B. In type for $n = 1$, *i.e.* D_2, the middle $W-$type in (6.4.1) decomposes further

$$\sigma[(2),(0)] + \sigma[(1),(1)]_I + \sigma[(1),(1)]_{II} + \sigma[(0),(2)]. \tag{6.4.5}$$ {eq:6.4.4}

The representations $\sigma[(1),(1)]_{I,II}$ are 1-dimensional with bases $\epsilon_1 \pm \epsilon_2$. The result is straightforward in this case as well.

We now do the induction step. We give details for type B. In the case $\sigma_e(1,n)$, embed X_M in the induced module from the characters corresponding to

$$(\nu)(-n)(-n+1,\ldots,-1). \tag{6.4.6}$$ {eq:6.4.5}

Write $M' = GL(1) \times GL(1) \times G(n-1)$ for the Levi component corresponding to these three strings. Then the intertwining operator $I : X_M((\nu)(\nu_0)) \longrightarrow$

$X_M((-\nu)(\nu_0))$ is the restriction of

$$I_{M',1,2}((-n),(-\nu)(\nu_0))\circ I_{M',2}((-n)(\nu)(\nu_0))\circ I_{M',1,2}(\nu,(-n),(\nu_0)). \quad (6.4.7)$$ {eq:6.4.6}

The r_σ have a corresponding decomposition

{eq:6.4.7}
$$(r_\sigma)_{M',1,2}((-\nu),(-n)(\nu_0))\circ(r_\sigma)_{M',2}((-n)(\nu)(\nu_0))\circ(r_\sigma)_{M',1,2}((\nu)(-n)(\nu_0)). \quad (6.4.8)$$

We need the restrictions of $\mu_e(1,n)$ and $\mu_o(1,n)$ to $W(M')$. We have

{eq:6.4.8}
$$\begin{aligned} Ind_{W(B_{n-1})}^{W(B_{n+1})}[\sigma[(n-1),(0)]] &= \sigma[(n+1),(0)]+2\sigma[(n),(1)]+2\sigma[(1,n),(0)]\\ &+2\sigma[(1,n-1),(1)]+\sigma[(n-1),(2)]+\sigma[(n-1),(1,1)]+\\ &+\sigma[(2,n-1),(0)]+\sigma[(1,1,n-1),(0)], \end{aligned} \quad \text{(a)}$$

$$Ind_{W(B_n)}^{W(B_{n+1})}[\sigma[(n),(0)]] = \sigma[(n+1),(0)]+\sigma[(n),(1)]+\sigma[(1,n),(0)], \quad \text{(b)}$$

(6.4.9)

$$Ind_{W(B_1)W(B_n)}^{W(B_{n+1})}[\sigma[(1),(0)]\otimes\sigma[(n),(0)]] = \sigma[(n+1),(0)]+\sigma[(1,n),(0)] \quad \text{(c)}$$

$$Ind_{W(B_1)W(B_n)}^{W(B_{n+1})}[\sigma[(0),(1)]\otimes\sigma[(n),(0)]] = \sigma[(n-1),(1)] \quad \text{(d)}$$

Thus $\mu_e(1,n)$ occurs with multiplicity 2 in $X_{M'}$. The $W(M')$ fixed vectors are the linear span of $\epsilon_1,\ \epsilon_2$. The intertwining operators $I_{M',1,2}$ and $I_{M',2}$ are induced from maximal parabolic subgroups whose Levi components we label M_1 and M_2. Then $\epsilon_1+\epsilon_2$ transforms like $triv\otimes triv$ under $W(M_1)$ and $\epsilon_1-\epsilon_2$ transforms like $sgn\otimes triv$. The vector ϵ_1 is fixed under $W(B_n)$ (which corresponds to M_2) and the vector ϵ_2 is fixed under $W(B_{n-1})$ and transforms like $\mu_o(1,n)$ under $W(B_n)$. The matrix r_σ is, according to (6.4.8),

{eq:6.4.9}
$$\begin{bmatrix}\frac{1}{2+\nu-n} & \frac{\nu-n+1}{2+\nu-n}\\ \frac{\nu-n+1}{1+\nu-n+1} & \frac{1}{2+\nu-n}\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & \frac{n-\nu}{n+\nu}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{1+\nu+n} & \frac{\nu+n}{1+\nu+n}\\ \frac{\nu+n}{1+\nu+n} & \frac{1}{2+\nu+n}\end{bmatrix}. \quad (6.4.10)$$

So the vector ϵ_1 is mapped into $\frac{n+1-\nu}{n+1+\nu}\epsilon_1$ as claimed. For $\sigma_o(1,n)$ we apply the same method. In this case the operator $I_{M',2}$ is the identity because in the representation $\mu_o(1,n)$ the element t_n corresponding to the short simple root acts by 1.

The calculation for type D is analogous, we sketch some details. We decompose the strings into

{eq:6.4.10}
$$(\nu)(-n+1,\dots,-1)(0), \quad (6.4.11)$$

and $M'=GL(1)\times GL(n-1)\times GL(1)$. Then

{eq:6.4.11}
$$\begin{aligned} &I_{M,1}((\nu)(\nu_0)) = \quad (6.4.12)\\ &I_{M',1,2}((-n+1,\dots,-1)(-\nu)(0))\circ I_{M',1}((-n+1,\dots,-1)(\nu)(0))\circ\\ &\qquad I_{M',1,2}((\nu)(-n+1,\dots,-1)(0)). \end{aligned}$$

□ {sec:6.5}

6.5. In this section we consider (6.3.2) for $k>1$, $n\ge 1$ and the $W-$types $\sigma_e(m,n+k-m)$ for $0\le m\le k$ (notation as in definition 4.8). These are the $W-$types which occur in X_M, with $M=GL(k)\times G(n)\subset G(k+n)$.

{p:6.5}

Proposition. *The* $r_\sigma(w_1,((\nu)(\nu_0))$ *for* $\sigma=\sigma_e(m,n+k-m)$ *are scalars. They equal*

Type B:

$$\prod_{0\le j\le m-1}\frac{n+1-(-\frac{k-1}{2}+\nu)-j}{n+1+(\frac{k-1}{2}+\nu)-j} \tag{6.5.1}$$ {eq:6.5.1}

Type C:

$$\prod_{0\le j\le m-1}\frac{n+1/2-(-\frac{k-1}{2}+\nu)-j}{n+1/2+(\frac{k-1}{2}+\nu)-j} \tag{6.5.2}$$ {eq:6.5.2}

Type D:

$$\prod_{0\le j\le m-1}\frac{n-(-\frac{k-1}{2}+\nu)-j}{n+(\frac{k-1}{2}+\nu)-j} \tag{6.5.3}$$ {eq:6.5.3}

Proof. The proof is by induction on k. The case $k=1$ was done in section 6.4 so we only need to do the induction step. For types B,C factor the intertwining operator as follows. Decompose the string

$$((\nu')(\frac{k-1}{2}+\nu)(\nu_0)):=((-\frac{k-1}{2}+\nu,\dots,\frac{k-3}{2}+\nu)(\frac{k-1}{2}+\nu)(\nu_0)) \tag{6.5.4}$$ {eq:6.5.4}

and let $M':=GL(k-1)\times GL(1)\times G(n)$, and $M''=GL(1)\times GL(k-1)\times G(n)$. Thus

$$\begin{aligned} I_{M,1}=\;&I_{M'',2}((-\frac{k-1}{2}-\nu)(\nu')(\nu_0))\circ\\ &I_{M',1,2}((\nu')(-\frac{k-1}{2}-\nu)(\nu_0))\circ\\ &I_{M',2}((\nu')(\frac{k-1}{2}+\nu)(\nu_0))\end{aligned} \tag{6.5.5}$$ {eq:6.5.5}

$I_{M',1,2}$ and $I_{M',2}$ were computed earlier, while $I_{M'',2}$ is known by induction. Then

$$\begin{aligned}&\sigma_e(m,n+k-m)\mid_{W(GL(k-1)\times W(G(n+1))}=\\ &triv\otimes[\sigma_e(1,n)+\sigma_e(0,n+1)+\dots\end{aligned} \tag{6.5.6}$$ {eq:6.5.6}

$$\begin{aligned}&\sigma_e(m,n+k-m)\mid_{W(GL(k)\times W(G(n+k-1))}=\\ &[(k)\otimes triv+(1,k-1)\otimes triv]+\dots\end{aligned} \tag{6.5.7}$$ {eq:6.5.7}

$$\begin{aligned}&\sigma_e(m,n+k-m)\mid_{W(GL(1)\times W(G(n+k-1))}=\\ &triv\otimes[\sigma_e(m-1,n+k-m)+\sigma_e(m,n+k-1-m)]+\dots\end{aligned} \tag{6.5.8}$$ {eq:6.5.8}

where $\dots$ denote $W-$types which are not spherical for $W(M)$, so do not matter for the computations.

The $W-$type $\sigma_e(m,n+k-m)\cong\bigwedge^m\sigma_e(1,n+k-1)$. It occurs with multiplicity 2 in $X_{M'}$ for $0<m<\min(k,n)$ and multiplicity 1 for $m=\min(k,n)$. We will write out an explicit basis for the invariant $S_1\times S_{k-1}\times W(B_n)$ vectors. Formulas (6.5.2)-(6.5.4) then come down to a computation with 2×2 matrices as in the case $k=1$. Let

{eq:6.5.9}
$$e:=\frac{1}{m!(k-m)!}\sum_{x\in S_k}x\cdot[\epsilon_1\wedge\dots\wedge\epsilon_m]. \tag{6.5.9}$$

This is the $S_k\times W(B_n)$ fixed vector of $\sigma_e(m,n+k-m)$. It decomposes as

{eq:6.5.10}
$$e=e_0+e_1=f_0+f_1 \tag{6.5.10}$$

where

{eq:6.5.11}
$$\begin{aligned}
e_0&=\frac{1}{m!(k-1-m)!}\sum_{x\in S_{k-1}\times S_1}x\cdot[\epsilon_1\wedge\dots\wedge\epsilon_m],\\
e_1&=\frac{1}{(m-1)!(k-m)!}\sum_{x\in S_{k-1}\times S_1}x\cdot[\epsilon_1\wedge\dots\wedge\epsilon_{m-1}]\wedge\epsilon_k,\\
f_0&=\frac{1}{m!(k-1-m)!}\sum_{x\in S_1\times S_{k-1}}x\cdot[\epsilon_2\wedge\dots\wedge\epsilon_{m+1}],\\
f_1&=\frac{1}{(m-1)!(k-m)!}\sum_{x\in S_1\times S_{k-1}}\epsilon_1\wedge x\cdot[\epsilon_2\wedge\dots\wedge\epsilon_m].
\end{aligned} \tag{6.5.11}$$

Let also

{eq:6.5.12}
$$\begin{aligned}
e_0'=e_0''&=\frac{1}{(m-1)!(k-m)!}\sum_{x\in S_k}x\cdot[\epsilon_1\wedge\dots\wedge\epsilon_m],\\
e_1'&=\sum_{x\in S_{k-1}\times S_1}x\cdot[\epsilon_1\wedge\dots\wedge\epsilon_{m-1}\wedge(\epsilon_m-\epsilon_k)],\\
e_1''&=\sum_{x\in S_1\times S_{k-1}}x\cdot[(-\epsilon_1+\epsilon_{m+1})\wedge\epsilon_2\wedge\dots\wedge\epsilon_{m+1}].
\end{aligned} \tag{6.5.12}$$

Then

{eq:6.5.13}
$$\begin{aligned}
e_0&=\frac{k-m}{k}e_0'+\frac{m}{k}e_1', &\qquad e_1&=\frac{m}{k}e_0'-\frac{m}{k}e_1',\\
e_0''&=f_0+f_1, & e_1''&=f_0-\frac{k-m}{m}f_1.
\end{aligned} \tag{6.5.13}$$

We now compute the action of the intertwining operators. The following relations hold:

{eq:6.5.14}
$$\begin{aligned} &I_{M',2}(e_0)=e_0, \qquad I_{M',2}(e_1)=\frac{n+\epsilon-(\frac{k-1}{2}+\nu)}{n+\epsilon+(\frac{k-1}{2}+\nu)}e_1,\\ &I_{M',12}(e_0')=e_0'', \qquad I_{M',12}(e_1')=\frac{2\nu-1}{2\nu+k-1}e_1'',\\ &I'_{M'',2}(f_0)=\prod_{0\le j\le m-2}\frac{n+\epsilon-(-\frac{k-1}{2}+\nu)-j}{n+\epsilon+(\frac{k-3}{2}+\nu)-j}f_0,\\ &I_{M'',2}(f_1)=\prod_{0\le j\le m-1}\frac{n+\epsilon-(-\frac{k-1}{2}+\nu)-j}{n+\epsilon+(\frac{k-3}{2}+\nu)-j}f_1, \end{aligned} \tag{6.5.14}$$

where $\epsilon=1$ in type B, $\epsilon=1/2$ in type C, and $\epsilon=0$ in type D. Then

$$I_{M',2}(e_0+e_1)=e_0+\frac{n+\epsilon-(\frac{k-1}{2}+\nu)}{n+\epsilon+(\frac{k-1}{2}+\nu)}e_1. \tag{6.5.15}$$ {eq:6.5.15}

Substituting the expressions of $e_0,\ e_1$ in terms of $e_0',\ e_1'$, we get

$$[\frac{k-m}{k}+\frac{m}{k}\frac{n+\epsilon-(\frac{k-1}{2}+\nu)}{n+\epsilon+(\frac{k-1}{2}+\nu)}]e_0'+\frac{m}{k}[1-\frac{n+\epsilon-(\frac{k-1}{2}+\nu)}{n+\epsilon+(\frac{k-1}{2}+\nu)}]e_1'. \tag{6.5.16}$$ {eq:6.5.16}

Applying $I_{M,2}$ to this has the effect that e_0' is sent to e_0'' and the term in e_1' is multiplied by $\frac{2\nu-1}{2\nu+k-1}$ and e_1' is replaced by e_1''. Substituting the formulas for e_0'' and e_1'' in terms of $f_0,\ f_1$, and applying $I_{M'',2}$, we get the claim of the proposition. □

{sec:6.6}

6.6. We now treat the case $\sigma=\sigma_o(m,n+k-m)$. We assume $n>0$ or else these $W-$types do not occur in the induced module X_M.

{p:6.6}

Proposition. *The $r_\sigma(w_1,((\nu)(\nu_0))$ are scalars. They equal*

$$\prod_{0\le j\le m-1}\frac{(\nu-\frac{k-1}{2})-(1-\epsilon)+j}{(\nu+\frac{k-1}{2})-(-n-\epsilon)-j}\cdot\frac{(-n-\epsilon)-(-\nu+\frac{k-1}{2})+j}{(1-\epsilon)-(-\nu-\frac{k-1}{2})-j} \tag{6.6.1}$$ {eq:6.6.1}

Proof. The intertwining operator $I_M(\nu)$ decomposes in the same way as (6.5.5). Furthermore, $\sigma_o(m,n+k-m)=\bigwedge^m\sigma_o(1,n+k-1)$. The difference from the cases σ_e is that while $\sigma_e(1,n+k-1)$ is the reflection representation, and therefore realized as the natural action on $\epsilon_1,\dots\epsilon_{n+k}$, $\sigma_o(1,n+k-1)$ occurs in $S^2\sigma_e(1,n+k-1)$, generated by $\epsilon_i^2-\epsilon_j^2$ with $i\ne j$. We can apply the same technique as for $\sigma_e(m,n+k-m)$, and omit the details. □

$\mathbf{GL(k)\subset G(k)}$ in types B, C. The formulas in proposition 6.5 and 6.6 hold with $n=0$. The proof is the same, but because $n=0$, (ν_0) is not present. The operator $I_{M',2}$ is an intertwining operator in $SL(2)$ and therefore simpler.

{sec:6.7}

6.7. $\mathbf{GL(k)} \subset \mathbf{G(k)}$ in type D. In this section we consider the maximal Levi components $M := GL(k) \subset G(k)$ and $M' := GL(k)' \subset G(k)$ for type D_n. The parameter corresponds to the string $(\nu) := (-\frac{k-1}{2}+\nu, \dots, \frac{k-1}{2}+\nu)$ or $(\nu') := (-\frac{k-1}{2}+\nu, \dots, -\frac{k-1}{2}-\nu)$.

k **even:** The $W-$structure of $X_M((\nu))$ and $X_{M'}((\nu)')$ is $\sigma_e[(n-r),(r)]$ for $0 \le r < k/2$, and $\sigma_e[(k/2),(k/2)]_I$, or $\sigma_e[k/2),(k/2)]_{II}$ respectively, with multiplicity 1. There are intertwining operators

{eq:6.7.1}

$$\begin{aligned} &I_M((\nu)) : X_M((\nu)) \longrightarrow X_M((-\nu)),\\ &I_{M'}((\nu)') : X_{M'}((\nu)') \longrightarrow X_{M'}((-\nu)'). \end{aligned} \tag{6.7.1}$$

corresponding to the shortest Weyl group element changing $((\nu))$ to $((-\nu))$. They determine scalars $r_\sigma((\nu))$ and $r_\sigma((\nu)')$.

k **odd:** The $W-$structure in this case is $\sigma_e[(n-r),(r)]$ with $0 \le r \le [k/2]$ for both X_M and $X_{M'}$, again with multiplicity 1. In this case there is a shortest Weyl group element which changes $((\nu))$ to $((-\nu)')$, and one which changes $((\nu)')$ to $((-\nu))$. These elements give rise to intertwining operators

{eq:6.7.2}

$$\begin{aligned} &I_M((\nu)) : X_M((\nu)) \longrightarrow X_{M'}((-\nu)'),\\ &I_{M'}((\nu)) : X_{M'}((\nu)') \longrightarrow X_M((-\nu)). \end{aligned} \tag{6.7.2}$$

Because the $W-$structure of X_M and $X_{M'}$ is the same, and $W-$types occur with multiplicity 1, these intertwning operators define scalars $r_\sigma(\nu)$ and $r_\sigma((\nu)')$.

{p:6.7}

Proposition. *The scalars $r_\sigma((\nu))$ and $r_\sigma((\nu)')$ are*

{eq:6.7.3}

$$r_{\sigma_e[(n-r),(r)]}((\nu)) = \prod_{0 \le j < r} \frac{(\frac{k-1}{2}-\nu)-j}{(\frac{k-1}{2}+\nu)-j}. \tag{6.7.3}$$

These numbers are the same for $((\nu))$ and $((\nu)')$, and representations with subscripts I, II they depend only on r.

{sec:6.8}

6.8. Proof of theorem 5.3. We use the results in the previous sections to prove the theorem in general. We give the details in the case of the group of type B and $W-$types σ_e. Thus the Hecke algebra is type C. There are no significant changes in the proof for the other cases. Recall the notation from section 2.3. Conjugate ν the middle element for the nilpotent orbit with partition $(2x_0, \dots, 2x_{2m})$ so that it is dominant (*i.e.* the coordinates are in decreasing order),

$$\nu = (x_{2m}-1/2, \dots, x_{2m}-1/2, \dots, x_0-1/2, \dots, x_0-1/2, \dots, 1/2, \dots, 1/2)$$

Then ν is dominant, so $X(\nu)$ has a unique irreducible quotient $L(\nu)$. We factor the long intertwining operator so that

{eq:6.8.1}

$$X(\nu) \xrightarrow{I_1} X_e(\nu) \xrightarrow{I_2} X(-\nu), \tag{6.8.1}$$

where X_e was defined in section 5.3. The claim will follow if the decomposition has the property that the operator I_1 is onto, and I_2 is into, when restricted to the σ_e isotypic component.

