UNITARY REPRESENTATIONS AND IWAHORI-HECKE ALGEBRAS

1. Notation

1.1. Let $\mathbb{G}$ be a linear algebraic connected reductive group defined over $\mathbb{R}$ and let $G := \mathbb{G}(\mathbb{R})$ be its rational points. A representation $(\pi, \mathcal{H})$ of a group is a continuous homomorphism

$$\pi : G \longrightarrow Aut(\mathcal{H}) \tag{1.1.1}$$

where $\mathcal{H}$ is a Hilbert space. The representation is called *irreducible* if it has no nontrivial closed G-invariant subspace. It is called *completely reducible* if any closed G invariant subspace V has a closed G-invariant complement. It is called *unitarizable* if $\mathcal{H}$ admits a G-invariant positive definite inner product.

2. Unitarity

2.1. One of the basic problems of representation theory is to classify unitary irreducible modules. This has applications in harmonic analysis and automorphic forms. The general problem is that if $H \subset G$ is a closed subgroup and G/H admits a G-invariant measure, one wants to decompose $L^2(G/H)$. For example $G = SL(2,\mathbb{R})$ and Γ any discrete subgroup so that $\mathrm{vol}(G/\Gamma) < \infty$ is related to the classical theory of modular forms on the upper half plane.

2.2. **Strategy.**

1. reduce to an algebraic problem,
2. find some constructions from smaller groups that preserve unitarity
3. for each G (with say simple Lie algebra) give a *cuspidal* set of representations such that the dual is obtained from them via the constructions in (2).

Part (1) is due to Harish-Chandra. Let $\mathfrak{g} := Lie(G)$ and $\mathfrak{g}_c$ its complexification. Then $\mathfrak{g}_c$ acts on

$$\mathcal{H}_\infty := \{v \in \mathcal{H} \mid g \mapsto \pi(g)v \text{ is } C^\infty\} \tag{2.2.1}$$

which is dense in $\mathcal{H}$. The action is by differentiation:

$$\pi(X)v := \frac{d}{dt}|_{t=0}\pi(\exp(-tX))v.$$

Then $\mathfrak{g}_c$ also acts on

$$\mathcal{H}_K := \{v \in \mathcal{H}_\infty \ : \ \dim \pi(K)v < \infty\} \tag{2.2.2}$$

This gives rise to a $(\mathfrak{g}_c, K)$-module (π_K, V), $(V = \mathcal{H}_K$ of course) *i.e.* a $\mathfrak{g}_c$ module with a compatible action of K in the sense that

1. $\dim \pi(K)v < \infty$ for all $v \in V$,
2. $\pi(k)\pi(X)v = \pi(\mathrm{Ad}kX)\pi(k)v$.

If in addition $\dim \mathrm{Hom}[V_\delta, V] < \infty$ for all $\delta \in \widehat{K}$, the module is called *admissible*. Irreducible modules always give rise to admissible $(\mathfrak{g}_c, K)$-modules.

Since $\mathfrak{g}_c$ is the complexification of a real algebra, it has a conjugation $\overline{}$. Thus it makes sense for a module to be *hermitian i.e.* to have a nondegenerate hermitian form $\langle\ ,\ \rangle$ satisfying

$$\langle \pi(X)v_1, v_2\rangle + \langle v_1, \pi(X)v_2\rangle = 0. \tag{2.2.3}$$

If the form is positive definite, then we say the module is unitary.

If $(\pi, \mathcal{H})$ is hermitian *i.e.* $\mathcal{H}$ admits hermitian form $\langle\ ,\ \rangle$ such that

$$\langle \pi(g)v_1, \pi(g)v_2\rangle = \langle v_1, v_2\rangle$$

then this form is hermitian for $(\pi_K, \mathcal{H}_K)$.

Theorem (Harish-Chandra). *The module $(\pi_K, \mathcal{H}_K)$ is unitary if and only if $(\pi, \mathcal{H})$ is unitary.*

Thus we can proceed as follows.

