Dirac Cohomology and Unitary Representations

Dan Barbasch, Cornell University

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Joint with Chao-Ping-Dong, Pavle Pandzic, and Daniel Wong

Background

If a Lie group G acts on a manifold X, then it also induces a representation on functions on X, via

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

Typically there is a G-invariant measure dx on X.

For example:

$$C^\infty(X) \text{ is a smooth representation of } G$$

$$L^2(X) \text{ is a unitary representation of } G$$

A representation of G is a complex topological vector space V, typically complete, with a continuous G-action by linear operators.

Harmonic analysis: *"decompose such representations into irreducible representations."*

Irreducible Representations: those with no closed invariant subspace.

Example: $G = \mathbb{T}$, the circle group. The irreducible modules are 1-dimensional, spanned by functions $e^{it} \mapsto e^{int}$ on $\mathbb{T}$, $n \in \mathbb{Z}$, and

$$L^2(\mathbb{T}) = \widehat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}\, f_n$$

Connection with differential equations

Let Δ be a G-invariant differential operator on X.
Then any eigenspace of Δ is $G-$invariant.

Conversely, (by some version of Schur's Lemma) Δ acts by scalars on irreducible G-subspaces.

So in the presence of such an operator, decomposing the representation is related to finding Δ-eigenspaces.

The representation of G gives extra structure to the eigenspace.

Real reductive groups

G: a real reductive Lie group (often assumed connected).

Main examples: closed (Lie) subgroups of $GL(n, \mathbb{C})$, stable under the Cartan involution $\Theta(g) =^t \bar{g}^{-1}$.

E.g., $SL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, $O(p, q)$.

$K = G^{\Theta}$: maximal compact subgroup

E.g., $SO(n) \subset SL(n, \mathbb{R})$; $U(p) \times U(q) \subset U(p, q)$;
$U(n) \subset Sp(2n, \mathbb{R})$, $O(p) \times O(q) \subset O(p, q)$.

We will focus mainly on complex groups viewed as real groups, $GL(n, \mathbb{C}), So(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$.

$(\mathfrak{g}, K)$-modules

It is always easier to study representations of the Lie algebra, and then derive properties of the representations of the Lie group.

For real reductive groups, these are the $(\mathfrak{g}, K)-$modules.

Following Harish-Chandra, one associates a $(\mathfrak{g}, K)-$module to each representation of the group. Let V be an admissible representation V of G, i.e., $\dim \mathrm{Hom}(V_\delta, V) < \infty$ for all irreducible K-representations V_δ.

Let V_K be the space of K-finite vectors in V. These vectors are smooth i.e. one can differentiate the group action to get an action of the Lie algebra. $\mathfrak{g} = (\mathfrak{g}_0)_\mathbb{C}$, the complexification of the real Lie algebra acts automatically.

Definition

A $(\mathfrak{g}, K)-$module is a vector space V, with a Lie algebra action of $\mathfrak{g}$ and a locally finite action of K, which are compatible, i.e., induce the same action of $\mathfrak{k}_0 =$ the Lie algebra of K. (If K is disconnected, one requires also that the action $\mathfrak{g} \otimes V \to V$ is $K-$equivariant). Such a V can be decomposed under K as

$$V = \bigoplus_{\delta \in \hat{K}} m_\delta V_\delta.$$

V is called a Harish-Chandra module if it is finitely generated and all $m_\delta < \infty$.

Example: $G = SU(1,1) \quad (\cong SL(2,\mathbb{R}))$.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ = 2x2 matrices of trace 0 has a basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with commutation relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Since ih spans the Lie algebra of K, h diagonalizes on $(\mathfrak{g}, K)$-modules and has integer eigenvalues. The possible irreducible modules are

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \dots \\ k & k+2 & k+4 & \dots \end{array} \tag{1}$$

$$\begin{array}{cccc} \dots & \bullet & \bullet & \bullet \\ \dots & -k-4 & -k-2 & -k \end{array} \tag{2}$$

$$\begin{array}{cccc} \bullet & \bullet & \dots & \bullet \\ -n & -n+2 & \dots & n \end{array} \tag{3}$$

$$\begin{array}{ccccc} \dots & \bullet & \bullet & \bullet & \dots \\ \dots & i-2 & i & i+2 & \dots \end{array} \tag{4}$$

where $k > 0$, $n \geq 0$ and i are integers.