Proof that $\mathbf{I_1}$ is onto. The operator I_1 is a composition of several operators. First take the long intertwining operator induced from the Levi component $GL(n)$,

$$X(x_{2m}-1/2,\dots,1/2)\longrightarrow X(1/2,\dots,x_{2m}-1/2), \qquad (6.8.2)$$ {eq:6.8.2}

corresponding to the shortest Weyl group element that permutes the entries of the parameter from decreasing order to increasing order. The image is the induced from the corresponding irreducible spherical module $L(1/2,\dots,x_{2m}-1/2)$ on $GL(n)$. In turn this is induced irreducible from 1-dimensional spherical characters on a $GL(x_0)\times\dots\times GL(x_{2m})$ Levi component corresponding to the strings

$$(1/2,\dots,x_0-1/2)\dots(1/2,\dots,x_{2m}-1/2)$$

or any permutation thereof. This is well known by results of Bernstein-Zelevinski in the $p-$adic case, [V1] for the real case.

Compose with the intertwining operator

$$X(\dots(1/2,\dots,x_{2m}-1/2))\longrightarrow X(\dots(-x_{2m}+1/2,\dots,-1/2)), \qquad (6.8.3)$$ {eq:6.8.3}

all other entries unchanged. This intertwining operator is induced from the standard long intertwining operator on $G(x_{2m})$ which has image equal to the trivial representation. The image is an induced module from characters on $GL(x_0)\times\dots\times GL(x_{2m-1})\times G(x_{2m})$. Now compose with the intertwining operator

$$\begin{aligned}&X(\dots(1/2,\dots,x_{2m-1}-1/2)(-x_{2m}+1/2,\dots,-1/2)) \qquad (6.8.4)\\ &\quad\longrightarrow X(\dots(-x_{2m-1}+1/2,\dots,-1/2)(-x_{2m}+1/2,\dots,-1/2))\end{aligned}$$ {eq:6.8.4}

(again all other entries unchanged). This is $I_{M,2m-1}$ defined in (6.1.8), so its restriction of (6.8.4) to the σ_e isotypic component is an isomorphism. Now compose this operator with the one corresponding to

$$\begin{aligned}&X(\dots(1/2,\dots,x_{2m-2}-1/2)(-x_{2m-1}+1/2,\dots,1/2)\dots) \qquad (6.8.5)\\ &\quad\longrightarrow X(\dots(-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots)\end{aligned}$$ {eq:6.8.5}

with all other entries unchanged. This is induced from

$$GL(x_0)\times\dots\times GL(x_{2m-3})\times GL(x_{2m-2}+x_{2m-1})\times G(x_{2m})$$

and the image is the representation induced from the character corresponding to the string

$$(-x_{2m-1}-1/2,\dots,-1/2,1/2,\dots,x_{2m-2}) \text{ on } GL(x_{2m-2}+x_{2m-1}).$$

Now compose further with the intertwining operator

$$\begin{aligned}&X(\dots(-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)(-x_{2m}-1/2,\dots,-1/2)) \qquad (6.8.6)\\ &\quad\longrightarrow X((-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots(-x_{2m}-1/2,\dots,-1/2))\end{aligned}$$ {eq:6.8.6}

from the representation induced from

$$GL(x_0)\times\cdots\times GL(x_{2m-3})\times GL(x_{2m-2}+x_{2m-1})\times G(x_{2m})$$

to the induced from

$$GL(x_{2m-2}+x_{2m-1})\times GL(x_0)\times\cdots\times GL(x_{2m-3})\times G(x_{2m}).$$

By lemma 6.2, this intertwining operator is an isomorphism on any σ_e isotypic component. In fact, because the strings are strongly nested, the irreducibility results for $GL(n)$, 3.3 imply that the induced modules are isomorphic.

We have constructed a composition of intertwining operators from the standard module $X(\nu)$ where the coordinates of ν are positive and in decreasing order (*i.e.* dominant) to a module induced from

$$GL(x_{2m-2}+x_{2m-1})\times GL(x_0)\times\cdots\times GL(x_{2m-3})\times G(x_{2m})$$

corresponding to the strings

$$((-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)(1/2,\dots,x_0-1/2),\dots \\ \dots,(1/2,\dots,x_{2m-3}-1/2)(-x_{2m}+1/2,\dots,-1/2))$$

so that the restriction to any σ_e isotypic component is onto. We can repeat the procedure with x_{2m-4},x_{2m-3} and so on to get an intertwining operator from $X(\nu)$ to the induced from

$$GL(x_{2m-1}+x_{2m-2})\times\cdots\times GL(x_1+x_0)\times G(x_{2m})$$

corresponding to the strings

$$((-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots(-x_1+1/2,\dots,x_0-1/2), \\ (-x_{2m}+1/2,\dots,-1/2)).$$

This is the operator I_1, and it is onto on the $\sigma_e(*)$ isotypic components.

Proof that I_2 is into. We now deal with I_2. Consider the group $G(x_1+x_0+x_{2m})$ and the Levi component $M=GL(x_1+x_0)\times G(x_{2m})$. Let M' be the Levi component

{eq:6.8.7}
$$M':=GL(x_{2m-1}+x_{2m-2})\times\cdots\times GL(x_3+x_2)\times GL(x_1)\times GL(x_0)\times G(x_{2m}). \tag{6.8.7}$$

Then X_e embeds in $X_{M'}(\dots(-x_1+1/2,\dots-1/2)(1/2,\dots,x_0-1/2)(-x_{2m}+1/2,\dots,-1/2))$. The intertwining operator $I_{M',m+1}$ which changes the string $(x_0-1/2,\dots,1/2)$ to $(-x_0+1/2,\dots,-1/2)$ is an isomorphism on the σ_e $W-$types, by the results in sections 6.1-6.5. Since the strings are strongly nested, the operators $I_{M,i,i+1}$ are all isomorphisms, so we can construct an intertwining operator to an induced module $X_{M''}(\nu'')$ where

{eq:6.8.8}
$$M''=GL(x_1)\times GL(x_0)\times GL(x_{2m-1}+x_{2m-2})\times\cdots\times G(x_{2m}), \\ \nu''=(-x_1+1/2,\dots-1/2)(-x_0+1/2,\dots,-1/2)\dots) \tag{6.8.8}$$

which is an isomorphism on the σ_e isotypic components. Repeating this argument for $x_3,\ x_2$ up to x_{2m-1}, x_{2m-2} we get an intertwining operator from X_e to an induced module $X_{M^{(3)}}(\nu^{(3)})$ where

{eq:6.8.9}
$$
\begin{aligned}
&M^{(3)} := GL(x_1)\times GL(x_0)\times\cdots\times GL(x_{2m-1})\times GL(x_{2m-2})\times G(x_{2m})\\
&\nu^{(3)} := (-x_1+1/2,\dots,1/2)(-x_0+1/2,\dots,-1/2)\dots\\
&(-x_{2m-1}+1/2,\dots,-1/2)(-x_{2m-2}+1/2,\dots,-1/2)(-x_{2m}+1/2,\dots,-1/2).
\end{aligned}
\tag{6.8.9}
$$

which is an isomorphism on the σ_e isotypic components. Let

$$M^{(4)} := GL(x_1)\times GL(x_0)\times\cdots\times GL(x_{2m-1})\times G(x_{2m})$$

and let $\nu^{(4)}$ be the same as $\nu^{(3)}$ but the last string is viewed as giving a parameter of a 1-dimensional representation on $GL(x_{2m})$. Then $M^{(4)}\subset M^{(3)}$, and $X_{M^{(3)}}(\nu^{(3)})$ is a submodule of $X_{M^{(4)}}(\nu^{(4)})$. The induced module from $M^{(4)}$ to $M^{(5)} := GL(x_{2m}+\cdots+x_0)$ is irreducible because the strings are strongly nested on the GL factors. Thus the intertwining operator which takes $\nu^{(4)}$ to $-\nu$ is an isomorphism on $X_{M^{(4)}}(\nu^{(4)})$. So the induced intertwining operator to G is therefore injective and maps to $X(-\nu)$. The composition of all these operators is I_1, and is therefore injective on the σ_e-isotypic components. The proof is complete in this case.

The case of σ_o is similar, and we omit the details.

7. Necessary conditions for unitarity {sec:7}1

7.1. We will need the following notions.

Definition. *We will say a spherical irreducible module $L(\chi)$ is **r-unitary** if the form is positive on all the relevant $W-$types. Similarly, an induced module $X(\chi)$ is **r-irreducible** if all relevant $W-$types occur with the same multiplicity in $X(\chi)$ as in $L(\chi)$.*

{sec:7.2}

7.2. We recall (6.1.4),

$$\epsilon = \begin{cases} 1/2 & \text{G of type } B, \quad (\mathbb{H} \text{ of type } C)\\ 0 & \text{G of type } C, \quad (\mathbb{H} \text{ of type } B)\\ 1 & \text{G of type } D.\end{cases}$$

{d:7.2}

Definition. *A string of the form $(f+\nu,\dots,F+\nu)$ with $f,F\in\epsilon+\mathbb{Z}$ is called adapted, if it is*

of even length for G of type B,

of odd length for G of types C,D.

Otherwise we say the string is not adapted.

We will consider the following case. Let $\check{\mathcal{O}}\subset\check{\mathfrak{g}}$ correspond to the partition

$$\check{\mathcal{O}}\longleftrightarrow((a_1,a_1),\dots(a_r,a_r);d_1,\dots,d_l) \tag{7.2.1}$$ {eq:7.2.1}

so that $\check{\mathcal{O}}$ meets the Levi component $\check{\mathfrak{m}} = gl(a_1)\times\cdots\times gl(a_r)\times\check{\mathfrak{g}}(n_0)$, with $2n_0+[1-\epsilon] = d_1+\cdots+d_l$, where $[x]$ is the integer part of x. The intersection of $\check{\mathcal{O}}$ with each $gl(a_i)$ is the principal nilpotent, and the intersection with $\check{\mathfrak{g}}(n_0)$ is the **even** nilpotent orbit $\check{\mathcal{O}}_0$ with partition $(d_1,\dots,d_l)$. Let

{eq:7.2.2}
$$\begin{aligned}&\chi_i=(f_i+\nu_i,\dots,F_i+\nu_i), \qquad 1\le i\le r,\\ &\chi_0=\check{h}_0/2, \text{where } \check{h}_0 \text{ is a neutral element for } (d_1,\dots,d_l).\end{aligned}\tag{7.2.2}$$

and χ be the parameter obtained by concatenating the χ_i. Then $L(\chi)$ is the spherical subquotient of

{eq:7.2.3}
$$Ind_M^G[\bigotimes_{1\le i\le r} L(\chi_i)\otimes L(\chi_0)]\tag{7.2.3}$$

The next theorem gives necessary conditions for the unitarity of $L(\chi)$.

{t:7.2}

Theorem. *In types B,C, the nilpotent orbit $\check{\mathcal{O}}_0$ is arbitrary. In type D assume that either $\check{\mathcal{O}}_0\ne(0)$, or else that the rank is even. The representation $L(\chi)$ is unitary* ***only if***

(1) *Any string that is not adapted can be written in the form*

{eq:7.2.4}
$$(-E+\tau,\dots,E-1+\tau) \qquad 0<\tau\le 1/2,\ E\equiv\epsilon(mod\ \mathbb{Z}).\tag{7.2.4}$$

(2) *Any string that is adapted can be written in the form*

{eq:7.2.5}
$$\begin{aligned}&(-E+\tau,\dots,E+\tau) \qquad 0<\tau\le 1/2,\ E\equiv\epsilon(mod\ \mathbb{Z}),\\ &\mathit{or}\\ &(-E-1+\tau,\dots,E-1+\tau) \qquad 0<\tau\le 1/2,\ E\equiv\epsilon(mod\ \mathbb{Z}).\end{aligned}\tag{7.2.5}$$

This is simply the fact that the ν_j satisfy $0<\nu_j<1/2$ or $1/2<\nu_j<1$ in theorem 3.1. The proof will be given in the next sections. It is by induction on the dimension of $\check{\mathfrak{g}}$, the number of strings with coordinates in an A_τ (definition after (3.3.6)) with $\tau\ne 0$, and by downward induction on the dimension of $\check{\mathcal{O}}$. The unitarity of the representation when there are no coordinates in any A_τ with $\tau\ne0$ is done in section 9.

{sec:7.3}

7.3. The proposition in this section is a restatement of theorem 7.2 for the case of a parameter of the form (7.3.1). It is the first case in the initial step of the induction proof of theorem 7.2. I have combined the two cases in (2) into a single string $(-E+\nu,\dots,E+\nu)$ with $0<\nu<1$ by changing $(-E-1+\tau,\dots,E-1+\tau)$ into $(-E+(1-\tau),\dots,E+(1-\tau))$. This notation seemed more convenient for the case when there is a single such string present.

Consider the representation $L(\chi)$ corresponding to the strings

{eq:7.3.1}
$$(a+\epsilon+\nu,\dots,A+\epsilon+\nu)(-x_0+\epsilon,\dots,-1+\epsilon), \quad |a|\le A,\ 0<\nu<1,\tag{7.3.1}$$

where $a,A\in\mathbb{Z}$, and ϵ is as in 7.2. This is the case when $L(\chi)$ is the spherical subquotient of an induced from a character on a maximal parabolic subalgebra of the form $gl(A-a+1)\times\mathfrak{g}(x_0)$. The second string may not be present; these are the cases $x_0=-1$ for $\mathfrak{g}$ of type $B,\ D$, $x_0=0$ for type C.

{p:7.3}

Proposition. *In type D, assume that if there is no string* $(-x_0+\epsilon,\dots,-1+\epsilon)$*, then* $A-a+1$ *is even. Let* $L(\chi)$ *correspond to (7.3.1). Then* $L(\chi)$ *is r-unitary if and only if* $a+\epsilon=-A-\epsilon$*, and the following hold.*

(1) *Assume that* $(a+\epsilon+\nu,\dots,A+\epsilon+\nu)$ *is adapted. If* $x_0=A-a+1$*, then* $0\le\nu<1$*, otherwise* $\nu=0$.
(2) *If* $(a+\epsilon+\nu,\dots,A+\epsilon+\nu)$ *is not adapted, then* $0\le\nu<1/2$.

Proof. This is a corollary of the formulas in section 6.2. □ {sec:7.4}

7.4. Initial Step. We do the case when there is a single A_τ with $0<\tau<1/2$, and the coordinates form a single string. We write the string as in (7.3.1), $(a+\epsilon+\nu,\dots,A+\epsilon+\nu)$ with $0<\nu<1$. We let $\check{\mathcal{O}}$ be the nilpotent orbit with partition $((A-a+1,A-a+1);d_1,\dots,d_l)$. Let $\mathfrak{m}:=gl(A-a+1)\times\check{\mathfrak{g}}(n_0)$, and let $\check{\mathcal{O}}_0$ be the intersection of $\check{\mathcal{O}}$ with $\check{\mathfrak{g}}(n_0)$. In type D, either $\check{\mathcal{O}}_0\ne(0)$ or else $A-a+1$ is even. The statement of theorem 7.2 is equivalent to the following proposition.

{p:7.4}

Proposition. *Assume* $\check{\mathcal{O}}_0$ *is even, and* χ *is attached to* $\check{\mathcal{O}}$*. Then* $L(\chi)$ *is r-unitary only if* $a+\epsilon=-A-\epsilon$*, and the following hold.*

(1) *If* $(a+\epsilon+\nu,\dots,A+\epsilon+\nu)$ *is adapted, then* $\nu=0$*, unless there is* $d_j=A-a+1$*, in which case* $0\le\nu<1$.
(2) *If* $(a+\epsilon+\nu,\dots,A+\epsilon+\nu)$ *is not adapted, then* $0\le\nu<1/2$.

Proof. We do the case of G of type C only, the others are similar. So $\epsilon=0$, and adapted means the length of the string is odd, not adapted means the length of the string is even. The nilpotent orbit $\check{\mathcal{O}}_0$ corresponds to the partition $(2x_0+1,\dots,2x_{2m}+1)$ and the parameter has strings

$$(1,\dots x_0)(0,1,\dots,x_1)\dots(1,\dots,x_{2m}).$$

The partition of $\check{\mathcal{O}}$ is $(A-a+1,A-a+1,2x_0+1,\dots,2x_{2m}+1)$.

We want to show that if $A+a>0$, or if $A+a=0$ and there is no $x_i=A$, then $L(\chi)$ is *not* r-unitary. We do an upward induction on the rank of $\check{\mathfrak{g}}$ and a downward induction on the dimension of $\check{\mathcal{O}}$. In the argument below with deformations of strings, we use implicitly the irreducibility results from section 2.6. So the first case is when $\check{\mathcal{O}}$ is maximal, *i.e.* the principal nilpotent ($m=0$). The claim follows from proposition 7.3. So we assume that m is strictly greater than 0.

Assume $\mathbf{x_{2i}<A\le x_{2i+1}}$ for some i. This case includes the possibility $x_{2m}<A$. We will show by induction on rank of $\check{\mathfrak{g}}$ and dimension of $\check{\mathcal{O}}$ that the form is negative on a W-type of the form $\sigma[(n-r),(r)]$. So we use the module X_e (notation as in 5.3). If there is any pair $x_{2j}=x_{2j+1}$, the module X_e is unitarily induced from $GL(2x_{2j}+1)\times G(n-2x_{2j}-1)$ and all $W-$types $\sigma[(n-r),(r)]$ have the same multiplicity in $L(\chi)$ as in X_e. We can *remove* the string corresponding to $(x_{2j}x_{2j+1})$ in X_e as explained in section 3.2, lemma (3). By induction on rank we are done. Similarly we can remove any pair (x_{2j},x_{2j+1}) such that either $x_{2j+1}\le|a|$ or $A\le x_{2j}$ as follows. Let

$M := GL(x_{2j}+x_{2j+1}+1)\times G(n-x_{2j}-x_{2j+1}-1)$. There is χ_M such that $L(\chi)$ is the spherical subquotient of

{eq:7.4.1} $$Ind_M^G[L(-x_{2j+1},\dots,x_{2j})\otimes L(\chi_M)]. \tag{7.4.1}$$

Precisely, χ_M is obtained from χ by removing the entries $(1,\dots,x_{2j}),(0,1,\dots,x_{2j+1})$. Write

{eq:7.4.2} $$\chi_t := (-x_{2j+1}+t,\dots,x_{2j}+t;\chi_M). \tag{7.4.2}$$

The induced module

{eq:7.4.3} $$X_e(\chi_t) := Ind_M^G[L(-x_{2j+1}+t,\dots,x_{2j}+t)\otimes L(\chi_M)]. \tag{7.4.3}$$

has $L(\chi_t)$ as its irreducible spherical subquotient. For $0\le t\le \frac{x_{2j+1}-x_{2j}}{2}$, the multiplicities of $\sigma[(n-r,r)]$ in $L(\chi_t)$ and $X_e(\chi_t)$ coincide. Thus the signatures on the $\sigma[(n-r),(r)]$ in $L(\chi_t)$ are constant for t in the above interval. At $t=\frac{x_{2j+1}-x_{2j}}{2}$, $X_e(\chi_t)$ is unitarily induced from $triv\otimes X_e'$ on $GL(x_{2j}+x_{2j+1}+1)\times G(n-x_{2j}-x_{2j+1}-1)$ and we can *remove* the string corresponding to $(x_{2j}x_{2j+1})$. The induction hypothesis applies to X_e'.

When $A+a=0$, by the above argument, we are reduced to the case

{eq:7.4.4} $$\check{\mathcal{O}}_0 \longleftrightarrow (2x_0+1,2x_1+1,2x_2+1),\qquad x_0<A<x_1\le x_2. \tag{7.4.4}$$

We reduce to (7.4.4) when $A+a>0$ as well. We assume $2m=2i+2$, since pairs (x_{2j},x_{2j+1}) with $A\le x_{2j}$ can be *removed.* Suppose there is a pair (x_{2j},x_{2j+1}) such that $|a|<x_{2j+1}$, and $j\ne i$. The assumption is that $x_{2i}<A\le x_{2i+1}$ so $x_{2j+1}\le x_{2i}<A$.