- **(A):** Classify all irreducible $(\mathfrak{g}_c, K)$ modules,
- **(B):** find the subset which is formed of hermitian modules,
- **(C):** find the unitary ones among the hermitian modules.

Example. Let $G := SL(2,\mathbb{R})$. The Iwasawa decomposition is $G = KAN$ where

$$K = \{r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\}$$

(2.2.4)

$$A = \{a(t) := \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}\} \qquad N = \{n(x) := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\}$$

The induced module $X(\nu)$ is $(\pi_\nu, \mathcal{H}_\nu)$ given by

(2.2.5)

$$\mathcal{H}_\nu := \{f : G \longrightarrow \mathbb{C} \mid f(gan) = e^{-(\nu+1/2)t} f(g)\}, \qquad \pi_\nu(g)f(x) := f(g^{-1}x).$$

The space $\mathcal{H}_\infty$ is the space of C^∞ functions as above and

$$\mathcal{H}_K = span\{f_{2n} \in \mathcal{H}_\nu \mid f_{2n}(r(\theta)) = e^{2in\theta}\}. \tag{2.2.6}$$

These are the *K-types* and one can check easily that the module is admissible. This module $X(\nu)$ is irreducible except when $\nu = n + 1/2$ with $n \in \mathbb{Z}$. It is unitary for $\nu \in i\mathbb{R}$ and the form is

$$\langle f_1, f_2\rangle = \int_K f_1(k)\overline{f_2(k)}\, dk. \tag{2.2.7}$$

The reason for the shift by 1/2 in (2.2.5) is so that the inner product has this simpler form. Another way of putting it is that G/B where $B = AN$ does not admit a G-invariant form; the group B is *not unimodular*. The module is hermitian for ν imaginary or real. It is also unitary for $-1/2 < \nu < 1/2$; but the form is not as easy to describe. In general $X(\nu)$ has finite composition series and there is a unique subquotient $\overline{X}(\nu)$ characterized by the fact that it contains the constant function, *i.e.* the trivial K-type. At $\nu = 1/2 + n$ the spherical subquotient is a finite dimensional module.

3. Spherical Representations

3.1. We consider the special case of spherical representations of a split group. We fix a Cartan involution θ and let K be its fixed points. We say G is *split* if it has a split maximal torus A. In this case fix a Borel subgroup $B = AN$.

Example. *The groups $GL(n,\mathbb{R})$, $Sp(n,\mathbb{R})$ and $SO(p,q)$ with $|p-q| \le 1$ are all split. More general, every simple Lie algebra has a split real form.*

Let $M := K \cap B$. Then $A = MA_0$, where A_0 is the connected component of the identity of A. To each character $\nu \in \widehat{A}$ which is trivial on M (also called unramified) we attach the principal series $X(\nu)$ with underlying space

$$\mathcal{H}_\nu := Ind_B^G[\nu\delta_B^{-1/2}]_{K-finite}. \tag{3.1.1}$$

and action given by left translations. By Frobenius reciprocity,

$$\mathrm{Hom}_K[V_\delta : X(\nu)] \cong \mathrm{Hom}_M[V_\delta|_M : \mathbb{C}]. \tag{3.1.2}$$

The trivial K-type occurs exactly once. We denote by $\overline{X}(\nu)$ the irreducible spherical subquotient.

Theorem. *Any spherical irreducible module is isomorphic to an $\overline{X}(\nu)$. Two such modules $\overline{X}(\nu)$ and $\overline{X}(\nu')$ are isomorphic if and only if there is $w \in W$ such that $w\nu = \nu'$.*

We need to identify $\overline{X}(\nu)$ a more precise manner. If $w \in N(A)/C(A)$ is an element of the Weyl group, then there is an intertwining operator

$$A_w(\nu) : X(\nu) \longrightarrow X(w\nu). \tag{3.1.3}$$

If $Re\nu$ is dominant, then $X(\nu)$ has a unique irreducible quotient namely $\overline{X}(\nu)$. If $Re\nu$ is anti-dominant, then $\overline{X}(\nu)$ is the unique irreducible submodule of $X(\nu)$. So if w_0 is the long Weyl group element, then $\overline{X}(\nu)$ is the image of $A_{w_0}(\nu)$.