Each dot represents a 1-dimensional eigenspace for h. Numbers are the corresponding eigenvalues.

In each picture, e raises the eigenvalue by 2, and f lowers the eigenvalue by 2.

Pictures (1), (2) and (3) define unique modules.

(We know $ef - fe = h$; we know ef or fe at one point; so we know ef and fe at all points.)

For picture (4), there are many modules; we do not know ef or fe at any point. But we would know them if we knew $ef + fe$. We use

$$\mathrm{Cas}_{\mathfrak{g}} = \frac{1}{2}h^2 + ef + fe,$$

which commutes with $\mathfrak{g}$ and so acts by a scalar on any irreducible module.

Fixing this scalar determines the module.

(Not all values are allowed, the module may break up.)

Casimir element

In general, can define $\mathrm{Cas}_{\mathfrak{g}}$ in the center of the enveloping algebra $U(\mathfrak{g})$:

Fix a nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$ (e.g. $\operatorname{tr} XY$).

Take dual bases b_i, d_i of $\mathfrak{g}$ with respect to B.

Write

$$\mathrm{Cas}_{\mathfrak{g}} = \sum b_i d_i.$$

Infinitesimal character

The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a polynomial algebra; one of the generators is $\mathrm{Cas}_{\mathfrak{g}}$.

All elements of $Z(\mathfrak{g})$ act as scalars on irreducible modules.

This defines the infinitesimal character of M, $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$.

Harish-Chandra proved that $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)^W$, so infinitesimal characters correspond to $\mathfrak{h}^*/W$.

($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$; in examples, the diagonal matrices. W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$; it is a finite reflection group.)

Dirac operator on $\mathbb{R}^n$

Look for D such that $D^2 = -\sum \partial_i^2$. (Or $D^2 = \sum \pm\partial_i^2$.)

If $D = \sum e_i \partial_i$, get

$$e_i^2 = -1; \qquad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

So the coefficients should belong to the Clifford algebra $C(\mathbb{R}^n)$.

Identifying $\partial_i \leftrightarrow e_i$, yields

$$D = \sum e_i \otimes e_i \in D_{cc}(\mathbb{R}^n) \otimes C(\mathbb{R}^n).$$

($D_{cc}(\mathbb{R}^n)$: the algebra of constant coefficient differential oprators on $\mathbb{R}^n$).

The Clifford algebra for G

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

($\mathfrak{k}$ and $\mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution;

$\mathfrak{k}$ is the complexified Lie algebra of the maximal compact subgroup K of G.)

Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B:

the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx + 2B(x, y) = 0.$$

The Dirac operator for G

Let b_i be any basis of $\mathfrak{p}$; let d_i be the dual basis with respect to B.
Dirac operator:

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

D is independent of b_i and K-invariant.

D^2 is the spin Laplacian (Parthasarathy):

$$D^2 = -\operatorname{Cas}_{\mathfrak{g}} \otimes 1 + \operatorname{Cas}_{\mathfrak{k}_\Delta} + \text{constant}.$$

Here $\operatorname{Cas}_{\mathfrak{g}}$, $\operatorname{Cas}_{\mathfrak{k}_\Delta}$ are the Casimir elements of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$;

$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \quad \text{and} \quad \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p}).$$

The constant is computed explicitly, $||\rho||^2 - ||\rho_{\mathfrak{k}}||^2$.

Dirac cohomology

Let M be an admissible $(\mathfrak{g}, K)$-module. Let S be a spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ a maximal isotropic subspace.

Then D acts on $M \otimes S$.

Dirac cohomology of M:

$$H_D(M) = \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D$$

$H_D(M)$ is a module for the spin double cover $\widetilde{K}$ of K. It is finite-dimensional if M is of finite length.

If M is unitary, then D is self adjoint wrt an inner product. So

$$H_D(M) = \operatorname{Ker} D = \operatorname{Ker} D^2,$$

and $D^2 \geq 0$ (Parthasarathy's Dirac inequality).

Example: $G = SU(1,1) \cong SL(2,\mathbb{R})$

The modules corresponding to pictures (1)-(3) have $H_D \neq 0$.

$H_D(M)$ is equal to highest weight$+1$ and/or lowest weight-1.