We consider the deformation χ_t in (7.4.2) with

$$\begin{aligned} 0\le t<\nu,&\qquad a<0,\\ -\nu<t\le 0,&\qquad a\ge 0.\end{aligned}$$

In either case $X_e(\chi_t)=X_e(\chi)$, so the multiplicities of the $\sigma[(n-r),(r)]$ do not change until t reaches ν in the first case, $-\nu$ in the second case. If the signature on some $\sigma[(n-r),(r)]$ isotypic component is positive semidefinite on $L(\chi)$, the same has to hold when $t=\nu$ or $-\nu$ respectively. The corresponding nilpotent orbit for this parameter is strictly larger, but it has two strings with coordinates which are not integers (so the induction hypothesis does not apply yet). For example, if $a<0$, the strings for $X_e(\chi_\nu)$ are (aside from the ones that were unchanged)

{eq:7.4.5} $$(-x_{2j+1}+\nu,\dots,A+\nu),\qquad (a+\nu,\dots,x_{2j}+\nu). \tag{7.4.5}$$

We can deform the parameter further by replacing the second string by $(a+\nu-t',\dots,x_{2j}+\nu-t')$ with $0\le t'<\nu$. The strings of the corresponding X_e do not change until t' reaches ν. At $t'=\nu$ the corresponding nilpotent orbit $\check{\mathcal{O}}'$ has partition

{eq:7.4.6} $$(\dots,2|a|+1,\dots,\widehat{2x_{2j+1}+1},\dots,A+x_{2j+1}+1,A+x_{2j+1}+1,\dots) \tag{7.4.6}$$

which contains $\check{\mathcal{O}}$ in its closure. Since $x_{2j+1} < A$, the induction hypothesis applies. The form is indefinite on a $W-$type $\sigma[(n-r),(r)]$, so this holds for the original χ as well.

We have reduced to case (7.4.4), *i.e.* the partition of $\check{\mathcal{O}}_0$ has just three terms $(2x_0+1, 2x_1+1, 2x_2+1)$. We now reduce further to the case

$$\check{\mathcal{O}}_0 \longleftrightarrow (2x_0+1), \qquad x_0 < A. \tag{7.4.7}$$ {eq:7.4.7}

which is the initial step.

Let $I(t)$ be the induced module coresponding to the strings

$$(-x_2+t,\dots,x_1+t)(a+\nu,\dots,A+\nu)(-x_0,\dots,-1). \tag{7.4.8}$$ {eq:7.4.8}

i.e. induced from

$$GL(x_1+x_2)\times GL(-a+A+1)\times G(x_0). \tag{7.4.9}$$ {eq:7.4.9}

Consider the irreducible spherical module for the last two strings in (7.4.8), inside the induced module from the Levi component $GL(-a+A+1)\times G(x_0)\subset G(-a+A+1+x_0)$. By section 7.1, the form is negative on $\sigma[(x_0-a+A),(1)]$ if $x_0<a$, negative on $\sigma[(A),(x_0+1-a)]$ if $a\le x_0$. In the second case the form is positive on all $\sigma[(A+r),(x_0+1-a-r)]$ for $1<r<x_0+1-a$. So let $r_0 := 1$ or x_0+1-a depending on these two cases. The multiplicity formulas from section 6.2 imply that

$$[\sigma[(n-r_0),(r_0)] \,:\, I(t)] = [\sigma[(n-r_0),(r_0)] \,:\, L(\chi)] \quad \text{for} \quad 0\le t\le \frac{x_2-x_1}{2}.$$

Thus signatures do not change when we deform t to $\frac{x_2-x_1}{2}$, where $I(t)$ is unitarily induced. We conclude that the form on $L(\chi)$ is negative on $\sigma[(n-r_0),(r_0)]$.

Assume $\mathbf{x_{2i-1} < A \le x_{2i}}$. In this case we can do the same arguments using X_o and $\sigma[(k,n-k),(0)]$. We omit the details. □

{7.5}

7.5. Induction step. The case when the parameter has a single string with coordinates in an A_τ with $0<\tau<1/2$ was done in section 7.4. So we assume there is more than one string. Again we do the case G of type C, and omit the details for the other ones.

Write the two strings as in (2.6),

$$(e+\tau_1,\dots,E+\tau_1), \qquad (f+\tau_2,\dots,F+\tau_2). \tag{7.5.1}$$ {eq:7.5.1}

where $0<\tau_1\le 1/2$ and $0<\tau_2\le 1/2$. Recall that because we are in type C, $e,E,f,F\in\mathbb{Z}$, and $\epsilon=0$.

We need to show that if $F+f>0$ or $F+f<-2$ when $F+f$ is even, or $F+f<-1$ when $F+f$ is odd, then the form is negative on a relevant $W-$type. Because $\tau_1,\tau_2>0$, and since r-reducibility and r-unitarity are not affected by small deformations, we may as well assume that $(f+\tau_2,\dots,F+\tau_2)$ is the only string with coordinates in A_{τ_2}, and $(e+\tau_1,\dots,E+\tau_1)$ the only one with coordinates in A_{τ_1}.

The strategy is as follows. Assume that $L(\chi)$ is r-unitary. We deform (one of the strings of) χ to a χ_t in such a way that the coresponding induced module is r-irreducible over a finite interval, but is no longer so at the endpoint, say t_0. Because of the continuity in t, the module $L(\chi_{t_0})$ is still r-unitary. The deformation is such that $L(\chi_{t_0})$ belongs to a larger nilpotent orbit than $L(\chi)$, so the induction hypothesis applies, and we get a contradiction. Sometimes we have to repeat the procedure before we arrive at a contradiction.

So replace the first string by

{eq:7.5.2}
$$(e+\tau_1+t,\dots,E+\tau_1+t). \tag{7.5.2}$$

If $\chi=(e+\tau_1,\dots E+\tau_1;\chi_M)$, then

$$\begin{aligned}&\chi_t=(e+\tau_1+t,\dots,E+\tau_1+t;\chi_M),\\&X(\chi_t):=Ind_M^G[L(e+\tau_1+t,\dots,E+\tau_1+t)\otimes L(\chi_M)],\end{aligned}$$

where $\mathfrak{m}=gl(E-e+1)\times\mathfrak{g}(n-E+e-1)$,

If $E<|e|$, we deform t in the negative direction, otherwise in the positive direction. If $t+\tau_1$ reaches 0 or 1/2, before the nilpotent orbit changes, we should rewrite the string to conform to the conventions (2.6.10) and (2.6.11). This means that we rewrite the string as $(e'+\tau_1',\dots,E'+\tau_1')$ with $0\le\tau_1'\le 1/2$, and continue the deformation with a t going in the direction $t<0$ if $E'<|e'|$, and $t>0$ if $E'\ge|e'|$. This is not essential for the argument. We may as well assume that the following cases occur.

(1) The nilpotent orbit changes at $t_0=-\tau_1$.
(2) the nilpotent orbit does not change, and at $t_0=-\tau_1$, either $e,E>x_{2m}+1$ or $-e,-E>x_{2m}+1$. This is the *easy* case when t can be deformed to ∞ without any r-reducibility occuring.
(3) The nilpotent orbit changes at a t_0 such that $0<\tau_1+t_0\le 1/2$.

In the first case, the induction hypothesis applies, and since the string $(f+\tau_2,\dots,F+\tau_2)$ is unaffected, we conclude that the signature is negative on a relevant $W-$type. In the second case we can deform the string so that either $e+\tau_1+t=x_{2m}+1$ or $E+\tau_1+t=-x_{2m}-1$. The induction hypothesis applies, and the form is negative definite on a W-relevant type. In the third case, the only way the nilpotent orbit can change is if the string $(e+\tau_1+t_0,\dots,E+\tau_1+t_0)$ can be combined with another string to form a strictly longer string. If $\tau_1+t_0\ne\tau_2$, the induction hypothesis applies, and since the string $(f+\tau_2,\dots,F+\tau_2)$ is unaffected, the form is negative on a relevant $W-$type. If the nilpotent does not change at $t=\tau_2-\tau_1$, continue the deformation in the same direction. Eventually either (1) or (2) are satisfied, or else we are in case (3), and the strings in (7.5.1) combine to give a larger nilpotent. There are four cases:

{eq:7.5.3}
$$\begin{aligned} &(1)\quad e < f \le E \le F,\ e \le f \le E < F\\ &(2)\quad f \le e \le F < E,\ f < e \le F \le E\\ &(3)\quad e \le E = f-1 < F,\\ &(4)\quad f \le F = e-1 < E. \end{aligned} \tag{7.5.3}$$

Assume $|e| \le E$. Then t is deformed in the positive direction so $\tau_1 < \tau_2$. If $e \le 0$, we look at the deformation (7.5.2) for $-\tau_1 \le t \le 0$. If the nilpotent changes for some $-\tau_1 < t < 0$, the string $(f+\tau_2, \dots, F+\tau_2)$ is not involved, the induction hypothesis applies, so the parameter is not r-unitary. Otherwise at $t = -\tau_1$ there is one less string with coordinates in an A_τ with $\tau \ne 0$, and again the induction hypothesis applies so the original parameter is not r-unitary. Thus we are reduced to the case $0 < e < E$. Then consider the nilpotent orbit for the parameter with $t = -\tau_1 + \tau_2$. In cases (1), (2) and (3) of (7.5.3), the new nilpotent is larger, and one of the strings is

$$(e+\tau_2, \dots, F+\tau_2), \tag{7.5.4}$$ {eq:7.5.4}

instead of (7.5.1), and $e + F > 0$. The induction hypothesis applies, so the parameter is not r-unitary, nor is the original one.

In case (4) of (7.5.3), the new nilpotent corresponds to the strings

$$(f+\tau_2, \dots, E+\tau_2) \tag{7.5.5}$$ {eq:7.5.7}

The induction hypothesis applies, so $f + E = 0, -2$ if $f + E$ is even or $f + E = -1$ if it is odd. If this is the case, consider a new deformation in 7.5.2, this time $-1 + \tau_2 < t \le 0$. We may as well assume that the parameter is r-irreducible in this interval, or else the argument from before gives the desired conclusion. So we arrive at the case when $t = -1 + \tau_2$. The new nilpotent corresponds to the strings

$$(f+\tau_2, \dots, E-1+\tau_2), \quad (e-1+\tau_1), \quad (F+\tau_2). \tag{7.5.6}$$ {eq:7.5.8}

Write the parameter as $(\chi'; e-1+\tau_2, F+\tau_2)$. Since $e - 1 = F$, the induced module

$$I = Ind_{GL(2)\times G(n-2)}^{G}[L(e-1+\tau_2, F+\tau_2) \otimes L(\chi')] \tag{7.5.7}$$ {eq:7.5.9}

is unitarily induced from a module which is hermitian and r-irreducible. But the parameter on $GL(2)$ is not unitary unless $e - 1 = F = 0$. Furthermore $f + E - 1 = 0, -2$ if $f + E$ is odd or $f + E - 1 = -1$ if $f + E$ is even. So the original parameter (7.5.1) is

$$\begin{aligned} &(1+\tau_1, \dots, E+\tau_1), \quad (1-E+\tau_2, \dots, \tau_2) \quad f+E=0,\\ &(1+\tau_1, \dots, E+\tau_1), \quad (-E+\tau_2, \dots, \tau_2) \quad f+E=-2,\\ &(1+\tau_1, \dots, E+\tau_1), \quad (-E-1+\tau_2, \dots, \tau_2) \quad f+E=-1. \end{aligned} \tag{7.5.8}$$ {eq:7.5.10}

Apply the deformation $t + \tau_2$ in the second string with $-\tau_2 < t \le 0$. We may as well assume that the parameter stays r-irreducible in this interval. But then the induction hypothesis applies at $t = -\tau_2$ because there is one less string with coordinates in A_τ with $\tau \neq 0$. However the first string does no satisfy the induction hypothesis.

Assume $|e| > E$. The same argument applies, but this time it is $e < E < 0$ that requires extra arguments, and in case (3) instead of case (4) of (7.5.3) we have to consider several deformations.

{sec:7.6} **7.6. Proof of necessary condition for unitarity in theorem 3.1.** We first reduce to the case of theorem 7.2. The difference is that the coordinates in A_0 may not form a $\check{h}/2$ for an even nilpotent orbit. However because of theorem 2.9, and properties of petite $K-$types, r-reducibility and r-unitary are unaffected by small deformations of the $\chi'_1, \dots, \chi'_r$ (notation as in (2.9.3)). So we can deform the strings corresponding to $\chi'_1, \dots, \chi'_r$ with coordinates in A_0, so that their coordinates are no longer in A_0. Then the assumptions in theorem 7.2 are satisfied.

The argument now proceeds by analyzing each size of strings separately. In the deformations that we will consider, strings of different sizes cannot combine so that the nilpotent orbit attached to the parameter changes.

Fix a size of strings with coordinates not in A_0. If the strings are not adapted, they can be written in the form

{eq:7.6.1}
$$(-E-1+\tau_i, \dots, E+\tau_i) \qquad 0 < \tau_i \leq 1/2,\ E \equiv \epsilon (mod\ \mathbb{Z}). \tag{7.6.1}$$

So there is nothing to prove. Now consider a size of strings that are adapted. Suppose there are **two** strings of the form

{eq:7.6.2}
$$(-E-1+\tau_i, \dots, E-1+\tau_i), \quad 0 < \tau_i \leq 1/2,\ E \equiv \epsilon (mod\ \mathbb{Z}). \tag{7.6.2}$$

Let $\mathfrak{m} := gl(2E+1) \times \mathfrak{g}(n-2E-1)$, ($\mathfrak{g}(a)$ means a subalgebra/Levi component of the same type as $\mathfrak{g}$ of rank a) and write

{eq:7.6.3}
$$\chi := ((-E-1+\tau_i, \dots, E-1+\tau_i; -E-1+\tau_i, \dots, E-1+\tau_i); \chi_M). \tag{7.6.3}$$

The module

{eq:7.6.4}
$$Ind_M^G[L(-E-1+\tau_i, \dots, E-1+\tau_i; -E-1+\tau_i, \dots, E-1+\tau_i) \otimes L(\chi_M)] \tag{7.6.4}$$

is r-irreducible, and unitarily induced from a hermitian module on M where the module on $GL(2E+1)$ **is not unitary**. Thus $L(\chi)$ is not unitary either. So $L(\chi)$ is unitary only if for each τ_i there is at most one string of the form $(-E-1+\tau_i, \dots, E-1+\tau_i)$.

Suppose there are two strings as in (7.6.1) with $\tau_1 < \tau_2$. If there is no string $(-E+\tau_3, \dots, E+\tau_3)$ with $\tau_1 < \tau_3 < \tau_2$, then when we deform $(-E-1+\tau_1+t, \dots, E-1+\tau_1+t)$ for $0 \leq t \leq \tau_2 - \tau_1$, $X(\chi_t)$ stays r-irreducible. At $t = \tau_2 - \tau_1$ we are in case (7.6.2), so the parameter is not unitary.

On the other hand suppose that there are **two** strings of the form

{eq:7.6.5}
$$(-E+\tau_i, \dots, E+\tau_i), \qquad \text{same } \tau_i. \tag{7.6.5}$$

Let $\mathfrak{m}$ be as before, and write

{eq:7.6.6}
$$\chi := ((-E+\tau_i, \dots, E+\tau_i; -E+\tau_i, \dots, E+\tau_i); \chi_M). \tag{7.6.6}$$

The module

{eq:7.6.7} $$Ind_M^G[L(-E+\tau_i,\dots,E+\tau_i;-E+\tau_i,\dots,E+\tau_i)\otimes L(\chi_M)] \tag{7.6.7}$$

is irreducible, and unitarily induced from a hermitian module on M where the module on $GL(2E+1)$ **is unitary**. Thus $L(\chi)$ is unitary if and only if $L(\chi_M)$ is unitary.

So we may assume that for each τ_i there is at most one string of the form $(-E+\tau_i,\dots,E+\tau_i)$.

Similarly if there are two strings of the form $(-E+\tau_1,\dots,E+\tau_1)$ and $(-E+\tau_2,\dots,E+\tau_2)$, such that there is no string of the form $(-E-1+\tau_3,\dots,E-1+\tau_3)$ with $\tau_1<\tau_3<\tau_2$ we reduce to the case (7.6.5).

Let τ_k be the largest such that a string of the form $(-E+\tau_k,\dots,E+\tau_k)$ occurs, and τ_{k+1} the smallest such that a string $(-E-1+\tau_{k+1},\dots,E-1+\tau_{k+1})$ occurs. If $\tau_k>\tau_{k+1}$, we can deform $(-E+\tau_k+t,\dots,E+\tau_k+t)$ with $0\le t\le 1-\tau_k-\tau_{k+1}$. No r-reducibility occurs, and we are again in case (7.6.2). The module is not unitary. If on the other hand $\tau_k<\tau_{k+1}$, the deformation $(-E-1+\tau_{k+1}+t,\dots,E-1+\tau_{k+1}-t)$ for $0\le t\le 1-\tau_k-\tau_{k+1}$ brings us to the case (7.6.5).

Together the above arguments show that conditions (1) and (2) of theorem 3.1 in types C,D must be satisfied. Remains to check that for the case of adapted strings, if there is an odd number of a given size $2E+1$, then there is a $d_j=2E+1$. This is condition (3) in theorem 3.1.

The arguments above (also the unitarity proof in the case $\check{\mathcal{O}}=(0)$) show that an $L(\chi)$ is unitary only if it is of the following form. There is a Levi component $\mathfrak{m}=gl(a_1)\times\dots\times gl(a_r)\times\mathfrak{g}(n-\sum a_i)$, and parameters $\chi_1,\dots,\chi_r,\chi_0$ such that,

$$L(\chi)=Ind_M^G[\bigotimes L(\chi_i)\otimes L(\chi_0)], \tag{7.6.8}$$ {eq:7.6.8}

with the following additional properties:

(1) The χ_i for $i>0$ are as in lemma (1) of section 3.2, with $0<\nu<1/2$, in particular unitary.
(2) χ_0 is such that there is at most one string for every A_τ with $\tau\ne 0$, and the strings are of different sizes.

In addition, conditions (1) and (2) of theorem 3.1 are satisfied for the strings. To complete the proof we therefore only need to consider the case of $L(\chi_0)$. We can deform the parameters of the strings in the A_τ with $\tau\ne 0$ to zero without r-reducibility occuring. If $L(\chi)$ is unitary, then so is the parameter where we deform all but one $\tau\ne 0$ to zero. But for a parameter with a single string belonging to an A_τ with $\tau\ne 0$, the necessary conditions for unitarity are given in section 7.4.

8. Real nilpotent orbits {sec:8}

In this section we review some well known results for real nilpotent orbits. Some additional details and references can be found in [CM].

{sec:8.1}

8.1. Fix a real form $\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{g}_c$. Let θ_c be the complexification of the Cartan involution θ of $\mathfrak{g}$, and write $^-$ for the conjugation. Let G be the adjoint group with Lie algebra $\mathfrak{g}_c$, and let

{eq:8.1.1}
$$\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{s}_c, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{s} \tag{8.1.1}$$

be the Cartan decomposition. Write $K_c \subset G_c$ for the subgroup corresponding to $\mathfrak{k}_c$, and G and K for the real Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{k}$.

{t:8.1}

Theorem (Jacobson-Morozov)**.**

(1) *There is a one to one correspondence between G_c-orbits $\mathcal{O}_c \subset \mathfrak{g}_c$ of nilpotent elements and G_c-orbits of* Lie triples $\{e,\ h,\ f\}$ *i.e. elements satisfying*

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

This correspondence is realized by completing a nilpotent element $e \in \mathcal{O}$ to a Lie triple.

(2) *Two Lie triples $\{e,\ h,\ f\}$ and $\{e',\ h',\ f'\}$ are conjugate if and only if the elements h and h' are conjugate.*

{sec:8.2}

8.2. Suppose $e \in \mathfrak{g}$ is nilpotent. Then one can still complete it to a Lie triple $e,\ h,\ f \in \mathfrak{g}$. Such a Lie triple is called *real* or $\rho-$stable. A Lie triple is called *Cayley* if in addition $\theta(h) = -h,\ \theta(e) = f$. Every real Lie triple is conjugate by G to one which is Cayley.

{t:8.2}

Theorem (Kostant-Rao)**.** *Two real Lie triples are conjugate if and only if the elements $e - f$ and $e' - f'$ are conjugate under G. Equivalently, two Cayley triples are conjugate if and only if $e - f$ and $e' - f'$ are conjugate under K.*

{sec:8.3}

8.3. Suppose $e \in \mathfrak{s}_c$ is nilpotent. Then e can be completed to a Lie triple satisfying

{eq:8.3.1}
$$\theta_c(e) = -e, \qquad \theta_c(h) = h, \qquad \theta_c(f) = -f. \tag{8.3.1}$$

We call such a triple θ-stable. To any Cayley triple one can associate a θ-stable triple as in (8.3.1), by the formulas

{eq:8.3.2}
$$\widetilde{e} := \frac{1}{2}(e + f + ih), \quad \widetilde{h} := i(e - f), \quad \widetilde{f} := \frac{1}{2}(e + f - ih). \tag{8.3.2}$$

A Lie triple is called *normal* if in addition to (8.3.1) it satisfies $\overline{e} = f,\ \overline{h} = -h$.

{t:8.3}

Theorem (Kostant-Sekiguchi)**.**

(1) *Any θ-stable triple is conjugate via K_c to a normal one.*
(2) *Two nilpotent elements $\widetilde{e},\ \widetilde{e}' \in \mathfrak{s}$ are conjugate by K_c, if and only if the corresponding Lie triples are conjugate by K_c. Two θ-stable triples are conjugate under K if and only if the elements $\widetilde{h},\ \widetilde{h}'$ are conjugate under K_c.*

(3) *The correspondence (8.3.2) is a bijection between G orbits of nilpotent elements in $\mathfrak{g}$ and K_c orbits of nilpotent elements in $\mathfrak{s}_c$.*

{t:8.3.1}

Proposition. *The correspondence between real and θ stable orbits preserves closure relations.*

Proof. This is the main result in [BS]. □

{sec:8.4}

8.4. Let $\mathfrak{p}_c = \mathfrak{m}_c + \mathfrak{n}_c$ be a parabolic subalgebra of $\mathfrak{g}_c$. Let $\mathfrak{c}_c := \operatorname{Ad} M_c \cdot e$ be the orbit of a nilpotent element $e \in \mathfrak{m}_c$. According to [LS], the induced orbit from $\mathfrak{c}_c$ is the unique G_c orbit $\mathfrak{C}_c$ which has the property that $\mathfrak{C}_c \cap [\mathfrak{c}_c + \mathfrak{n}_c]$ is dense (and open) in $\mathfrak{c} + \mathfrak{n}_c$.

Proposition (1)**.** *Let $E = e + n \in e + \mathfrak{n}_c$.*

(1) $\dim Z_{M_c}(e) = \dim Z_{G_c}(E)$.
(2) $\mathfrak{C}_c \cap [\mathfrak{c}_c + \mathfrak{n}_c]$ *is a single P_c orbit.*

This is theorem 1.3 in [LS]. In particular, an element $E' = e' + n' \in \mathfrak{c}_c + \mathfrak{n}_c$ is in $\mathfrak{C}_c$ if and only if the map

$$\operatorname{ad} E' : \mathfrak{p}_c \longrightarrow T_{e'}\mathfrak{c} + \mathfrak{n}_c, \qquad \operatorname{ad} E'(y) = [E', y] \tag{8.4.1}$$ {eq:8.4.1}

is onto.

Another characterization of the induced orbit is the following.

Proposition. *The orbit $\mathfrak{C}_c$ is the unique open orbit in* $\operatorname{Ad} G_c(e + \mathfrak{n}_c) = \operatorname{Ad} G_c(\mathfrak{c}_c + \mathfrak{n}_c)$, *as well as in the closure* $\overline{\operatorname{Ad} G_c(e + \mathfrak{n}_c)} = \operatorname{Ad} G_c(\overline{\mathfrak{c}_c} + \mathfrak{n}_c)$.

We omit the proof, but note that the statements about the closures follow from the fact that G_c/P_c is compact.

Proposition (2)**.** *The orbit $\mathfrak{C}_c$ depends on $\mathfrak{c}_c \subset \mathfrak{m}_c$, but not on $\mathfrak{n}_c$.*

Proof. This is proved in section 2 of [LS]. We give a different proof which generalizes to the real case. Let $\xi \in \mathfrak{h}_c \subset \mathfrak{m}_c$ be an element in the center of $\mathfrak{m}_c$ such that $\langle \xi, \alpha \rangle \neq 0$ for all roots $\alpha \in \Delta(\mathfrak{n}_c, \mathfrak{h}_c)$. Then by a standard argument,

$$\operatorname{Ad} P_c(\xi + e) = \xi + \mathfrak{c}_c + \mathfrak{n}_c. \tag{8.4.2}$$ {eq:8.4.2}

Again because G_c/P_c is compact,

$$\overline{\bigcup_{t>0} \operatorname{Ad} G_c(t\xi + e)} \setminus \bigcup_{t>0} \operatorname{Ad} G_c(t\xi + e) = \operatorname{Ad} G_c(\overline{\mathfrak{c}_c} + \mathfrak{n}_c). \tag{8.4.3}$$ {eq:8.4.3}

Formula 8.4.3 is valid for any parabolic subgroup with Levi component M_c. The claim follows because the left hand side of (8.4.3) only depends on M_c and the orbit $\mathfrak{c}_c$. □

We now consider the case of real induction. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a real parabolic subalgebra, $e \in \mathfrak{m}$ a nilpotent element, and $\mathfrak{c} := \operatorname{Ad} Me$.

{d:8.4}

Definition. *The $\rho-$induced set from $\mathfrak{c}$ to $\mathfrak{g}$ is the finite union of orbits $\mathfrak{C}_i := \operatorname{Ad} GE_i$ such that one of the following equivalent conditions hold.*

(1) $\mathfrak{C}_i$ *is open in* $\operatorname{Ad} G(\mathfrak{c}+\mathfrak{n})$ *and* $\overline{\bigcup \mathfrak{C}_i} = \overline{\operatorname{Ad} G(e+\mathfrak{n})}$.
(2) *The intersection* $\mathfrak{C}_i \cap [\mathfrak{c}+\mathfrak{n}]$ *is open in* $\mathfrak{c}+\mathfrak{n}$, *and the union of the intersections is dense in* $\mathfrak{c}+\mathfrak{n}$.

We write

{eq:8.4.4}
$$ind_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{c}) = \bigcup \mathfrak{C}_i. \tag{8.4.4}$$

and we say that each E_i *is real or* $\rho-$*induced from* e. *Some times we will write* $ind_{\mathfrak{p}}^{\mathfrak{g}}(e)$.

We omit the details of the proof of the equivalence of the two statements.

{p:8.4}
Proposition (3). *The* $\rho-$*induced set depends on the orbit* $\mathfrak{c}$ *of* e *and the Levi component* $\mathfrak{m}$, *but not on* $\mathfrak{n}$.

Proof. The proof is essentially identical to the one in the complex case. We omit the details. □

In terms of the θ-stable versions $\widetilde{e}$ of e, and $\widetilde{E}_i$ of E_i, $\rho-$induction is computed in [BB]. This is as follows. Let $\mathfrak{h}_c \subset \mathfrak{m}_c$ be the complexification of a maximally split real Cartan subalgebra $\mathfrak{h}$, and $\xi \in \mathcal{Z}(\mathfrak{m}_c) \cap \mathfrak{s}_c$ an element of $\mathfrak{h}$ such that

$$\alpha \in \Delta(\mathfrak{n}, \mathfrak{h}) \text{ if and only if } \alpha(\xi) > 0.$$

Then

{eq:8.4.5}
$$\overline{\bigcup \operatorname{Ad} K_c(\widetilde{E}_i)} = \overline{\bigcup_{t>0} \operatorname{Ad} K_c(t\xi + \widetilde{e})} \setminus \bigcup_{t>0} \operatorname{Ad} K_c(t\xi + \widetilde{e}). \tag{8.4.5}$$

{sec:8.5}
8.5. Let $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be a θ-stable parabolic subgroup, and write $\overline{\mathfrak{q}_c} = \mathfrak{l}_c + \overline{\mathfrak{u}_c}$ for its complex conjugate. Let $e \in \mathfrak{l}_c \cap \mathfrak{s}_c$ be a nilpotent element.

{p:8.5}
Proposition. *There is a unique* K_c-*orbit orbit* $\mathcal{O}_{K_c}(E)$ *so that its intersection with* $\mathcal{O}_{L_c \cap K_c}(e) + (\mathfrak{u}_c \cap \mathfrak{s}_c)$ *is open and dense.*

Proof. This follows from the fact that $e + (\mathfrak{u}_c \cap \mathfrak{s}_c)$ is formed of nilpotent orbits, there are a finite number of nilpotent orbits, and being complex, the K_c-orbits have even real dimension. □

{d:8.5}
Definition. *The orbit* $\mathcal{O}_{K_c}(E)$ *as in the proposition above is called* $\theta-$*induced from* e, *and we write*

$$ind_{\mathfrak{q}_c}^{\mathfrak{g}_c}(\mathcal{O}_{\mathfrak{l}_c}(e)) = \mathcal{O}(E),$$

and say that E *is* $\theta-$*induced from* e.

Remark. The induced orbit is characterized by the property that it is the (unique) largest dimensional one which meets $e + \mathfrak{u}_c \cap \mathfrak{s}_c$. It depends on e as well as $\mathfrak{q}_c$, not just e and $\mathfrak{l}_c$.

{sec:8.7}

8.6. u(p,q). Let V be a complex finite dimensional vector space of dimension n. There are two inner classes of real forms of $gl(V)$. One is such that θ is an outer automorphism. It consists of the real form $GL(n,\mathbb{R})$, and when n is even, also $U^*(n)$. The other one is such that θ is inner, and consists of the real forms $U(p,q)$ with $p+q=n$. In sections 8.6-8.12, we investigate ρ and θ induction for the forms $u(p,q)$, and then derive the corresponding results for $so(p,q)$ and $sp(n,\mathbb{R})$ from them in sections 8.13-8.14. The corresponding results for the other real forms area easier, the case of $GL(n,\mathbb{R})$ is well known, and we will not need $u(n)^*$. The usual description of $u(p,q)$ is that V is endowed with a hermitian form $(\ ,\ )$ of signature (p,q), and $u(p,q)$ is the Lie algebra of skew hermitian matrices with respect to this form. Fix a positive definite hermitian form $\langle\ ,\ \rangle$. We will identify the complexification of $\mathfrak{g} := u(p,q)$ with $\mathfrak{g}_c := gl(V)$, and the complexification of $U(p,q)$ with $GL(V)$. Up to conjugacy by $GL(V)$,

$$(v,w) = \langle \theta v, w\rangle, \qquad \theta^2 = 1, \tag{8.6.1}$$ {eq:8.7.1}

The eigenspaces of θ on V will be denoted $V^{\pm}$. The Cartan decomposition is $\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{s}_c$, where $\mathfrak{k}$ is the $+1$ eigenspace, and $\mathfrak{s}$ the -1 eigenspace of $\operatorname{Ad}\theta$.

The classification of nilpotent orbits of $u(p,q)$ is by signed tableaus as in theorem 9.3.3 of [CM]. The same parametrization applies to θ stable orbits under K_c. We need some results about closure relations between nilpotent orbits. For a θ-stable nilpotent element e, we write $a_\pm(e^k)$ for the signature of θ on the kernel of e^k, and $a(e^k) = a_+(e^k) + a_-(e^k)$ for the dimension of the kernel. If it is clear what nilpotent element they refer to, we will abbreviate them as $a_\pm(k)$. The interpretation of these numbers in terms of signed tableaus is as follows. $(a_+(k), a_-(k))$ are the numbers of $+$'s respectively $-$'s in the longest k columns of the tableau.

{t:8.7.2}

Theorem. *Two $\theta-$stable nilpotent elements e and e' are conjugate by K_c if and only if e^k and e'^k have the same signatures. The relation $\mathcal{O}_{K_c}(e') \subset \overline{\mathcal{O}_{K_c}(e)}$ holds if and only if for all k,*

$$a_+(e'^k) \geq a_+(e^k), \qquad a_-(e'^k) \geq a_-(e^k).$$

Proof. For real nilpotent orbits, the analogue of this result is in [D]. The theorem follows by combining [D] with proposition 8.3. We sketch a direct proof, ommitting most details except those we will need later.

Decompose

$$V = \bigoplus V_i$$

into $sl(2)$ representations which are also stabilized by θ. Let ϵ_i be the eigenvalue of θ on the highest eigenweight of V_i (also the kernel of e). We encode the information about e into a *tableau* with rows equal to the dimensions of V_i and alternate signs $+$ and $-$ starting with the sign of ϵ_i. The number of $+$'s and $-$'s in the first column gives the signature of θ on the kernel of e.

Then the number of $\pm$ in the first two columns gives the signature of θ on the kernel of e^2 and so on. The total number of $+$'s equals p, the number of $-$'s equals q. Write $V = V_+ + V_-$, where $V_\pm$ are the ± 1 eigenspaces of θ. The element e is given by a pair (A, B), where $A \in \mathrm{Hom}[V_+, V_-]$, and $B \in \mathrm{Hom}[V_-, V_+]$. Then e^k is represented by $(ABAB\dots, BABA\dots)$, k factors each, and $a_\pm(k)$ is the dimension of the kernel of the corresponding composition of A and B. The fact that the condition in the theorem is necessary, follows from this interpretation. $\square$

{sec:8.8}

8.7. A parabolic subalgebra of $gl(V)$ is the stabilizer of a generalized flag

{eq:8.8.1}
$$(0) = W_0 \subsetneq W_1 \subsetneq \dots \subsetneq W_k = V. \tag{8.7.1}$$

Fix complementary spaces V_i,

{eq:8.8.2}
$$W_i = W_{i-1} + V_i, \qquad i > 0. \tag{8.7.2}$$

They determine a Levi component

{eq:8.8.1.2}
$$\mathfrak{l} \cong gl(V_1) \times \dots \times gl(V_k). \tag{8.7.3}$$

{sec:8.9}

8.8. Conjugacy classes under K_c of θ-stable parabolic subalgebras are parametrized by ordered pairs $(p_1, q_1), \dots, (p_k, q_k)$ such that the sum of the p_i is p, and the sum of the q_i is q. A realization in terms of flags is as follows. Choose the W_i to be stable under θ, or equivalently that the restriction of the hermitian form to each W_i is nondegenerate. In this case we may assume that the V_i are θ-stable as well, and let $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be the corresponding parabolic subalgebra of $gl(V)$. The signature of the form restricted to V_i is (p_i, q_i), so that

{eq:8.9.1}
$$\mathfrak{l}_c \cap \mathfrak{g} \cong u(p_1, q_1) \times \dots \times u(p_k, q_k). \tag{8.8.1}$$

{sec:8.10}

8.9. Conjugacy classes under G of real parabolic subalgebras are given by ordered subsequences $n_1, \dots, n_k$ and a pair (p_0, q_0) such that $\sum n_i + p_0 = p$ and $\sum n_i + q_0 = q$. The complexification of the corresponding *real* parabolic subalgebra is given as follows. Start with a partial flag

{eq:8.10.1}
$$(0) = W_0 \subsetneq \dots \subsetneq W_k \tag{8.9.1}$$

such that the hermitian form is trivial when restricted to W_k, and complete it to

{eq:8.10.2}
$$(0) = W_0 \subsetneq \dots \subsetneq W_k \subsetneq W_k^* \subsetneq \dots \subsetneq W_0^* = V \tag{8.9.2}$$

Choose transverse spaces

{eq:8.10.3}
$$W_i = W_{i-1} + V_i, \qquad W_i^* = W_{i-1}^* + V_i^*, \qquad W_k^* = W_k + V_0. \tag{8.9.3}$$

They determine a Levi component

{eq:8.8.10.4}
$$\mathfrak{l}_c = gl(V_1) \times \dots \times gl(V_k) \times gl(V_0) \times gl(V_k^*) \times \dots \times gl(V_1^*), \tag{8.9.4}$$

so that

{eq:8.10.5}
$$\mathfrak{l}_c \cap \mathfrak{g} = gl(V_1, \mathbb{C}) \times \dots \times gl(V_k, \mathbb{C}) \times u(p_0, q_0). \tag{8.9.5}$$

Then $n_i = \dim V_i$, and (p_0, q_0) is the signature of V_0.

{sec:8.11} **8.10.** Let now $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be a maximal θ stable parabolic subalgebra corresponding to the flag $W_1 = V_1 \subsetneq W_2 = V_1 + V_2 = V$. Then

{eq:8.11.1}
$$\mathfrak{l}_c \cong \mathrm{Hom}[V_1, V_1] + \mathrm{Hom}[V_2, V_2] = gl(V_1) \times gl(V_2), \qquad \mathfrak{u}_c \cong \mathrm{Hom}[V_2, V_1]. \tag{8.10.1}$$

Write $n_i := \dim V_i$, and $\theta = \theta_1 + \theta_2$ with $\theta_i \in End(V_i)$. A nilpotent element $e \in gl(V_2)$ satisfying $\theta_2 e = -e\theta_2$, can be viewed as a $\theta-$stable nilpotent element in $\mathfrak{l}_c$ by making it act by 0 on V_1. Let $E = e + X$, with $X \in \mathfrak{u}_c$ (so $X \in gl(V)$ acts by 0 on V_1). Then $X\theta_2 = -\theta_1 X$. Decompose

$$V_2 = \bigoplus W_i^{\epsilon_i} \tag{8.10.2}$$ {eq:8.11.2}

where $W_i^{\epsilon_i}$ are irreducible $sl(2, \mathbb{C})$ representations stabilized by θ such that the eigenvalue of θ_2 on the highest weight v_i is ϵ_i. Order the $W_i^{\epsilon_i}$ so that $\dim W_i \geq \dim W_{i+1}$. Write $A_\pm(k)$ for the signatures of E^k and $a_\pm(k)$ for the signatures of e^k.

{p:8.11}

Proposition. *The signature* $(A_+(k),\ A_-(k))$ *of* E^k *satisfies*

$$\begin{aligned}
A_+(k) \geq & \dim V_{1,+} + a_+(k-1) + \\
& + \max\left[0\ ,\ \#\{\ i \mid \ \dim W_i^{\epsilon_i} \geq k, \epsilon_i = (-1)^{k-1}\} - \dim V_1^{(-1)^k}\right],\\
A_-(k) \geq & \dim V_{1,-} + a_-(k-1) + \\
& + \max\left[0\ ,\ \#\{\ i \mid \ \dim W_i^{\epsilon_i} \geq k, \epsilon_i = (-1)^{k}\} - \dim V_1^{(-1)^{k-1}}\right].
\end{aligned}$$

Proof. Since $E^k = e^k + Xe^{k-1}$, an element $v \in V_2$, is in the kernel of E^k if and only if $e^{k-1}v$ is in the kernel of X as well as e. Thus $V_1 \subset \ker E$. This accounts for the terms $\dim V_1^{\pm}$. Since $\ker e^{k-1} \subset \ker Xe^{k-1} \cap \ker e^k$, this accounts for the terms $a_\pm(k-1)$.