The modules corresponding to picture (4) have $H_D = 0$.

Vogan's conjecture

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. View $\mathfrak{t}^* \subset \mathfrak{h}^*$ via extension by 0 over $\mathfrak{a}$.

The following was conjectured by Vogan in 1997, and proved by Huang-Pandzic in 2002.

Theorem

Assume M has infinitesimal character and $H_D(M)$ contains a $\widetilde{K}$-type E_γ of highest weight $\gamma \in \mathfrak{t}^*$.

Then the infinitesimal character of M is $\gamma + \rho_{\mathfrak{k}}$ (up to the Weyl group $W_{\mathfrak{g}}$).

Vogan's conjecture - structural version

Let $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}) \cong Z(\mathfrak{k}_\Delta)$ be the homomorphism corresponding under Harish-Chandra isomorphism to the restriction map $P(\mathfrak{h}^*)^{W_\mathfrak{g}} \to P(\mathfrak{t}^*)^{W_\mathfrak{k}}$. For any $z \in Z(\mathfrak{g})$, there is $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ such that

$$z \otimes 1 = \zeta(z) + Da + aD.$$

This implies the above module version, since $Da + aD$ acts as zero on $H_D(M)$.

Motivation

- unitarity: Dirac inequality and its improvements.
- irreducible unitary M with $H_D \neq 0$ are interesting (discrete series, $A_{\mathfrak{q}}(\lambda)$ modules, unitary highest weight modules, some unipotent representations...) They should form a nice part of the unitary dual.
- H_D is related to classical topics like generalized Weyl character formula, generalized Bott-Borel-Weil Theorem, construction of discrete series, multiplicities of automorphic forms
- There are nice constructions of representations with $H_D \neq 0$; e.g., Parthasarthy and Atiyah-Schmid constructed the discrete series representations using spin bundles on G/K.

Complex Groups

Let G be a complex reductive group viewed as a real group. Let K be a maximal compact subgroup of G. Let θ be the corresponding Cartan involution, and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G. Let $H = TA$ be a $\theta-$stable Cartan subgroup of G, with Lie algebra $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$, a $\theta-$stable Cartan subalgebra of $\mathfrak{g}_0$. We assume that $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{s}_0$.

Let $B = HN$ be a Borel subgroup of G. Let $(\lambda_L, \lambda_R) \in \mathfrak{h}_0 \times \mathfrak{h}_0$ be such that $\mu := \lambda_L - \lambda_R$ is integral. Write $\nu := \lambda_L + \lambda_R$. We can view μ as a weight of T and ν as a character of A. Let

$$X(\lambda_L, \lambda_R) := \mathrm{Ind}_B^G[\mathbb{C}_\mu \otimes \mathbb{C}_\nu \otimes \mathbb{1}]_{K-finite}.$$

Then the $K-$type with extremal weight μ occurs in $X(\lambda_L, \lambda_R)$ with multiplicity 1. Let $L(\lambda_L, \lambda_R)$ be the unique irreducible subquotient containing this $K-$type.

Admissible Representations

Theorem ([Zh], [BV])

1. *Every irreducible admissible* $(\mathfrak{g}, K)$ *module is of the form* $L(\lambda_L, \lambda_R)$.
2. *Two such modules* $L(\lambda_L, \lambda_R)$ *and* $L(\lambda'_L, \lambda'_R)$ *are equivalent if and only if the parameters are conjugate by* $\Delta(W) \subset W_c \cong W \times W$. *In other words, there is* $w \in W$ *such that* $w\mu = \mu'$ *and* $w\nu = \nu'$.
3. $L(\lambda_L, \lambda_R)$ *admits a nondegenerate hermitian form if and only if there is* $w \in W$ *such that* $w\mu = \mu$, $w\nu = -\overline{\nu}$.

This result is a special case of the more general Langlands classification, which can be found for example in [Kn], Theorem 8.54.

Spin Representation

We next describe the spin representation of the group $\widetilde{K}$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{b},\mathfrak{h})}\alpha$. Let r denote the rank of $\mathfrak{g}$.

Lemma

The spinor representation Spin viewed as a $\widetilde{K}$-module is a direct sum of $[\frac{r}{2}]$ copies of the irreducible representation $E(\rho)$ of $\widetilde{K}$ with highest weight ρ.