The representation theory of $sl(2, \mathbb{C})$ implies

$$\ker e \cap \mathrm{Im}\, e^{k-1} = span\{v_i^{\epsilon_i} \mid e \cdot v_i^{\epsilon_i} = 0,\ \ dim W_i^{\epsilon_i} \geq k\} \tag{8.10.3}$$ {eq:8.11.3}

If the sign of v_i is ϵ_i, and $v_i = e^{k-1}w_j$, then the sign of θ on w_j is $\epsilon_j(-1)^{k-1}$. Then $X : V_2^{\epsilon_i} \longrightarrow V_1^{-\epsilon_i}$, and the minimum possible dimension of the kernel of X on the space in (8.10.3) is the last term in the inequalities of the proposition. The claim follows. □

{sec:8.12}

8.11. We now construct an E such that the inequalities in proposition 8.10 are equalities.

For any integers $a,\ b$, let

$$\begin{aligned}
\mathbb{K}_a^+ &:= span\{\text{ first } a\ v_i \text{ with } \epsilon_i = 1\},\\
\mathbb{K}_b^- &:= span\{\text{ first } b\ v_j \text{ with } \epsilon_j = -1\}
\end{aligned} \tag{8.11.1}$$ {eq:8.12.1}

Note that

$$X(\mathbb{K}_a^+) \subset V_1^-, \qquad X(\mathbb{K}_b^-) \subset V_1^+. \tag{8.11.2}$$ {eq:8.11.4}

{t:8.12}

Theorem. *Let $E = e + X$ with notation as in 8.11.2. Choose X such that it is nonsingular on $\mathbb{K}^{\pm}_{a,b}$ for as large an a and b as possible. Then* $\operatorname{Ad} K_c(E) = ind_{\mathfrak{q}}^{\mathfrak{g}_c} e$.

Proof. ¿From the proposition it follows that the $A^k_{\pm}$ of any element in $e + (\mathfrak{u}_c \cap \mathfrak{s}_c)$ are minimal when they are equal to the RHS of proposition 8.10. Theorem 8.6 implies that if a nilpotent element achieves this minimum, its orbit contains any other $e + X$ in its closure. This minumum is achieved by the choice of X in the proposition, bering in mind that the W_i were ordered in decreasing order of their dimension. Thus $\operatorname{Ad} K_c(E)$ has maximal dimension among all orbits meeting $e + (\mathfrak{u} \cap \mathfrak{s})$ and so the claim follows from the observation at the end of 8.5. □

This theorem implies the following algorithm for computing the induced orbit in the case $\mathfrak{g} \cong u(p, q)$:

Suppose the signature of V_1 is (a_+, a_-). Then add a_+ $+$'s to the beginning of largest possible rows of e starting with a $-$ and a_- $-$'s to the largest possible rows of e starting with a $+$. If a_+ is larger than the number of rows starting with $-$, add a new row of size 1 starting with $+$. The similar rule applies to a_-.

If $e \in gl(V_1)$, the analogous procedure applies, but the a_+ $+$'s are added at the end of the largest possible rows finishing in $-$ and a_- $-$'s to the end of the largest possible rows finishing in $+$.

Because induction is transitive, the above algorithm can be generalized to compute the θ-induced of any nilpotent orbit. We omit the details.

{sec:8.13}

8.12. Suppose $\mathfrak{p}_c = \mathfrak{m}_c + \mathfrak{n}_c$ is the complexification of a real parabolic subalgebra corresponding to the flag $(0) \subset V_1 \subset V_1 + V_0 \subset V_1 + V_0 + V_1^*$, and let $e \subset gl(V_0)$ be a real nilpotent element. The rest of the notation is as in section 8.4.

{t:8.13}

Theorem. *The tableau of an orbit $\operatorname{Ad} G(E_i)$ which is in the $\rho-$induced set $ind_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{c})$, is obtained from the tableau of e as follows.*

Add two boxes to the end of each of $\dim V_1$ of the largest rows such that the result is still a signed tableau.

Proof. We use (8.4.2) and (8.4.3). Let $\alpha \in \operatorname{Hom}[V_1, V_1^*] \oplus \operatorname{Hom}[V_1^*, V_1]$ be nondegenerate such that $\alpha^2 = Id \oplus Id$, and extend it to an endomorphism $\xi \in gl(V)$ so that its restriction to V_0 is zero. This is an element such that the centralizer of $\operatorname{ad} \xi$ is $\mathfrak{m}$, in particular, $[\xi, e] = 0$. Let

{eq:8.13.1}

$$P(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0 \tag{8.12.1}$$

be any polynomial in $X \in gl(V)$. Suppose $t_i \in \mathbb{R}$ are such that $t_i \to 0$, and assume there are $g_i \in K$ such that $t_i g_i(\xi + e)g_i^{-1} \to E$. Then

{eq:8.13.2}

$$\ker t_i^m P(g_i(\xi + e)g_i^{-1}) \cong \ker P(\xi + e). \tag{8.12.2}$$

On the other hand,

{eq:8.13.3}
$$t_i^m P(g_i(\xi+e)g_i^{-1}) = [t_i g_i(\xi+e)g_i^{-1}]^m + \\ + a_{m-1}t_i[t_i g_i(\xi+e)g_i^{-1}]^{m-1} + \cdots + t_i^m Id \to E^m, \tag{8.12.3}$$

as $t_i \to 0$. Thus

$$\dim\ker E^m \mid_{V_\pm} \geq \dim\ker P(\xi+e) \mid_{V_\pm} . \tag{8.12.4}$$ {eq:8.13.4}

Choosing $P(X) = (X^2-1)X^n$, we conclude that E must be nilpotent. Choosing $P(X) = X^m$, $(X \pm 1)X^{m-1}$ or $P(X) = (X^2-1)X^{m-2}$, we can bound the dimensions of $\ker E^m \mid_{V_\pm}$ to conclude that it must be in the closure of one of the nilpotent orbits given by the algorithm of the theorem. The fact that these nilpotent orbits are in (8.4.3) follows by a direct calculation which we omit. □

{sec:8.14}

8.13. sp(V). Suppose $\mathfrak{g}_c \cong sp(V_0)$, where $(V_0, \langle\ ,\ \rangle)$ is a real symplectic vector space of dimension n. The complexification $(V, \langle\ ,\ \rangle)$ admits a complex conjugation $^-$, and we define a nondegenerate hermitian form

$$(v,w) := \langle v, \overline{w}\rangle \tag{8.13.1}$$ {eq:8.14.1}

which is of signature (n,n). Denote by $u(n,n)$ the corresponding unitary group. Since $sp(V_0)$ stabilizes $(\ ,\ )$, it embeds in $u(n,n)$, and the Cartan involutions are compatible. The results of sections 8.1-8.3 together with section 8.6 imply the following classification of nilpotent orbits of $sp(V_0)$ or equivalently θ-stable nilpotent orbits. See chapter 9 of [CM] for a more detailed explanation.

Each orbit corresponds to a signed tableau so that every odd part occurs an even number of times. Odd sized rows occur in pairs, one starting with + the other with −.

A real parabolic subalgebra of $sp(V)$ is the stabilizer of a flag of isotropic subspaces

$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k, \tag{8.13.2}$$ {eq:8.14.2}

so that the symplectic form restricts to 0 on $\mathcal{W}_k$. As before, complete this to a flag

$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k \subset \mathcal{W}_k^* \subset \cdots \subset \mathcal{W}_0^* = V. \tag{8.13.3}$$ {eq:8.14.3}

We choose transverse spaces

$$\mathcal{W}_i = \mathcal{W}_{i-1} + V_i, \quad \mathcal{W}_k^* = \mathcal{W}_k + \mathcal{W}, \quad \mathcal{W}_{i-1}^* = \mathcal{W}_i^* + V_i^* \tag{8.13.4}$$ {eq:8.14.4}

in order to fix a Levi component. We get

$$\mathfrak{l} \cong gl(V_1) \times \cdots \times gl(V_k) \times sp(\mathcal{W}). \tag{8.13.5}$$ {eq:8.14.5}

If we assume that V_i, $\mathcal{W}$ are θ-stable, then the corresponding parabolic subalgebra is θ-stable as well, and the real points of the Levi component are

$$\mathfrak{l}_0 \cong u(p_1,q_1) \times \cdots \times u(p_k,q_k) \times sp(\mathcal{W}_0). \tag{8.13.6}$$ {eq:8.14.6}

where (p_i, q_i) is the signature of V_i. The parabolic subalgebra corresponding to (8.13.4) in $gl(V)$ satisfies

$$\mathfrak{l}' \cong u(p_1, q_1)\times\cdots\times u(p_k, q_k)\times u(n_0, n_0)\times u(q_k, p_k)\times\cdots\times u(q_1, p_1). \quad (8.13.7)$$ {eq:8.14.7}

For a maximal θ-stable parabolic subalgebra, the Levi component $\mathfrak{l}$ satisfies $\mathfrak{l} \cong u(p_1, q_1) \times sp(\mathcal{W}_0)$. Let $e \in sp(W)$ be a θ-stable nilpotent element. The algorithm for induced nilpotent orbits in section 8.8 implies the following algorithm for $ind_{\mathfrak{l}}^{\mathfrak{g}_c}(e)$.

(1) add p boxes labelled $+$'s to the beginning of the longest rows starting with $-$'s, and q $-$'s to the beginning of the longest rows starting with $+$'s.
(2) add q $+$'s to the ending of the longest possible rows starting with $-$'s, and p $-$'s to the beginning of the longest possible rows starting with $+$'s.

Unlike in the complex case, the result is automatically the signed tableau corresponding to a nilpotent element in $sp(V)$.

For a maximal real parabolic subalgebra, we must assume that $\overline{V_1} = V_1$, $\overline{\mathcal{W}} = \mathcal{W}$. Let $V_{1,0}$ and $\mathcal{W}_0$ be their real points. The Levi component satisfies

{eq:8.14.8}
$$\mathfrak{l} \cong gl(V_{1,0}) \times sp(\mathcal{W}_0). \quad (8.13.8)$$

The results in section 8.12 imply the following algorithm for real induction.

(1) add two boxes to the largest $\dim V_1$ rows of e so that the result is still a signed tableau for a nilpotent orbit.
(2) Suppose $\dim V_1$ is odd and the last row that would be increased by 2 is odd size as well. In this case there is a pair of rows of this size, one starting with $+$ the other with $-$. In this case increase these two rows by one each.

{sec:8.15}

8.14. so(p,q). Suppose $\mathfrak{g}_c \cong so(V_0)$, where $(V_0, \langle\ ,\ \rangle)$ is a real nondegenerate quadratic space of signature (p, q). The complexification admits a hermitian form $\langle\ ,\ \rangle$ with signature (p, q) as well as a complex nondegenerate quadratic form $(\ ,\ )$, which restrict to $\langle\ ,\ \rangle$ on V_0. The form $\langle\ ,\ \rangle$ gives an embedding of $o(p, q)$ into $u(p, q)$ with compatible Cartan involutions. The results of sections 8.1-8.3 together with section 8.6 imply the following classification of nilpotent orbits of $so(V_0)$ or equivalently θ-stable nilpotent orbits. See chapter 9 of [CM] for more details.

Orbits correspond to signed tableaus so that every even part occurs an even number of times. Even sized rows occur in pairs, one starting with $+$ the other with $-$. When all the rows have even sizes, there are two nilpotent orbits denoted I and II.

A parabolic subalgebra of $so(V)$ is the stabilizer of a flag of isotropic subspaces

{eq:8.15.1}
$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k, \quad (8.14.1)$$

so that the quadratic form restricts to 0 on $\mathcal{W}_k$. As before, complete this to a flag

{eq:8.15.2} $$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k \subset \mathcal{W}_k^* \subset \cdots \subset \mathcal{W}_0^* = V. \qquad (8.14.2)$$

We choose transverse spaces

$$\mathcal{W}_i = \mathcal{W}_{i-1} + V_i, \quad \mathcal{W}_k^* = \mathcal{W}_k + \mathcal{W}, \quad \mathcal{W}_{i-1}^* = \mathcal{W}_i^* + V_i^* \qquad (8.14.3)$$ {eq:8.15.3}

in order to fix a Levi component,

$$\mathfrak{l} \cong gl(V_1) \times \cdots \times gl(V_k) \times so(W). \qquad (8.14.4)$$ {eq:8.15.4}

To get a θ-stable parabolic subalgebra we must assume $V_i,\ W$ are θ-stable and so $\overline{V_i} = V_i^*,\ \overline{W} = W$. If the signature of V_i with respect to $\langle\ ,\ \rangle$ is (p_i, q_i), and that of W is (p_0, q_0), then

$$\mathfrak{l}_0 \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times so(p_0, q_0). \qquad (8.14.5)$$ {eq:8.15.5}

The parabolic subalgebra corresponding to (8.14.2) in $gl(V)$ satisfies

$$\mathfrak{l}' \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times u(p_0, q_0) \times u(p_k, q_k) \times \cdots \times u(p_1, q_1). \qquad (8.14.6)$$ {eq:8.15.6}

For a maximal θ-stable parabolic subalgebra, the Levi component $\mathfrak{l}$ satisfies $\mathfrak{l} \cong u(p_1, q_1) \times so(\mathcal{W}_0)$. Let $e \in so(W)$ be a θ-stable nilpotent element. The algorithm for induced nilpotent orbits in section 8.8 implies the following algorithm for $ind_{\mathfrak{l}}^{\mathfrak{g}_c}(e)$.

(1) add p_1 +'s to the beginning of the longest possible rows starting with −'s, and q_1 −'s to the beginning of the longest possible rows starting with +'s.
(2) add p_1 +'s to the ending of the longest possible rows starting with −'s, and q_1 −'s to the beginning of the longest possible rows starting with +'s.

Unlike in the complex case, the result is automatically a signed tableau for a nilpotent element in $so(V)$.

For a maximal real parabolic subalgebra, we must assume that $\overline{V_1} = V_1,\ \overline{\mathcal{W}} = \mathcal{W}$. Let $V_{1,0}$ and $\mathcal{W}_0$ be their real points. The Levi component satisfies

$$\mathfrak{l} \cong gl(V_{1,0}) \times so(W_0). \qquad (8.14.7)$$ {eq:8.15.7}

The results in section 8.12 imply the following algorithm for real induction.

Add two boxes to $\dim V_1$ *of the largest possible rows so that the result is still a signed tableau for a nilpotnet orbit. Suppose* $\dim V_1$ *is even and the last row that would be increased by 2 is even size as well. In this case there is a pair of rows of this size, one starting with + the other with −. Increase these two rows by one each so that the result is still a signed tableau. When there are only even sized rows and* $\dim V_1$ *is even as well, type I goes to type I and type II goes to type II.*

9. Unitarity

{sec:9}

In this section we prove the unitarity of the representations of the form $L(\chi)$ where $\chi = \check{h}/2$. As already mentioned, in the $p-$adic case this is done in [BM1]. It amounts to the observation that the Iwahori-Matsumoto involution preserves unitarity, and takes such an $L(\chi)$ into a tempered representation.

The idea of the proof in the real case is described in [B2]. We will do an induction on rank. We rely heavily on the properties of the wave front set, asymptotic support and associated variety, and their relations to primitive ideal cells and Harish-Chandra cells. We review the needed facts in sections 9.1-9.4. Details are in [BV1], [BV2], and [B3]. In sections 9.5-9.7 we give details of the proof of the unitarity in the case of $SO(2n+1)$. The proof is simpler than in [B2].

{sec:9.1}

9.1. Let π be an admissible $(\mathfrak{g}_c, K)$ module. We review some facts from [BV1]. The distribution character Θ_π lifts to an invariant eigendistribution θ_π in a neighborhood of the identity in the Lie algebra. For $f \in C_c^\infty(U)$, where $U \subset \mathfrak{g}$ is a small enough neighborhood of 0, let $f_t(X) := t^{-\dim \mathfrak{g}_c} f(t^{-1}X)$. Then

{eq:9.1.1}

$$\theta_\pi(f_t) = t^{-d} \sum_j c_j \widehat{\mu_{\mathcal{O}_j(\mathbb{R})}}(f) + \sum_{i>0} t^{d+i} D_{d+i}(f)]. \tag{9.1.1}$$

The D_i are homogeneous invariant distributions (each D_i is tempered and the support of its Fourier transform is contained in the nilpotent cone). The $\mu_{\mathcal{O}_j}$ are invariant measures supported on real forms $\mathcal{O}_j$ of a single complex orbit $\mathcal{O}_c$, and $\mu_{\mathcal{O}_j(\mathbb{R})}$ is the Liouville measure on the nilpotent orbit associated to the symplectic form induced by the Cartan-Killing form. Furthermore $d = \dim_{\mathbb{C}} \mathcal{O}_c/2$, and the number c_j is called the multiplicity of $\mathcal{O}_j(\mathbb{R})$ in the leading term of the expansion. The closure of the union of the supports of the Fourier transforms of all the terms occuring in (9.1.1) is called the *asymptotic support*, denoted $AS(\pi)$. The leading term in (9.1.1) will be called $AC(\pi)$. We will use the fact that the nilpotent orbits in the leading term are contained in the *wave front set* of θ_π at the origin, denoted $WF(\pi)$.

Alternatively, [V3] attaches to each π a combination of θ-stable orbits with integer coefficients

{eq:9.1.2}

$$AV(\pi) = \sum a_j \mathcal{O}_j, \tag{9.1.2}$$

where $\mathcal{O}_j$ are nilpotent K_c-orbits in $\mathfrak{s}_c$. The main result of [SV] is that $AC(\pi)$ in (9.1.1) and $AV(\pi)$ in (9.1.2) correspond via theorem 8.3. Precisely, the leading term in formula (8.3.2), and (9.1.2) are the same, when we identify real and θ stable nilpotent orbits via the Kostant-Sekiguchi correspondence. The algorithms in section 8 compute the associated variety of an induced representation as a set, which we also denote by $AV(\pi)$ when there is no possibility of confusion. The multiplicities are computed in the real setting in [B4] theorem 5.0.7. The formula is as follows. Let $v_j \in \mathcal{O}_j$ and $v_{ij} = v_j + X_{ij}$ be representatives of the induced orbits $\mathcal{O}_{ij}$ from $\mathcal{O}_{j,\mathfrak{m}}$. If $AV(\pi) = \sum c_j \mathcal{O}_{j,\mathfrak{m}}$, then

$$AV(ind_{\mathfrak{p}}^{\mathfrak{g}_c}(\pi)) = \sum_{i,j} c_j \frac{|C_G(v_{ij})|}{|C_P(v_{ij})|} \mathcal{O}_{ij}. \tag{9.1.3}$$ {eq:9.1.3}

We use [SV] to compare multiplicities of real and θ induced modules. Formula (9.1.3) is straightforward for real induction and $AC(\pi)$. Its analogue for θ stable induction and $AV(\pi)$ is also straightforward. It is the passage from $AC(\pi)$ to $AV(\pi)$ that is nontrivial.

{sec:9.2}

9.2. Fix a regular integral infinitesimal character χ_{reg}. Let $\mathcal{G}(\chi_{reg})$ be the set of parameters of irreducible admissible $(\mathfrak{g}_c, K)$ modules with infinitesimal character χ_{reg}, and denote by $\mathbb{Z}\mathcal{G}(\chi_{reg})$ the corresponding Grothendieck group of characters. Recall from [V2] (and references therein) that there is an action of the Weyl group on $\mathbb{Z}\mathcal{G}(\chi_{reg})$, called the *coherent continuation action.* As a set, $\mathcal{G}(\chi_{reg})$ decomposes into a disjoint union of blocks $\mathcal{B}$, and so $\mathbb{Z}\mathcal{G}$ decomposes into a direct sum

$$\mathbb{Z}\mathcal{G}(\chi_{reg}) = \bigoplus \mathbb{Z}\mathcal{G}_{\mathcal{B}}(\chi_{reg}). \tag{9.2.1}$$ {eq:9.2.1}

Each $\mathbb{Z}\mathcal{G}(\chi_{reg})$ is preserved by the coherent continuation action. We give the explicit description of the $\mathbb{Z}\mathcal{G}_{\mathcal{B}}$ in all classical cases. We will suppress the $\mathbb{Z}$ whenever there is no danger of confusion.

Type B: In order to conform to the duality between type B and type C in [V2], we only count the real forms with $p > q$. The representation $\mathcal{G}(\chi_{reg})$ equals

$$\mathbb{Z}\mathcal{G}(\chi_{reg}) = \sum_{a,b,\tau} Ind_{W_a\times W_b\times W_{2s}\times S_t}^{W_n}[sgn \otimes sgn \otimes \sigma[\tau,\tau] \otimes triv], \tag{9.2.2}$$ {eq:9.2.2}

where τ is a partition of s, and $a+b+2s+t=n$. The multiplicity of a $\sigma[\tau_L,\tau_R]$ in one of the induced modules in (9.2.2) is as follows. Choose a τ that fits inside both τ_L and τ_R, and label it by $\bullet$'s. Add "a" r and "b" r' to τ_R, at most one to each row, and "t" c, at most one to each column, to τ_L or τ_R. The multiplicity of σ in the induced module for a given (τ, a, b) is then the number of ways that τ_l, τ_R can be filled in this way. This procedure uses induction in stages, and the well known formula

$$Ind_{S_n}^{W_n}(triv) = \sum_{k+l=n} \sigma[(k),(l)]. \tag{9.2.3}$$ {eq:9.2.3}

Example. Let $\mathfrak{g}_c = so(5)$. The real forms are $so(3,2), so(4,1), so(5)$. The choices of (τ, a, b, t) are

$$\begin{aligned} &(1,1,0,0), \quad (1,0,1,0), \quad (1,0,0,1),\\ &(0,2,0,0), \quad (0,1,1,0)), \quad (0,1,0,1), \quad (0,0,2,0), \quad (0,0,1,1), \quad (0,0,0,2). \end{aligned} \tag{9.2.4}$$ {eq:9.2.4}

Let $\sigma = \sigma[(1),(1)]$. Then its multiplicity is given by the number of labelings

{eq:9.2.5}
$$\begin{aligned} &(\bullet,\bullet) \quad \emptyset, \quad \emptyset,\\ &\emptyset, \qquad \emptyset, \quad (c,r), \quad \emptyset, \quad (c,r'), \quad (c,c). \end{aligned} \tag{9.2.5}$$

For $\sigma = \sigma[(0),(2)]$ we get

{eq:9.2.6}
$$\begin{aligned} &\emptyset, \quad \emptyset, \qquad \emptyset\\ &\emptyset, \quad (0,rr'), \quad (0,rc), \quad \emptyset, \quad (0,r'c), \quad (0,cc). \end{aligned} \tag{9.2.6}$$

□

The following formula sorts the representations according to the various real forms of $SO(p,q)$ with $p+q = 2n+1$. Each real form gives a single block. A representation occuring in $\mathcal{G}$, labelled as above, occurs in $\mathbb{Z}\mathcal{G}_{SO(p,q)}$ with

{eq:9.2.7}
$$p = n+1+ \mid \#r' - \#r \mid -\epsilon, \text{ where } \epsilon = \begin{cases} 0 & \text{if } \#r' \geq \#r,\\ 1 & \text{otherwise.} \end{cases} \tag{9.2.7}$$

In the above example, $(\bullet,\bullet)$, (c,c), $(0,rr')$, $(0,rc)$ and $(0,cc)$ belong to $so(3,2)$ while (c,r') and $(0,r'c)$ belong to $so(4,1)$.

To each pair of partitions parametrizing a representation of W,

{eq:9.2.8}
$$\tau_L = (r_0, \dots, r_{2m}), \qquad \tau_R = (r_1, \dots, r_{2m-1}), \qquad r_i \leq r_{i+2}, \tag{9.2.8}$$

Lusztig attaches a *symbol*

{eq:9.2.9}
$$\begin{pmatrix} r_0 & & r_2+1 & & & \dots & & r_{2m}+m\\ & r_1 & & r_3+1 & \dots & & r_{2m-1}+m-1 & \end{pmatrix}. \tag{9.2.9}$$

The symbol is called special if

{eq:9.2.10}
$$r_0 \leq r_1 \leq r_2+1 \leq r_3+1 \leq \dots \leq r_{2m}+m. \tag{9.2.10}$$

Two representations belong to the same double cell if and only if their symbols have the same entries. Given a special symbol of the form (9.2.9), the corresponding nilpotent orbit $\mathcal{O}_c$ has partition obtained as follows. Form the set

{eq:9.2.11}
$$\{2r_{2i}+2i+1, 2r_{2j-1}+2j-2\}, \tag{9.2.11}$$

and order the numbers in increasing order, $x_0 \leq \dots \leq x_{2m}$. The partition of $\mathcal{O}_c$ is

{eq:9.2.12}
$$(x_0, x_1-1, \dots, x_i-i, \dots, x_{2m}-2m). \tag{9.2.12}$$

Type C: The representation $\mathcal{G}(\chi_{reg})$ is obtained from the one in type B by tensoring with sign. Thus

{eq:9.2.13} $$\mathbb{Z}\mathcal{G}(\chi_{reg}) = \sum_{a,b,\tau} Ind_{S_t\times W_{2s}\times W_a\times W_b}^{W_n}[sgn\otimes\sigma[\tau,\tau]\otimes triv\otimes triv], \quad (9.2.13)$$

where τ is a partition of s, and $a+b+2s+t=n$. This takes into account the duality in [V2] of types B and C. We write r for the sign representation of S_t, and c and c' for the trivial representations of W_a, W_b. A representation of W is parametrized by a pair of partitions (τ_L,τ_R), with

$$\tau_L=(r_0,\dots,r_{2m}),\qquad \tau_R=(r_1,\dots,r_{2m-1}),\qquad r_i\le r_{i+2}. \quad (9.2.14)$$ {eq:9.2.14}

The associated symbol is

$$\begin{pmatrix} r_0 & & r_2+1 & & \dots & & r_{2m}+m \\ & r_1 & & r_3+1 \quad \dots & & r_{2m-1}+m-1 & \end{pmatrix}, \quad (9.2.15)$$ {eq:9.2.15}

and it is called special if

$$r_0\le r_1\le r_2+1\le r_3+1\le\dots\le r_{2m}+m. \quad (9.2.16)$$ {eq:9.2.16}

Two representations belong to the same double cell if their symbols have the same entries. Given a special symbol as in (9.2.15), the nilpotent orbit $\mathcal{O}_c$ attached to the double cell has partition obtained as follows. Order the set

$$\{2r_{2i}+2i, 2r_{2j-1}+2j-1\} \quad (9.2.17)$$ {eq:9.2.17}

in increasing order, $x_0\le\dots\le x_{2m}$. Then the partition of $\mathcal{O}_c$ is

$$(x_0,\dots,x_j-j,\dots,x_{2m}-2m). \quad (9.2.18)$$ {eq:9.2.18}

The decomposition into blocks is obtained from the one for type B by tensoring with sgn.

Type D: Since in this case $\sigma[\tau_L,\tau_R]$ and $\sigma[\tau_R,\tau_L]$ parametrize the same representation, (except of course when $\tau_L=\tau_R$ which corresponds to two nonisomorphic representations), we assume that the size of τ_L is the larger one. The Cartan subgroups are parametrized by integers $(t,u,2s,p,q)$, $p+q+2s+t+u=n$. There are actually two Cartan subgroups for each $s>0$, related by the outer automorphism of order 2. Then $\mathcal{G}(\chi_{reg})$ equals

$$\mathbb{Z}\mathcal{G}(\chi_{reg}) = \\ = \sum_{p+q+2s+t+u=n} Ind_{W_a\times W_b\times W'_{2s}\times W_t\times W_u}^{W'_n}[sgn\otimes sgn\otimes\sigma[\tau,\tau]_{I,II}\otimes triv\otimes triv]. \quad (9.2.19)$$ {eq:9.2.19}

The sum is also over τ which is a partition of s. We label the σ by $\bullet$'s, trivial representations by c and c' and the sgn representations by r and r'. These are added to τ_L when inducing. In this case we count all the real forms $SO(p,q)$ with $p+q=2n$. Each real form gives riswe to a single block. A represntation labelled as above belongs to the block $\mathbb{Z}\mathcal{G}_{SO(p,q)}$ with $p=n+\#r'-\#r$.

If

$$\tau_L = (r_0, \dots, r_{2m-2}), \qquad \tau_R = (r_1, \dots, r_{2m-1}), \tag{9.2.20}$$ {eq:9.2.20}

then the associated symbol is

{eq:9.2.21}
$$\begin{pmatrix} r_0 & r_2+1 & \dots & r_{2m-2}+m-1 \\ r_1 & r_3+1 & \dots & r_{2m-1}+m-1 \end{pmatrix}. \tag{9.2.21}$$

A representation is called special if the symbol satisfies

{eq:9.2.22}
$$r_0 \le r_1 \le r_2+1 \le r_3+1 \le \dots \le r_{2m-1}+m-1. \tag{9.2.22}$$

Two representations belong to the same double cell if their symbols have the same entries. The nilpotent orbit $\mathcal{O}_c$ attached to the special symbol is given by the same procedure as for type B.

{sec:9.3}
9.3. We follow section 14 of [V2]. We say that $\pi' \le \pi$ if π' is a factor of $\pi \otimes F$ with F a finite dimensional representation with highest weight equal to an integer sum of roots. Two irreducible representations $\pi,\ \pi'$ are said to be in the same Harish-Chandra cell if $\pi' \le \pi$ and $\pi \le \pi$. The Harish-Chandra cell of π is denoted $\mathcal{C}(\pi)$. Recall the relation $\overset{LR}{<}$ from definition 14.6 of [V2]. The *cone above* π is defined to be

{eq:9.3.1}
$$\overline{\mathcal{C}}^{LR}(\pi) := \{\pi' \ : \ \pi' \overset{LR}{<} \pi\}. \tag{9.3.1}$$

The subspace in $\mathbb{Z}\mathcal{G}$ generated by the elements in $\overline{\mathcal{C}}^{LR}(\pi)$ is a representation of W denoted $\overline{\mathcal{V}}^{LR}(\pi)$. The equivalence $\overset{LR}{\approx}$ is defined by

{eq:9.3.2}
$$\gamma \overset{LR}{\approx} \phi \Longleftrightarrow \gamma \overset{LR}{<} \phi \overset{LR}{<} \gamma. \tag{9.3.2}$$

The Harish-Chandra cell $\mathcal{C}(\pi)$ is then

{eq:9.3.3}
$$\mathcal{C}(\pi) = \{\pi' \ : \ \pi' \overset{LR}{\approx} \pi\}. \tag{9.3.3}$$

Define

{eq:9.3.5}
$$\mathcal{C}_+^{LR}(\pi) = \overline{\mathcal{C}}^{LR}(\pi) \setminus \mathcal{C}(\pi) \tag{9.3.4}$$

and $\overline{\mathcal{V}}(\pi)$ and $\mathcal{V}^{LR}(\pi)$ in analogy with $\overline{\mathcal{V}}^{LR}(\pi)$. Thus there is a representation of W on $\mathcal{V}(\pi)$ by the natural isomorphism

{eq:9.3.6}
$$\mathcal{V}(\pi) \cong \overline{V}^{LR}(\pi)/\mathcal{V}(\pi)_+. \tag{9.3.5}$$

Let $\mathcal{O}_c \subset \mathfrak{g}_c$ be a nilpotent orbit. We say that a Harish-Chandra cell is attached to a complex orbit $\mathcal{O}_c$ if

$$\overline{\operatorname{Ad} G_c(AS(\pi))} = \overline{\mathcal{O}_c}.$$

The sum of the Harish Chandra cells attached to $\mathcal{O}_c$ is denoted $\mathcal{V}(\mathcal{O}_c)$.

Let $\mathfrak{h}_a \subset \mathfrak{g}_c$ be an abstract Cartan subalgebra and let Π_a be a set of (abstract) simple roots. For each irreducible representation $\mathcal{L}(\gamma)$, denote by $\tau(\gamma)$ the tau-invariant as defined in [V2]. Given a block $\mathcal{B}$ and disjoint orthogonal sets $S_1,\ S_2 \subset \Pi_a$, define

{eq:9.2.23}
$$\mathcal{B}(S_1, S_2) = \{\gamma \in \mathcal{B} \mid S_1 \subset \tau(\gamma),\ S_2 \cap \tau(\gamma) = \emptyset\}\ . \tag{9.3.6}$$

If in addition we are given a nilpotent orbit $\mathcal{O}_c \subset \mathfrak{g}_c$, we can also define

{eq:9.2.24}
$$\mathcal{B}(S_1,S_2,\mathcal{O}_c)=\{\gamma\in\mathcal{B}(S_1,S_2)|\ AS(\mathcal{L}(\gamma))\subset\overline{\mathcal{O}_c}\}\ . \qquad (9.3.7)$$

The convention is that if f $S_i=\emptyset$ then no condition is imposed on the parameter. Let $W_i = W(S_i)$, and define

$$\begin{aligned} m_S(\sigma) &= [\sigma : Ind_{W_1\times W_2}^{W}(Sgn\otimes Triv)],\\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{G}_{\mathcal{B}}(\chi_{reg})]\ . \end{aligned} \qquad (9.3.8)$$

In the case of G_c viewed as a real group, the cones defined by (9.3.1) are parametrized by nilpotent orbits in $\mathfrak{g}_c$. In other words, $\pi' \overset{LR}{<} \pi$ is and only if $AS(\pi')\subset\overline{AS(\pi)}$. So let $\mathcal{C}(\mathcal{O}_c)$ be the cone corresponding to $\mathcal{O}_c$. Note that in this case $W_c \cong W\times W$, and the representations are of the form $\sigma\otimes\sigma$.

Theorem.
$$|\mathcal{B}(S_1,S_2,\mathcal{O}_c)| = \sum_{\sigma\otimes\sigma\in\mathcal{C}(\mathcal{O}_c)} m_{\mathcal{B}}(\sigma)m_S(\sigma)\ .$$

Proof. Consider $\mathbb{Z}\mathcal{B}(S_1,S_2,\mathcal{O}_c)\subset\mathbb{Z}\mathcal{B}(\emptyset,\emptyset,\mathcal{O}_c)$. Then $\mathcal{B}(\emptyset,\emptyset,\mathcal{O}_c)$ is a representation of W which consists of the representations in $\mathcal{V}(\pi)$ with $AS(\pi)\subset\overline{\mathcal{O}_c}$. The fact that the representation $\mathcal{V}(\pi)$ is formed of σ with $\sigma\otimes\sigma\in\mathcal{C}(\mathcal{O}_c)$ follows from the argument before theorem 1 of [McG]. This accounts for $m_{\mathcal{B}}(\sigma)$ in the sum. The expressions of the action of W given by lemma 14.7 in [V2] and Frobenius reciprocity imply that the dimension of $\mathbb{Z}\mathcal{B}(S_1,S_2,\mathcal{O}_c)$ equals the left hand side of the formula in the theorem, and it equals the cardinality of $\mathcal{B}(S_1,S_2,\mathcal{O}_c)$. ☐

{sec:9.4}

9.4. Assume that $\check{\mathcal{O}}$ is even. Then $\lambda := \check{h}/2$ is integral, and it defines a set S_2 by

$$S_2 = S(\lambda)=\{\alpha\in\Pi_a|(\alpha,\lambda)=0\}\ . \qquad (9.4.1)$$ {eq:9.2.25}

Let $\mathcal{O}_c$ be the nilpotent orbit attached to $\check{\mathcal{O}}$ by the duality in [BV3]. Then the *special unipotent representations attached to* $\check{\mathcal{O}}$, called $Unip(\check{\mathcal{O}})$, are defined to be the representations π with infinitesimal character λ and $AS(\pi)\subset\mathcal{O}_c$. Via translation functors they are in 1-1 correspondence with the set

$$\bigcup_{\mathcal{B}}\mathcal{B}(\emptyset,S(\lambda),\mathcal{O}_c). \qquad (9.4.2)$$ {eq:9.2.26}

So we can use theorem 9.4 to count the number of unipotent representations. In the classical groups case, $m_{\mathcal{B}}(\sigma)$ is straightforward to compute. For the special unipotent case, $m_S(\sigma)$ equals 0 except for the representations occuring in a particular left cell sometimes also called the *Lusztig cell*, which we denote $\mathcal{C}^L(\mathcal{O}_c)$. The multiplicities of the representations occuring in $\overline{\mathcal{C}}^L(\mathcal{O}_c)$ are all 1. These representations are in 1-1 correspondence with the conjugacy classes in Lusztig's quotient of the component group $\overline{A}(\check{\mathcal{O}})$. See [BV2] for details.

{t:9.2.2}

Theorem (1).
$$|Unip(\check{\mathcal{O}})| = \sum_{\mathcal{B}} \sum_{\sigma \in \overline{\mathcal{C}}^L(\mathcal{O}_c)} m_{\mathcal{B}}(\sigma) \ .$$

{t:9.3} **Theorem** (2, [McG]). *In the classical groups $Sp(n)$, $SO(p,q)$, each Harish-Chandra cell is of the form $\overline{\mathcal{C}}^L(\mathcal{O}_c)$.*

{d:9.2} **Definition.** *We say that a nilpotent orbit $\mathcal{O}_c$ is* smoothly cuspidal *if it satisfies*

Type B, D: *all odd sizes occur an even number of times,*
Type C: *all even sizes occur an even number of times.*

For $\mathcal{O}(\mathbb{R})$, a real form of $\mathcal{O}_c$, write $A(\mathcal{O}(\mathbb{R}))$ for its (real) component group.

{p:9.2} **Proposition.** *For smoothly cuspidal orbits, $A(\check{\mathcal{O}}) = \overline{A}(\check{\mathcal{O}})$. In particular, $|\mathcal{C}^L(\mathcal{O}_c)| = |A(\check{\mathcal{O}})|$. Furthermore,*
$$|Unip(\check{\mathcal{O}}))| = \sum_{\mathcal{O}(\mathbb{R})} |A(\mathcal{O}(\mathbb{R}))|$$
where the sum is over all real forms $\mathcal{O}(\mathbb{R})$ of $\mathcal{O}_c$.

Proof. This is theorem 5.3 in [B2]. The proof consists of a direct calculation of multiplicities in the coherent continuation representation using the results developed earlier in this section. □

{sec:9.5} **9.5.** We now return to type $G = SO(2n+1)$. Consider the spherical irreducible representation $L(\chi_{\check{\mathcal{O}}})$ with $\chi_{\check{\mathcal{O}}} = \check{h}/2$ corresponding to a nilpotent orbit $\check{\mathcal{O}}$ in $sp(n)$. If the orbit $\check{\mathcal{O}}$ meets a proper Levi component $\check{\mathfrak{m}}$, then $L(\check{\mathcal{O}})$ is a subquotient of a representation which is unitarily induced from a unipotent representation on $\mathfrak{m}$. By induction, $L(\chi_{\check{\mathcal{O}}})$ is unitary. Thus we only consider the cases when $\check{\mathcal{O}}$ does not meet any proper Levi component. This means

{eq:9.5.1}
$$\check{\mathcal{O}} = (2x_0, \dots, 2x_{2m}), \qquad 0 \le x_0 < \dots < x_i < x_{i+1} < \dots < x_{2m}, \quad (9.5.1)$$

so these orbits are even.