Lemma 4 implies that in calculating $H_D(\pi)$ for unitary π, one can replace *Spin* by $E(\rho)$ and then in the end simply multiply the result by multiplicity $[\frac{r}{2}]$.

So a unitary representation $L(\lambda_L,\lambda_R)$ has nonzero Dirac cohomology if and only if there is $(w_1,w_2)\in W_c$ such that

$$w_1\lambda_L + w_2\lambda_R = 0, \qquad w_1\lambda_L - w_2\lambda_R = \tau+\rho \tag{5}$$

where τ is the highest weight of a $\widetilde{K}-$type which occurs in $L(\lambda_L,\lambda_R)\otimes E(\rho)$. More precisely

$$[H_D(\pi):E(\tau)] = \sum_{\mu}[\frac{r}{2}]\,[\pi:E(\mu)]\,[E(\mu)\otimes E(\rho):E(\tau)], \tag{6}$$

Dirac Cohomology for Unitary Representations

Write $\lambda := \lambda_L$. The first equation in (5) implies that $\lambda_R = -w_2^{-1} w_1 \lambda$. The second one says that $2w_1\lambda = \tau + \rho$, so that $w_1\lambda$ must be regular, and $2w_1\lambda$ regular integral. Replace $w_1\lambda$ by λ. Thus we can write the parameter of π as $(\lambda, -s\lambda)$ with λ dominant, and $s \in W$. Since $L(\lambda, -s\lambda)$ is assumed unitary, it is hermitian. So there is $w \in W$ such that

$$w(\lambda + s\lambda) = \lambda + s\lambda, \quad w(\lambda - s\lambda) = -\lambda + s\lambda. \tag{7}$$

This implies that $w\lambda = s\lambda$, so $w = s$ since λ is regular, and $ws\lambda = s^2\lambda = \lambda$. So s must be an involution.

Thus to compute $H_D(\pi)$ for π that are unitary, we need to know

1. $L(\lambda, -s\lambda)$ that are unitary with

$$2\lambda = \tau + \rho, \tag{8}$$

   in particular 2λ is regular integral,
2. The multiplicity of τ in $H_D(\pi)$ is

$$\left[\ L(\lambda, -s\lambda) \otimes E(\rho)\ :\ E(\tau)\ \right]. \tag{9}$$

Unitary Dual

Theorem (Classical Groups, [B])

A hermitian module with infinitesimal character $(\lambda, -\lambda)$ with 2λ integral is unitary if and only if it is unitarily induced from a unipotent representation.

The full unitary dual for complex classical groups is computed in [B]. A self contained/different proof using petite $K-$types is now available. The unipotent representations in the Theorem for all complex groups are listed in [BP].

Theorem (B-D-W)

For classical groups, a unitary module has nontrivial Dirac cohomology if and only if it is unitarily induced from a unitary representation with nontrivial Dirac cohomology (implicitly 2λ is assumed regular integral). In all cases, $H_D(\pi)$ is irreducible.

Sketch of Proof I

We sketch the fact that the Dirac cohomology of a unitarily induced module is nonzero if the induced data have Dirac cohomology. This is already in [BP]. The theorem above is a refinement taking into account the shape of the unitary dual.

Theorem

Let $P = MN$ be a parabolic subalgebra of G and let $\Delta = \Delta_{\mathfrak{m}} \cup \Delta(\mathfrak{n})$ be the corresponding system of positive roots. Let $\pi_{\mathfrak{m}} := L_{\mathfrak{m}}(\lambda, -s\lambda)$ be an irreducible unitary representation of M with nonzero Dirac cohomology such that its parameter is zero on the center of $\mathfrak{m}$. Let ξ be a unitary character of M which is dominant with respect to Δ. Suppose that twice the infinitesimal character of $\pi = Ind_P^G[\pi_{\mathfrak{m}} \otimes \xi]$ is regular and integral. Then π has nonzero Dirac cohomology. □