Because of assumption (9.5.1), the AS-set of $L(\chi_{\check{\mathcal{O}}})$ satisfies the property that
$$\overline{\mathrm{Ad}\, G_c(AS(L(\chi_{\check{\mathcal{O}}})))}$$
is the closure of the special orbit $\mathcal{O}_c$ dual to $\check{\mathcal{O}}$. This is the orbit $\mathcal{O}_c$ with partition

{eq:9.5.2}
$$(\underbrace{1,\dots,1}_{r_1}, \underbrace{2,\dots,2}_{r_2}, \dots, \underbrace{2m,\dots,2m}_{r_{2m}}, \underbrace{2m+1,\dots,2m+1}_{r_{2m+1}}), \qquad (9.5.2)$$

where

$$r_{2i+1} = 2(x_{2m-2i} - x_{2m-2i-1} + 1),$$
$$r_{2i} = 2(x_{2m-2i+1} - x_{2m-2i} - 1),$$
$$r_{2m+1} = 2x_0 + 1.$$

The columns of $\mathcal{O}_c$ are $(2x_{2m}+1, 2x_{2m-1}-1, \dots, 2x_0+1)$.

Definition. *Given an orbit $\mathcal{O}_c$ with partition (9.5.2) or more generally a smoothly cuspidal orbit, we call the* split real form $\mathcal{O}_{spl}$ *the one which satisfies for each row size,*

Type C,D: *the number of rows starting with + and − is equal,*

Type B: *in addition to the condition in types C,D for rows of size less than $2m+1$, for size $2m+1$, the number of starting with + is one more than those starting with −.*

Theorem. *The WF-set of the spherical representation $L(\chi_{\check{\mathcal{O}}})$ with $\check{\mathcal{O}}$ satisfying (9.5.1) is the closure of the split real form $\mathcal{O}_{spl}$ of the (complex) orbit $\mathcal{O}_c$ given by (9.5.2).*

Proof. The main idea is outlined in [B2]. We use the fact that if π is a factor of π', then $WF(\pi) \subset WF(\pi')$. We do an induction on m. The claim amounts to showing that if E occurs in $AS(L(\chi_{\check{\mathcal{O}}}))$, then the signatures of $E,\ E^2, \dots$ are greater than the pairs

$$\begin{aligned}&(x_{2m}+1, x_{2m}), (x_{2m}+x_{2m-1}, x_{2m}+x_{2m-1}), \dots,\\ &\dots(x_{2m}+\dots+x_1, x_{2m}+\dots+x_1),\\ &(x_{2m}+\dots+x_1+x_0+1, x_{2m}+\dots+x_1+x_0).\end{aligned} \tag{9.5.3}$$ {eq:9.5.3}

The statement is clear when $m=0$; $L(\chi_{\check{\mathcal{O}}})$ is the trivial representation. Let $\check{\mathcal{O}}_1$ be the nilpotent orbit corresponding to

$$(2x_0, \dots, 2x_{2m-2}). \tag{9.5.4}$$ {eq:9.5.4}

By induction, $WF(L(\check{\mathcal{O}}_1))$ is the split real form of the nilpotent orbit corresponding to the partition

$$(\underbrace{1,\dots,1}_{r_1'}, \underbrace{2,\dots,2}_{r_2'}, \dots, \underbrace{2m-2,\dots,2m-2}_{r_{2m-2}'}, \underbrace{2m-1,\dots,2m-1}_{r_{2m-1}'}), \tag{9.5.5}$$ {eq:9.5.5}

where the columns are $(2x_{2m-2}+1, 2x_{2m-3}-1, \dots, 2x_0+1)$. Let $\mathfrak{p}$ be the real parabolic subalgebra with Levi component $\mathfrak{g}(n-x_{2m}-x_{2m-1}) \times gl(x_{2m}+x_{2m-1})$. There is a character χ of $gl(x_{2m}+x_{2m-1})$ such that $\pi := L(\chi_{\check{\mathcal{O}}})$ is a factor of $\pi' := Ind_{\mathfrak{p}}^{\mathfrak{g}_c}[L(\chi_{\check{\mathcal{O}}_1}) \otimes \chi]$. But by section 8, $WF(\pi')$ is in the closure of nilpotent orbits corresponding to partitions

$$(\underbrace{2,\dots,2}_{(r_1+r_2)/2}, \dots, \underbrace{2m,\dots,2m}_{r_{2m}}, \underbrace{2m+1,\dots,2m+1}_{r_{2m+1}}),\ r_1+r_2 \text{ even}, \tag{9.5.6}$$ {eq:9.5.6}

$$(1,1, \underbrace{2,\dots,2}_{(r_1+r_2-1)/2}, \dots, \underbrace{2m,\dots,2m}_{r_{2m}}, \underbrace{2m+1,\dots,2m+1}_{r_{2m+1}}),\ r_1+r_2 \text{ odd}. \tag{9.5.7}$$ {eq:9.5.7}

It follows that the signatures for E^k in $WF(L(\chi_{\check{\mathcal{O}}}))$ are greater than the pairs

$$(a_+, a_-),\ (x_{2m} + x_{2m-1}, x_{2m} + x_{2m-1}), \dots , \tag{9.5.8}$$ {eq:9.5.8}

for some $a_+ + a_- = x_{2m} + 1$. Also, each row size greater than two and less than $2m+1$ has an equal number that start with $+$ and $-$. For size $2m+1$ there is one more row starting with $+$ than $-$.

The same argument with $\check{\mathcal{O}}_2$ corresponding to

$$(2x_0, \dots \widehat{2x_{2m-2}}, \widehat{2x_{2m-1}}, 2x_{2m})$$

shows that $WF(L(\chi_{\check{\mathcal{O}}}))$ is also contained in the closure of the nilpotent orbits with signatures

{eq:9.5.9}
$$\begin{aligned}&(x_{2m}+1, x_{2m}),\ (x_{2m}+1+a_+, x_{2m}+a_-),\\ &(x_{2m}+1+x_{2m-1}+x_{2m-2}, x_{2m}+1+x_{2m-1}+x_{2m-2}),\ \dots ,\end{aligned} \tag{9.5.9}$$

for some $a_+ + a_- = x_{2m-1}$. The claim follows. □

{sec:9.6}

9.6. Consider the special case when

{eq:9.6.1}
$$x_0 = x_1 - 1 \le x_2 = x_3 - 1 \le \dots \le x_{2m-2} = x_{2m-1} - 1 \le x_{2m}. \tag{9.6.1}$$

The cell $\mathcal{C}^L(\mathcal{O}_c)$ has size 2^m. We produce 2^m distinct irreducible representations with AS equal to the closure of $\mathcal{O}_{spl}$. So $\mathfrak{g}$ is $so(2p+1, 2p)$. Let $\mathfrak{h}$ be the compact Cartan subalgebra. We write the coordinates

{eq:9.6.2}
$$(a_1, \dots, a_p \mid b_1, \dots, b_p) \tag{9.6.2}$$

where the first p coordinates before the $|$ are in the Cartan subalgebra of $so(2p+1)$ the last p coordinates are in $so(2p)$. The roots $\epsilon_i \pm \epsilon_j, \epsilon_i$ with $1 \le i, j \le p$ are all compact and so are $\epsilon_{p+k} \pm \epsilon_{p+l}$ with $1 \le k, l \le p$. The roots $\epsilon_i \pm \epsilon_{p+k},\ \epsilon_{p+k}$ are noncompact. Let $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be a θ-stable parabolic subalgebra with Levi component

{eq:9.6.3}
$$\mathfrak{l} = u(x_{2i_1+1}, x_{2i_1}) \times u(x_{2i_2}, x_{2i_2+1}) \times \dots \times \mathfrak{g}(x_{2m}), \tag{9.6.3}$$

where the i_j are the numbers $0, \dots, m-1$ in some order. The parabolic subalgebra $\mathfrak{q}_c$ corresponds to the weight

{eq:9.6.4}
$$\begin{aligned}&\xi = (m^{x_{2i_1+1}}, \dots, 1^{x_{2i_{m-1}}+1}, 0^{x_{2m}} \ \mid\ m^{x_{2i_1}}, \dots, 1^{x_{2i_{m-1}}}, 0^{x_{2m}}),\\ &\text{or}\\ &\xi = (m^{x_{2i_1+1}}, \dots, 1^{x_{2i_{m-1}}}, 0^{x_{2m}} \ \mid\ m^{x_{2i_1}}, \dots, 1^{x_{2i_{m-1}}+1}, 0^{x_{2m}}),\end{aligned} \tag{9.6.4}$$

depending whether m is odd or even.

The derived functor modules $\mathcal{R}^i_{\mathfrak{q}_c}(\xi)$ from characters on $\mathfrak{l}_c$ have AC-set contained in $\mathcal{O}_{spl}$. To get infinitesimal character $\chi_{\check{\mathcal{O}}}$, these characters can only be

{eq:9.6.5}
$$\xi^{\pm}_{i_j} := \pm(1/2, \dots, 1/2), \tag{9.6.5}$$

on the unitary factors $u(x_{2i_j+1}, x_{2i_j})$ or $u(x_{2i_j}, x_{2i_j+1})$, and trivial on $\mathfrak{g}(x_{2m})$. We need to show that there are choices of parabolic subalgebras $\mathfrak{q}_c$ as in

(9.6.3) and characters as in (9.6.5) so that we get 2^m nonzero and distinct representations. For this we have to specify the Langlands parameters.

For each subset $A := \{k_1, \dots, k_r\} \subset \{0, \dots, m-1\}$, k_j in decreasing order, label the complement $A^c := \{\ell_1, \dots, \ell_t\}$, and consider the $\theta-$stable parabolic subalgebra $\mathfrak{q}_{c,A}$ as in (9.6.3) and (9.6.4) corresponding to

$$\{i_1, \dots, i_{m-1}\} = \{k_1, \dots, k_r, \ell_1, \dots, \ell_t\}. \tag{9.6.6}$$ {eq:9.6.6}

We will consider the representations $\mathcal{R}_{\mathfrak{q}_{c,A}}(\xi_A)$, where ξ_A is the concatenation of the $\xi^{\pm}_{i_j}$ with $+$ for the first r, and $-$ for the last t.

{l:9.6}

Lemma.

$$\mathcal{R}^i_{\mathfrak{q}_{c,A}}(\xi_A) = \begin{cases} 0 \text{ } \textit{if } i \neq \dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c), \\ \textit{nonzero irreducible if } i = \dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c). \end{cases}$$

Proof. For the vanishing part we check that the conditions in proposition 5.93 in [KnV] chapter V section 7 are satisfied. It is sufficient to show that

$$ind^{\mathfrak{g},K}_{\overline{\mathfrak{q}}_{c,A},L\cap K}(Z^{\#}_{\overline{\mathfrak{q}}_{c,A}}) := U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}_{c,A}} Z^{\#}_{\overline{\mathfrak{q}}_{c,A}} \tag{9.6.7}$$ {eq:9.6.7}

is irreducible. Here $Z^{\#}_{\overline{\mathfrak{q}}_{c,A}}$ is the $1-$dimensional module corresponding to $\xi_A - \rho(\mathfrak{u}_{c,A})$, with

$$\rho(\mathfrak{u}_{c,A}) := \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}_{A,c})} \alpha.$$

The derived functors are normalized so that if W has infinitesimal character χ, then so do $\mathcal{R}^i_{\mathfrak{q}_c}(W)$.

For generalized Verma modules of this kind we can apply the notions and results about associated cycles. The associated cycle of (9.6.8) is $\mathcal{O}_c$ from (9.5.2), and the multiplicity is 1. Any composition factor cannot have associated cycle formed of nilpotent orbits of strictly smaller dimension than $\mathcal{O}_c$ because the results in [BV2] apply. So if there is more than one factor, the multiplicity of $\mathcal{O}_c$ would be strictly larger than 1.

To show that $\mathcal{R}^{\dim(\mathfrak{u}_{c,A}\cap\mathfrak{k}_c)}_{\mathfrak{q}_{c,A}}(\xi_A) \neq 0$, we use the bottom layer $K-$ types defined in chapter V section 6 of [KnV]. To simplify the notation slightly, we write

$$\begin{aligned} &a_1 = x_{2k_1+1},\ b_1 = x_{2k_1}, \dots, a_r = x_{2k_r},\ b_r = x_{2k_r+1} \ r \text{ even}, \\ &a_1 = x_{2k_1+1},\ b_1 = x_{2k_1}, \dots, a_r = x_{2k_r+1},\ b_r = x_{2k_r} \ r \text{ odd}. \end{aligned} \tag{9.6.8}$$ {eq:9.6.8}

Let also $a := \sum a_j,\ b := \sum b_j$. Note that $|a_j - b_j| = 1$, and also $|a - b| = 1$. Then

$$\mu := \xi + 2\rho(\mathfrak{u} \cap \mathfrak{s}) - \rho(\mathfrak{u}) = (1^a, 0^{p-a} \mid 1^b, 0^{p-a}) \tag{9.6.9}$$ {eq:9.6.9}

is dominant, therefore bottom layer. The aforementioned results then imply the nonvanishing. Finally the derived functor module is irreducible because the multiplicity in AV of the nilpotent orbit is 1. □

We now show that there are 2^m distinct representations. We will need to use the intermediate parabolic subalgebras

{eq:9.6.10}
$$\mathfrak{q}_{c,A} \subset \mathfrak{q}'_{c,A} \subset \mathfrak{q}''_{c,A} \subset \mathfrak{g}_c \tag{9.6.10}$$

with Levi components

{eq:9.6.11}
$$\begin{aligned} \mathfrak{l}'_{c,A} &= u(a_1,b_1) \times \cdots \times u(a_r,b_r) \times \mathfrak{g}(n-a-b), \\ \mathfrak{l}''_{c,A} &= u(a,b) \times \mathfrak{g}(n-a-b), \end{aligned} \tag{9.6.11}$$

Apply induction in stages from $\mathfrak{q}_{c,A}$ to $\mathfrak{q}'_{c,A}$ first. On the factor $\mathfrak{g}(n-a-b)$ the $K-$type μ in (9.6.9) is trivial, so the Langlands parameter is that of the spherical principal series. Similarly on the $u(a_j,b_j)$ assume the infinitesimal character is $\chi_j := (\max(a_j,b_j),\dots,\min(a_j,b_j))$ with the coordinates going down by 1, and the Langlands parameter is that of a principal series with the appropriate 1-dimensional Langlands subquotient. Let $\mathfrak{h}_A \subset \mathfrak{l}'_{c,A}$ be the the most split Cartan subalgebra. In particular the real roots are

{eq:9.6.12}
$$\alpha_d := \epsilon_d + \epsilon_{d+p}, \quad \sum_{j\le s} a_j < d < \sum_{j\le s} a_j + \min(a_j,b_j), \quad 0 \le s \le r-1. \tag{9.6.12}$$

For each factor $u(a_j,b_j)$ the Langlands parameter is of the form (λ_j,ν_j) where $\lambda_j \in \mathfrak{h}_A \cap \mathfrak{k}_c$, and $\nu_j \in \mathfrak{h}_A \cap \mathfrak{s}_c$. Then

{eq:9.6.13}
$$\lambda_j = (1/2^{a_j} \mid 1/2^{b_j}), \tag{9.6.13}$$

while

{eq:9.6.14}
$$\langle \nu_j, \alpha_d\rangle = \max(a_j,b_j) - (d - \sum_{j\le s} a_j) \tag{9.6.14}$$

{p:9.6}

Proposition. *The representations $\mathcal{R}^{\dim(\mathfrak{u}_{c,A}\cap\mathfrak{k}_c)}_{\mathfrak{q}_{c,A}}(\xi_A)$ have Langlands parameters (λ^G,ν) where λ^G is obtained by concatenating the λ_j in (9.6.13) and ν satisfies (9.6.14).*

Proof. There is a nonzero map $X_{\mathfrak{l}'_c}(\lambda^G,-\nu) \longrightarrow L_{\mathfrak{l}'}(\lambda^G,-\nu)$ given by the Langlands classification. Thus there is a map

{eq:9.6.15}
$$\begin{aligned} &\mathcal{R}^{\dim \mathfrak{k}_c\cap\mathfrak{u}'_c}_{\mathfrak{q}'_c, L'\cap K}[X_{\mathfrak{l}'}(\lambda^G,-\nu)] \longrightarrow \\ &\longrightarrow \mathcal{R}^{\dim \mathfrak{k}_c\cap\mathfrak{u}'_c}_{\mathfrak{q}'_c,L'\cap K}(L_{\mathfrak{l}'_c}(\lambda^G,-\nu)) = \mathcal{R}^{\dim \mathfrak{k}_c\cap\mathfrak{u}_c}_{\mathfrak{q}_c,L\cap K}(\xi_A), \end{aligned} \tag{9.6.15}$$

which is nonzero on the bottom layer $K-$type (9.6.9). On the other hand, because these are standard modules,

{eq:9.6.16}
$$\mathcal{R}^i_{\mathfrak{q}}(X_{\mathfrak{l}'}(\lambda^G,\nu)) = \begin{cases} X(\lambda^G,\nu) & \text{if } i = \dim \mathfrak{k}_c \cap \mathfrak{u}_c, \\ 0 & \text{otherwise.} \end{cases} \tag{9.6.16}$$

The proof follows. □

{sec:9.7}
9.7.

Theorem. *The spherical unipotent representations $L(\chi_{\check{\mathcal{O}}})$ are unitary.*

Proof. Write $\mathfrak{g}(n)$ for the Lie algebra containing $\mathcal{O}$. There is a (real) parabolic subalgebra $\mathfrak{p}^+$ with Levi component $\mathfrak{m}^+ := gl(n_1)\times\cdots\times gl(n_k)\times\mathfrak{g}(n)$ in $\mathfrak{g}^+$ of rank $n_1+\cdots+n_k+n$, such that the split form $\mathcal{O}^+_{spl}$ of

$$\mathcal{O}^+_c := (1,1,3,3,\dots,2m-1,2m-1,2m+1)$$

is induced from $\mathcal{O}$ on $\mathfrak{g}(n)$, trivial on the gl's. We will consider the representation

$$I(\pi) := Ind_{\mathfrak{m}^+}^{\mathfrak{g}^+}[triv\otimes\cdots\otimes triv\otimes\pi]. \qquad (9.7.1)$$ {eq:9.7.1}

We show that the form on $I(\pi)$ induced from π is positive definite; this implies that the form on π is definite. We do this by showing that the possible factors of $I(\pi)$ have to be unitary, and the forms on their lowest $K-$types are positive definite.

Combining proposition 9.4 with (9.2.3), we conclude that there are $3^m\cdot 2^m$ unipotent representations in the block of the spherical irreducible representation; all the factors of $I(\pi)$ are in this block. The number 3^m also equals the number of real forms of $\mathcal{O}^+$. We describe how to get $3^m\cdot 2^m$ representations. For each $\mathcal{O}^+_j$, we produce one representation π such that $AC(\pi)=\overline{\mathcal{O}^+_j}$. Then theorem 9.4 implies that there is a Harish-Chandra cell with 2^m representations with this property. Since these cells must be disjoint, this gives the required number.

¿From section 9.1, each such form $\mathcal{O}^+_j$ is θ-stable induced from the trivial nilpotent orbit on a parabolic subalgebra with Levi component a real form of $gl(1)\times gl(3)\times\cdots\times gl(2m-1)\times\mathfrak{g}_c(m)$. Using the results in [KnV], for each such parabolic subalgebra, we can find a derived functor induced module from an appropriate 1-dimensional character, that is nonzero and has associated variety equal to the closure of the given real form. Actually it is enough to construct this derived functor module at regular infinitesimal character where the fact that it is nonzero irreducibile is considerably easier.