Sketch of Proof II

We consider the Dirac cohomology of a representation π which is unitarily induced from a unitary representation of the Levi component M of a parabolic subgroup $P = MN$. We write $\pi := \mathrm{Ind}_P^G[\mathbb{C}_\xi \otimes \pi_\mathfrak{m}]$ where ξ is a unitary character of M, and $\pi_\mathfrak{m}$ is a unitary representation of M such that the center of M acts trivially. It is straightforward that $\pi_\mathfrak{m}$ has Dirac cohomology if and only if $\mathbb{C}_\xi \otimes \pi_\mathfrak{m}$ has Dirac cohomology.
The representation $\pi_\mathfrak{m} = L_\mathfrak{m}(\lambda_\mathfrak{m}, -s\lambda_\mathfrak{m})$ satisfies

$$\lambda_\mathfrak{m} + s\lambda_\mathfrak{m} = \mu_\mathfrak{m}, \qquad 2\lambda_\mathfrak{m} = \mu_\mathfrak{m} + \nu_\mathfrak{m}, \tag{10}$$

$$\lambda_\mathfrak{m} - s\lambda_\mathfrak{m} = \nu_\mathfrak{m}, \qquad 2s\lambda_\mathfrak{m} = \mu_\mathfrak{m} - \nu_\mathfrak{m}, \tag{11}$$

with $s \in W_\mathfrak{m}$.
Assume that $\pi_\mathfrak{m}$ has Dirac cohomology. So

$$2\lambda_\mathfrak{m} = \mu_\mathfrak{m} + \nu_\mathfrak{m} = \tau_\mathfrak{m} + \rho_\mathfrak{m} \tag{12}$$

Sketch of Proof III

is regular integral for a positive system $\Delta_{\mathfrak{m}}$. Here $\tau_{\mathfrak{m}}$ is dominant with respect to $\Delta_{\mathfrak{m}}$, and $\rho_{\mathfrak{m}}$ is the half sum of the roots in $\Delta_{\mathfrak{m}}$. Also,

$$\left[\pi_{\mathfrak{m}} \otimes F(\rho_{\mathfrak{m}}) \ : \ F(\tau_{\mathfrak{m}})\right] \neq 0. \tag{13}$$

For a dominant $\mathfrak{m}$-weight χ, let $F(\chi)$ denote the finite-dimensional $\mathfrak{m}$-module with highest weight χ. For a dominant $\mathfrak{g}$-weight η, let $E(\eta)$ denote the finite-dimensional $\mathfrak{g}$-module with highest weight η. We are also going to use analogous notation when χ and η are not necessarily dominant, but any extremal weights of the corresponding modules.

The lowest $K-$type subquotient of π is $L(\lambda, -s\lambda)$. It has parameters

$$\begin{aligned} \lambda &= \xi/2 + \lambda_{\mathfrak{m}}, & \mu &= \xi + \mu_{\mathfrak{m}}, \\ s\lambda &= \xi/2 + s\lambda_{\mathfrak{m}}, & \nu &= \nu_{\mathfrak{m}}. \end{aligned} \tag{14}$$

Sketch of Proof IV

We assume that ξ is dominant for $\Delta(\mathfrak{n})$ the roots of N. This is justified in view of the results in [V1] and [B] which say that any unitary representation is unitarily induced irreducible from a representation $\pi_{\mathfrak{m}}$ on a Levi component with these properties.

In order to have Dirac cohomology, 2λ must be regular integral; so assume this is the case. Let Δ' be the positive system such that λ is dominant. Then

$$2\lambda = \xi + \mu_{\mathfrak{m}} + \nu_{\mathfrak{m}} = \tau' + \rho'. \tag{15}$$

Here ρ' is the half sum of the roots in Δ', and τ' is dominant with respect to Δ'.

Sketch of Proof V
Lemma

$$E(\rho)\mid_{\mathfrak{m}} = F(\rho_{\mathfrak{m}}) \otimes \mathbb{C}_{-\rho_{\mathfrak{n}}} \otimes \bigwedge^{*} \mathfrak{n},$$

where $F(\rho_{\mathfrak{m}})$ *denotes the irreducible* $\mathfrak{m}$*-module with highest weight* $\rho_{\mathfrak{m}}$*, and* $\rho_{\mathfrak{n}}$ *denotes the half sum of roots in* $\Delta(\mathfrak{n})$*.*

Sketch of Proof VI

Proof.

Since $\mathfrak{g}$ and $\mathfrak{m}$ have the same rank, we can use Lemma 4 to replace $E(\rho)$ and $F(\rho_{\mathfrak{m}})$ by the corresponding spin modules. Recall that the spin module $Spin_{\mathfrak{m}}$ can be constructed as $\bigwedge^* \mathfrak{m}^+$, where $\mathfrak{m}^+$ is a maximal isotropic subspace of $\mathfrak{m}$. We can choose $\mathfrak{m}^+$ so that it contains all the positive root subspaces for $\mathfrak{m}$, as well as a maximal isotropic subspace $\mathfrak{h}^+$ of the Cartan subalgebra $\mathfrak{h}$. To construct $Spin_{\mathfrak{g}}$, we can use the maximal isotropic subspace $\mathfrak{g}^+ = \mathfrak{m}^+ \oplus \mathfrak{n}$ of $\mathfrak{g}$. It follows that $Spin_{\mathfrak{g}} = Spin_{\mathfrak{m}} \otimes \mathbb{C}_{-\rho_{\mathfrak{n}}} \otimes \bigwedge^* \mathfrak{n}$. The ρ-shift comes from the fact that the highest weight of $Spin_{\mathfrak{m}}$ is $\rho_{\mathfrak{m}}$ and the highest weight of $Spin_{\mathfrak{g}}$ is ρ, while the highest weight of $Spin_{\mathfrak{m}} \otimes \bigwedge^* \mathfrak{n}$ is $\rho_{\mathfrak{m}} + 2\rho_{\mathfrak{n}} = \rho + \rho_{\mathfrak{n}}$. □

Sketch of Proof VII

Since π is unitary, the computation for its Dirac cohomology is

$$\begin{aligned}&\left[\pi\otimes E(\rho)\ :\ E(\tau')\right]=\left[\pi_{\mathfrak{m}}\otimes\mathbb{C}_{\xi}\otimes E(-\tau')\mid_{\mathfrak{m}}:\ E(\rho)\mid_{\mathfrak{m}}\right]=\\&\left[\pi_{\mathfrak{m}}\otimes\mathbb{C}_{\xi}\otimes E(-\tau')\mid_{\mathfrak{m}}:\ F(\rho_{\mathfrak{m}})\otimes\mathbb{C}_{-\rho_{\mathfrak{n}}}\otimes\bigwedge\mathfrak{n}^{\star}\right]=\\&\left[\mathbb{C}_{\xi+\rho_{\mathfrak{n}}}\otimes\pi_{\mathfrak{m}}\otimes F(\rho_{\mathfrak{m}})\otimes E(-\tau')\mid_{\mathfrak{m}}:\ \bigwedge\mathfrak{n}^{\star}\right].\end{aligned}\tag{16}$$

Here the first equality used Frobenius reciprocity, while the second equality used Lemma 8. Note that the dual of $E(\tau')$ is the module $E(-\tau')$ which has lowest weight $-\tau'$ with respect to Δ'.
Using (15) and (12), we can write

$$-\tau'=-2\lambda+\rho'=-\xi-\mu_{\mathfrak{m}}-\nu_{\mathfrak{m}}+\rho'=-\xi-\tau_{\mathfrak{m}}-\rho_{\mathfrak{m}}+\rho'.\tag{17}$$

We have assumed ξ to be dominant for $\Delta(\mathfrak{n})$, and $2\lambda_{\mathfrak{m}}$ is dominant for $\Delta(\mathfrak{m})$. Thus $\Delta_{\mathfrak{m}}\subset\Delta,\Delta'$. Because of (13), the LHS of the last line of (16) contains the representation

$$\mathbb{C}_{\xi+\rho_{\mathfrak{n}}}\otimes F(\tau_{\mathfrak{m}})\otimes E(-\tau')\mid_{\mathfrak{m}}\supseteq\mathbb{C}_{\xi+\rho_{\mathfrak{n}}}\otimes F(\tau_{\mathfrak{m}}-\tau').$$