So in this block, there is a cell for each real form of $\mathcal{O}^+$, and each cell has 2^m irreducible representations with infinitesimal character $\chi_{\check{\mathcal{O}}}$. In particular for $\mathcal{O}_{spl}$, the Levi component is $u(1,0)\times u(1,2)\times u(3,2)\times\cdots\times so(m,m+1)$. For this case, section 9.6 produced exactly 2^m parameters; their lowest $K-$types are of the form $\mu_e(n-k,k)$. These are the only possible constituents of the induced from $L(\chi_{\check{\mathcal{O}}})$. Since the constituents of the restriction of a $\mu_e(n-k,k)$ to a Levi component are again $\mu_e(m-l,l)$'s, the only way $L(\chi_{\check{\mathcal{O}}})$ can fail to be unitary is if the form is negative on one of the $K-$types $\mu_e(n-k,k)$. But sections 5 and 6.2 show that the form is positive on the $K-$types μ_e of $L(\chi_{\check{\mathcal{O}}})$. □

10. Irreducibility {sec:10}1}

10.1. To complete the classification of the unitary dual we also need to prove the following irreducibility theorem. It is needed to show that the regions in theorem 3.1 are indeed unitary in the real case.

{t:10.1}

Theorem. *Assume $\check{\mathcal{O}}$ is even, and such that $x_{i-1} = x_i = x_{i+1}$ for some i. Let $\mathfrak{m} = gl(x_i) \times \mathfrak{g}(n-x_i)$, and $\check{\mathcal{O}}_1 \subset \mathfrak{g}(n-x_i)$ be the nilpotent orbit obtained from $\check{\mathcal{O}}$ by removing two rows of size x_i. Then*

$$L(\chi_{\check{\mathcal{O}}}) = Ind_{GL(x_i)\times G(n-x_i)}^{G(n)}[triv \otimes L(\chi_{\check{\mathcal{O}}_1})].$$

In the $p-$adic case this follows from the work of Kazhdan-Lusztig ([BM1]). In the real case, it follows from the following proposition.

{p:10.1}

Proposition. *The associated variety of a spherical representation $L(\chi_{\check{\mathcal{O}}})$ is given by the sum with multiplicity one of the following nilpotent orbits.*

- **Type B, D:** *On the odd sized rows, the difference between the number of $+$'s and number of $-$'s is 1, 0 or -1.*
- **Type C:** *On the even sized rows, the difference between the number of $+$'s and number of $-$'s is 1, 0 or -1.*

The proof of the proposition is lengthy, and follows from more general results which are unpublished ([B5]). We will give a different proof of theorem 10.1 in the next sections.

Remark. When $\check{\mathcal{O}}_1$ is even, but $\check{\mathcal{O}}$ is not, and just $x_i = x_{i+1}$, the proof follows from [BM1] in the $p-$adic case, and the Kazhdan-Lusztig conjectures for nonintegral infinitesimal character in the real case. We have already used these results in the course of the paper. □

The outline of the proof is as follows. In section 2, we prove some auxiliary reducibility results in the case when $\check{\mathcal{O}}$ is induced from the trivial nilpotent orbit of a maximal Levi component. In section 3, we combine these results with intertwining operator techniques to complete the proof of theorem 10.1.

{sec:10.2}

10.2. We need to study the $\rho-$induced modules from the trivial module on $\mathfrak{m} \subset \mathfrak{g}(n)$ where $\mathfrak{m} \cong gl(n)$, or in some cases $\mathfrak{m} \cong gl(a) \times \mathfrak{g}(b)$ with $a+b = n$.

Type B. Assume $\check{\mathcal{O}}$ corresponds to the partition $2x_0 = 2x_1 = 2a$ in $sp(n, \mathbb{C})$. The infinitesimal character is $(-a+1/2, \dots, a-1/2)$ and the nilpotent orbit $\mathcal{O}_c$ corresponds to $(1, 1, \underbrace{2, \dots, 2}_{2a-2}, 3)$. We are interested in the composition series of

{eq:10.2.1o}
$$Ind_{GL(2a)}^{G(2a)}[triv]. \qquad (10.2.1)$$

There are three real forms of $\mathcal{O}_c$ in $so(2a+1, 2a)$,

{eq:10.2.1}
$$\begin{array}{ccc} + & - & + \\ + & - & \\ - & + & \\ \vdots & \vdots & \\ + & - & \\ - & + & \\ + & & \\ + & & \end{array} \qquad \begin{array}{ccc} + & - & + \\ + & - & \\ - & + & \\ \vdots & \vdots & \\ + & - & \\ - & + & \\ + & & \\ - & & \end{array} \qquad \begin{array}{ccc} - & + & - \\ + & - & \\ - & + & \\ \vdots & \vdots & \\ + & - & \\ - & + & \\ + & & \\ + & & \end{array} \tag{10.2.2}$$

The associated cycle of (10.2.1) is the middle nilpotent orbit in (10.2.2) with multiplicity 2. Section 6 shows that there are at least two factors characterized by the fact that they contain the petite $K-$types which are the restrictions to $S[O(2a+1) \times O(2a)]$ of

$$\begin{aligned} &\mu(0^a; +) \otimes \mu(0^a; +) \\ &\mu(1, 0^{a-1}; -) \otimes \mu(0^a; -). \end{aligned} \tag{10.2.3}$$ {eq:10.2.3}

Thus because of multiplicity 2, there are exactly two factors. One of the factors is spherical. The nonspherical factor has Langlands parameter

$$\begin{aligned} \lambda^G &= (1/2, 0, \dots, 0 \mid 0, \dots, 0), \\ \nu &= (0, a-1/2, a-1/2, \dots, 3/2, 3/2, 1/2). \end{aligned} \tag{10.2.4}$$ {eq:10.2.1p}

The Cartan subalgebra for the nonspherical parameter is such that the root ϵ_1 is noncompact imaginary, $\epsilon_i, \epsilon_i \pm \epsilon_j$ with $j > i \geq 2$, are real. The standard module $X(\lambda^G, \nu)$ which has $\overline{X}(\lambda^G, \nu)$ as quotient is the one for which ν is dominant. Thus we conjugate the Cartan subalgebra such that ϵ_{2a} is noncompact imaginary, $\epsilon_i, \epsilon_i \pm \epsilon_j$ with $i < j < 2a$ are real, and the usual positive system $\Delta^+ = \{\epsilon_i, \epsilon_i \pm \epsilon_j\}_{i<j}$.

Type C. Consider $\check{\mathcal{O}}$ which corresponds to the partition $2x_0 = 2x_1 = 2a + 1 < 2x_2 = 2b + 1$ in $so(n, \mathbb{C})$. The infinitesimal character is

$$(-a, \dots, a)(-b, \dots, -1) \tag{10.2.5}$$ {eq:10.2.2}

The nilpotent orbit $\mathcal{O}_c$ is induced from the trivial one on $gl(2a+1) \times \mathfrak{g}(b)$ and corresponds to

$$(\underbrace{1, \dots, 1}_{2b-2a-2}, 2, 2, \underbrace{3, \dots, 3}_{2a}). \tag{10.2.6}$$ {eq:10.2.3a}

We are interested in the composition series of

$$Ind_{GL(2a+1) \times G(b)]}^{G(2a+b+1)}[triv]. \tag{10.2.7}$$ {eq:10.2.4a}

There are three real forms of (10.2.6),

$$
\begin{array}{ccc}
+ & - & + \\
- & + & - \\
\vdots & & \vdots \\
+ & - & + \\
- & + & - \\
+ & - & \\
+ & - & \\
+ & & \\
- & & \\
\vdots & & \\
+ & & \\
- & &
\end{array}
\qquad
\begin{array}{ccc}
+ & - & + \\
- & + & - \\
\vdots & & \vdots \\
+ & - & + \\
- & + & - \\
+ & - & \\
- & + & \\
+ & & \\
- & & \\
\vdots & & \\
+ & & \\
- & &
\end{array}
\qquad
\begin{array}{ccc}
+ & - & + \\
- & + & - \\
\vdots & & \vdots \\
+ & - & + \\
- & + & - \\
- & + & \\
- & + & \\
+ & & \\
- & & \\
\vdots & & \\
+ & & \\
- & &
\end{array}
\tag{10.2.8}
$$
{eq:10.2.4}

The AC cycle of (10.2.7) consists of the middle nilpotent orbit in (10.2.8) with multiplicity 2. By a similar argument as for type B, we conclude that the composition series consists of two representations containing the petite $K-$types

{eq:10.2.4c}
$$
\begin{aligned}
&\mu(0^n),\\
&\mu(1^{a+1}, 0^{b-1}, (-1)^{a+1}).
\end{aligned}
\tag{10.2.9}
$$

These are also the lowest $K-$types of the represntations. The nonspherical representation has parameter

{eq:10.2.4b}
$$
\begin{aligned}
&\lambda^G = (1/2^a, 0^b, -1/2)^a),\\
&\nu = (1/2^a, 0^b, 1/2^a).
\end{aligned}
\tag{10.2.10}
$$

Type D. Let $\check{\mathcal{O}}$ correspond to the partition $2x_0 = 2x_1 = 2a+1$ in $so(n,\mathbb{C})$. The infinitesimal character is $(-a,\dots,a)$. The real forms of the nilpotent orbit $\mathcal{O}$ are

{eq:10.2.5}
$$
\begin{array}{cc}
+ & - \\
- & + \\
\vdots & \vdots \\
+ & - \\
- & +
\end{array}
\tag{10.2.11}
$$

There are two nilpotent orbits with this partition labelled $I,\ II$. Each of them is induced from $\mathfrak{m} \cong gl(2a)$, there are two such Levi components. We are interested in the induced modules

{eq:10.2.6}
$$
Ind_{GL(2a)}^{G(2a)}[triv]. \tag{10.2.12}
$$

The multiplicity of the nilpotent orbit (10.2.11) in the AC cycle of (10.2.12) is 1, so the representations are irreducible.

We summarize these calculations in a proposition.

{p:10.2}

Proposition. *The composition factors of the induced module from the trivial representation on $\mathfrak{m}$ all have relevant lowest $K-$types. In particular, the induced module is generated by spherically relevant $K-$types. Precisely,*
Type B: *the representation is generated by the $K-$types of the form μ_e,*
Type C: *the representation is generated by the $K--$types of the form μ_o,*
Type D: *the representation is generated by $\mu_e(0) = \mu_o(0)$.*

{sec10:3}

10.3. We now prove the irreducibility result mentioned at the beginning of the section in the case of $\mathfrak{g}$ of type B; the other cases are similar. Let $\check{\mathcal{O}}_1$ be the nilpotent orbit where we have removed one string of size $2a$. Let $\mathfrak{m} := gl(2a) \times \mathfrak{g}(n-2a)$. Then $L(\chi_{\check{\mathcal{O}}})$ is the spherical subquotient of the induced representation

$$I(a, L(\chi_{\check{\mathcal{O}}_1})) := Ind_{\mathfrak{m}}^{\mathfrak{g}}[(-a+1/2, \dots, a-1/2) \otimes L(\chi_{\check{\mathcal{O}}_1})]. \qquad (10.3.1)$$ {eq:10.3.1}

It is enough to show that if a parameter is unipotent, and satisfies $x_{i-1} = x_i = x_{i+1} = a$, then $I(a, L(\chi_{\check{\mathcal{O}}_1}))$ is generated by its $K-$types μ_e. This is because by theorem 5.3, the $K-$types of type μ_e in (10.3.1) occur with full multiplicity in the spherical irreducible subquotient, and the module is unitary.

First, we reduce to the case when there are no $0 < x_j < a$. Let ν be the dominant parameter of $L(\chi_{\check{\mathcal{O}}})$, and assume i is the smallest index so that $x_{i-1} = a$. There is an intertwining operator

$$X(\nu) \longrightarrow I(1/2, \dots, x_0 - 1/2; \dots ; 1/2, \dots, x_{i-2} - 1/2; \nu') \qquad (10.3.2)$$ {eq:10.3.2}

where I is induced from $gl(x_0) \times \dots \times gl(x_{i-2}) \times g(n - \sum_{j<i-1} x_j)$ with characters on the gl's corresponding to the strings in (10.3.2) and the irreducible module $L(\nu')$ on $\mathfrak{g}(n - \sum_{j<i-1} x_j)$. The intertwining operator is onto, and thus the induced module is generated by its spherical vector. By the induction hypothesis, the induced module from $(-a+1/2, \dots, a-/2) \otimes L(\nu'')$ on $gl(2a) \times \mathfrak{g}(n - \sum_{j \le i} x_j)$ is irreducible. But

$$I(1/2, \dots, x_0 - 1/2; \dots ; 1/2, \dots, x_{i-2} - 1/2; -a+1/2, \dots, a-1/2; \nu'') \cong$$
$$I(-a+1/2, \dots, a-1/2; 1/2, \dots, x_0 - 1/2; x \dots ; 1/2, \dots, x_{i-2} - 1/2, \nu'') \qquad (10.3.3)$$ {eq:10.3.3}

This module maps by an intertwining operator onto $I(a, L(\chi_{\check{\mathcal{O}}_1}))$, so $I(a, L(\chi_{\check{\mathcal{O}}_1}))$ is generated by its spherical vector.

So we have reduced to the case when

$$\begin{aligned} x_0 = x_1 = x_2 = a, \qquad &\text{or} \\ x_0 = 0 < x_1 = x_2 = x_3 = a. \end{aligned} \qquad (10.3.4)$$ {eq:10.3.4}

Suppose we are in the first case of (10.3.4) and $m = 1$. The infinitesimal character is

$$(a-1/2, a-1/2, a-1/2, \dots, 1/2, 1/2, 1/2),$$

each coordinate occuring three times. The induced module

$$I(-a+1/2,\dots,a-1/2) \tag{10.3.5}$$ {eq:10.3.5}

of $\mathfrak{g}(2a)$ is a direct sum of irreducible factors computed in section 10.2; in particular it is generated by $K-$types of the form $\mu_e(2a-k,k)$ (with $k=0,1$). Consider the module

{eq:10.3.6} $$I(a-1/2;\dots;1/2;-a+1/2,\dots,a-1/2), \tag{10.3.6}$$

induced from characters on $GL(1)\times\dots\times GL(1)\times GL(2a)$. It is a direct sum of induced modules from the two factors of (10.3.5). Each such induced module is a homomorphic image of the corresponding standard module with dominant parameter. So (10.3.6) is also generated by its μ_e isotypic components. But then

{eq:10.3.7} $$\begin{aligned}&I(a-1/2;\dots;1/2;-a+1/2,\dots,a-1/2)\cong\\&I(-a+1/2,\dots,a-1/2;a-1/2;\dots;1/2)\end{aligned} \tag{10.3.7}$$

so the latter is also generated by its μ_e isotypic components. Finally, the intertwining operator

{eq:10.3.8} $$I(a-1/2;\dots;1/2)\longrightarrow I(1/2,\dots,a-1/2) \tag{10.3.8}$$

is onto, and the image of the intertwining operator

{eq:10.3.9} $$I(1/2,\dots,a-1/2)\longrightarrow I(-a+1/2,\dots,-1/2) \tag{10.3.9}$$

is onto $L(-a+1/2,\dots,-1/2)$. Thus the module induced from $gl(2a)\times\mathfrak{g}(a)$,

{eq:10.3.10} $$I(-a+1/2,\dots,a-1/2;L(-a+1/2,\dots,-1/2)), \tag{10.3.10}$$

is generated by its μ_e isotypic components. Since the multiplicity of these $K-$types in (10.3.10) is the same as in the irreducible spherical module, it follows that they must be equal.

Now suppose that we are in the first case of (10.3.4) and $m>1$, or in the second case, and $m>2$. The parameter has another $x_{2m-1}\le x_{2m}$. We use an argument similar to the one above to show that the module

{eq:10.3.11} $$I(-x_{2m-1}+1/2,\dots,x_{2m}-1/2,L(\chi_{\check{\mathcal{O}}_2})), \tag{10.3.11}$$

where $\check{\mathcal{O}}_2$ is the nilpotent orbit with partition obtained from $\check{\mathcal{O}}$ by removing $2x_{2m-1},2x_{2m}$, is generated by its μ_e isotypic components. The claim then follows because the induced module is a homomorphic image of (10.3.11). Precisely, $X(\nu)$ maps onto

{eq:10.3.12} $$\begin{aligned}I(x_{2m-1}+1/2,\dots,&x_{2m}-1/2;1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2;\\&L(-x_{2m-1}+1/2,-x_{2m-1}+1/2,\dots,-1/2,-1/2))\end{aligned} \tag{10.3.12}$$

So this module is generated by its spherical vector. Replace $L(-x_{2m-1}+1/2,-x_{2m-1}+1/2,\dots,-1/2,-1/2)$ by $I(-x_{2m-1}+1/2,\dots,x_{2m-1}-1/2)$. The ensuing module is a direct sum of two induced modules by section 10.2.

They are both homomorphic images of standard modules, so generated by their lowest $K-$types, which are of type μ_e. Next observe that the map

$$\begin{aligned}&I(x_{2m-1}+1/2,\dots,x_{2m}-1/2;1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2;\\ &\qquad\qquad -x_{2m-1}+1/2,\dots,x_{2m-1}-1/2)\longrightarrow\\ &I(-x_{2m-1}+1/2,\dots x_{2m}-1/2;1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2)\end{aligned} \tag{10.3.13}$$

{eq:10.3.13}

is onto. So the target module is generated by its μ_e isotypic components. The module

$$I(1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2) \tag{10.3.14}$$

{eq:10.3.14}

(the string $(-x_{2m-1}+1/2,\dots,x_{2m}-1/2)$ has been removed) has $L(-x_{2m-2}+1/2,\dots,1/2)$ as its unique irreducible quotient, because it is the homomorphic image of an $X(\nu)$ with ν dominant. Therefore it is generated by its spherical vector. Combining this with the induction assumption, we conclude that

$$I(-x_{2m-1}+1/2,\dots,x_{2m}-1/2;-a+1/2,\dots,a-1/2;L(\check{\mathcal{O}}_3)) \tag{10.3.15}$$

{eq:10.3.15}

is generated by its μ_e isotypic components. It is isomorphic to

$$I(-a+1/2,\dots,a-1/2;-x_{2m-1}+1/2,\dots,x_{2m}-1/2;L(\check{\mathcal{O}}_3)). \tag{10.3.16}$$

{eq:10.3.16}

Finally, the multiplicities of the μ_e isotypic components of $I(-x_{2m-1}+1/2,\dots,x_{2m}-1/2;L(\check{\mathcal{O}}_3))$ are the same as for the irreducible subquotient $L(\check{\mathcal{O}}_1))$. This completes the proof of the claim in this case.

Remains to consider the case when $m=2$ and $x_0=0<x_1=x_2=x_3=a\le x_4$. In this case, the module

$$I(a+1/2,\dots,x_4-1/2;-a+1/2,\dots,a-1/2;-a+1/2,\dots,a-1/2) \tag{10.3.17}$$

{eq:10.3.17}

is generated by its μ_e isotypic components because of proposition 10.2, and arguments similar to the above. Therefore the same holds for

$$I(-a+1/2,\dots,x_4-1/2;-a+1/2,\dots,a-1/2), \tag{10.3.18}$$

{eq:10.3.18}

which is a homomorphic image via the intertwining operator which interchanges the first two strings. But this is isomorphic to

$$I(-a+1/2,\dots,a-1/2,-a+1/2,\dots,x_4-1/2). \tag{10.3.19}$$

{eq:10.3.19}

Then $I(-a+1/2,\dots,a-1/2,L(-x_4+1/2,\dots,-1/2,-1/2)$ is a homomorphic image of (10.3.19) so it is generated by its μ_e isotypic components. By section 5.3, the multiplicities of the μ_e isotypic components are the same in $I(-a+1/2,\dots,a-1/2,L(-x_4+1/2,\dots,-1/2,-1/2))$ as in $L(\chi_{\check{\mathcal{O}}})$.

This completes the proof of theorem 10.1. □

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