Sketch of Proof VIII

Namely, $F(\tau_{\mathfrak{m}} - \tau')$ is the PRV component of $F(\tau_{\mathfrak{m}}) \otimes F(-\tau') \subseteq F(\tau_{\mathfrak{m}}) \otimes E(-\tau') \mid_{\mathfrak{m}}$. By (15) and (12), $\tau_{\mathfrak{m}} - \tau' = -\xi - \rho_{\mathfrak{m}} + \rho'$, so

$$\mathbb{C}_{\xi+\rho_{\mathfrak{n}}} \otimes F(\tau_{\mathfrak{m}} - \tau') \supseteq F(\rho_{\mathfrak{n}} - \rho_{\mathfrak{m}} + \rho') = F(w_{\mathfrak{m}}\rho + \rho'),$$

where $w_{\mathfrak{m}}$ is the longest element of the Weyl group of $\mathfrak{m}$. Namely, $w_{\mathfrak{m}}$ sends all roots in $\Delta_{\mathfrak{m}}$ to negative roots for $\mathfrak{m}$, while permuting the roots in $\Delta(\mathfrak{n})$, so $w_{\mathfrak{m}}\rho = -\rho_{\mathfrak{m}} + \rho_{\mathfrak{n}}$.
So we see that the LHS of the last line of (16) contains the $\mathfrak{m}$-module $F(w_{\mathfrak{m}}\rho + \rho') = F(w_{\mathfrak{m}}\rho' + \rho)$. Namely, both $w_{\mathfrak{m}}\rho + \rho'$ and $w_{\mathfrak{m}}\rho' + \rho = w_{\mathfrak{m}}(w_{\mathfrak{m}}\rho + \rho')$ are extremal weights for the same module.
We will show that

$$\left[F(w_{\mathfrak{m}}\rho' + \rho) : \bigwedge \mathfrak{n}^*\right] \neq 0. \qquad (18)$$

Sketch of Proof IX

This will prove that (16) is nonzero, and consequently that π has nonzero Dirac cohomology.
Note that $w_{\mathfrak{m}}\rho' + \rho$ is a sum of roots in $\Delta(\mathfrak{n})$, and antidominant for $\Delta_{\mathfrak{m}}$, because for any simple $\gamma \in \Delta_{\mathfrak{m}}$, $\langle \rho', \check{\gamma} \rangle \in \mathbb{N}^+$ and $\langle \rho, \check{\gamma} \rangle = 1$. Moreover,

$$w_{\mathfrak{m}}\rho' + \rho = \sum_{\langle \alpha, w_{\mathfrak{m}}\rho' \rangle > 0,\ \langle \alpha, \rho \rangle > 0} \alpha. \tag{19}$$

To show that (18) holds, it is enough to show that

$$v := \bigwedge_{\langle \alpha, \rho \rangle > 0,\ \langle \alpha, w_{\mathfrak{m}}\rho' \rangle > 0} e_\alpha \quad \in \bigwedge \mathfrak{n}^\star \tag{20}$$

is a lowest weight vector for $\Delta_{\mathfrak{m}}$. Here e_α denotes a root vector for the root α.

Sketch of Proof X

Let $\gamma \in \Delta_{\mathfrak{m}}$. Then, up to constant factors,

$$\operatorname{ad} e_{-\gamma} e_{\alpha} = \begin{cases} 0 & \text{if } \alpha - \gamma \text{ is not a root,} \\ e_{-\gamma+\alpha} & \text{if } \alpha - \gamma \text{ is a root.} \end{cases} \tag{21}$$

But $\langle -\gamma, w_{\mathfrak{m}}\rho' \rangle > 0$, and $\langle \alpha, w_{\mathfrak{m}}\rho' \rangle > 0$ by assumption, so

$$\langle -\gamma + \alpha, w_{\mathfrak{m}}\rho' \rangle > 0 + 0 = 0. \tag{22}$$

Also, if $-\gamma + \alpha$ is a root, then it is in $\Delta(\mathfrak{n})$, since $\alpha \in \Delta(\mathfrak{n})$ and $\mathfrak{n}$ is an $\mathfrak{m}$-module. So $\langle -\gamma + \alpha, \rho \rangle > 0$. Thus every $e_{-\gamma+\alpha}$ appearing in (21) is one of the factors in (20).

The claim now follows from the formula

$$\operatorname{ad} e_{-\gamma} \bigwedge e_{\alpha} = \sum e_{\alpha_1} \wedge \cdots \wedge \operatorname{ad} e_{-\gamma} e_{\alpha_i} \wedge \ldots . \tag{23}$$

In each summand either $\operatorname{ad} e_{-\gamma} e_{\alpha_i}$ equals 0, or is a multiple of one of the root vectors already occurring in the same summand. So $\operatorname{ad} e_{-\gamma} v = 0$. We have proved the following theorem.

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