3.2. **Definition of Intertwining operators.** There are two definitions of intertwining operators which are essentially equivalent. One is as follows Suppose $B' = AN'$ is another Borel subgroup containing A. There are two induced modules $X(B,\nu)$ and $X(B',\nu)$. Formally we can pass from one to the other by integration:

$$Af(g) = \int_{N'/(N\cap N')} f(gx)\ dx. \tag{3.2.1}$$

It is easy to check that this is an intertwining operator. The convergence is a little harder.

The second version is to compose A with the "canonical" isomorphism $X(B',\nu) \cong X(B, w\nu)$. Let $w \in W$ be such that $wBw^{-1} = B'$. Then a function $f' \in X(B',\nu)$ corresponds to one in $X(B,w\nu)$ namely $f(g) := f'(gw^{-1})$.

We normalize these operators to be 1 on the spherical K-type.

3.3. **Properties.**

1. if $w = s_1 \cdot \dots \cdot s_k$ is a reduced decomposition, then $A_w = A_{s_1} \circ \dots \circ A_{s_k}$.
2. $\overline{X}(\nu)$ is hermitian if and only if there is $w \in W$ such that $w\nu = -\overline{\nu}$.
3. $X(\nu)^h \cong X(-\overline{\nu})$ so if (2) holds, the hermitian form on $\overline{X}(\nu)$ is

   $$\langle v, w\rangle = (v, A_w(\nu)w).$$

   Implicit in this statement is the fact that the radical is the largest proper submodule of $X(\nu)$.

Let $\delta \in \widehat{K}$. Since it is an intertwining operator, $A_w(\nu)$ it induces a map

$$A_w(\nu) : Hom_K[V_\delta : X(\nu)] \longrightarrow Hom_K[V_\delta : X(\nu)] \tag{3.3.1}$$

Using Frobenius reciprocity this can be interpreted as a map

$$a_w(\delta,\nu) : (V_\delta^*)^M \longrightarrow (V_\delta^*)^M. \tag{3.3.2}$$

This is hermitian and its signature determines whether $\overline{X}(\nu)$ is unitary or not.

Now $(V_\delta^*)^M$ has a structure of W module. Under some conditions the W-modules structure determines $a_w(\delta,\nu)$ completely. I will call these K-types *spherically relevant.*

Example. Consider again $G = SL(2,\mathbb{R})$. The K-types are the functions f_{2n} The action of $W = S_2$ is trivial for n even and sgn for n odd. The $a(\delta,w)$ is a scalar. With the normalization that we use,

$$\frac{1-2\nu}{1+2\nu}\frac{3-2\nu}{3+2\nu}\cdots\frac{2n-1-2\nu}{2n-1+2\nu}. \tag{3.3.3}$$

Only $n = 0, \pm 1$ are called *spherically relevant.* It is easy to see in this case that a module is unitary if and only if the form is definite on the spherically relevant types.

The general condition is as follows. For each root α of $\mathfrak{a}$ on $\mathfrak{n}$, there is a map

$$\phi_\alpha : SL(2,\mathbb{R}) \longrightarrow G_\alpha \tag{3.3.4}$$

where G_α is the span of the root vectors for $\pm\alpha$. This map is such that the diagonal group of $SL(2)$ maps into A and it intertwines the usual Cartan involution of $SL(2)$ with the Cartan involution of G. Let $d_\alpha := d\phi_\alpha(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$. The condition of *spherically relevant* is that δ should take values $0, \pm 2$ only on the d_α. For such a K-type

$$a_{s_i,\delta}(\nu) = \begin{cases} 1 & \text{on the } +1 \text{ eigenspace of } s_i, \\ \frac{1-\langle\nu,\alpha\rangle}{1+\langle\nu,\alpha\rangle} & \text{on the -1 eigenspace of } s_i. \end{cases} \tag{3.3.5}$$

Note that $\phi_\alpha(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$ is the reflection about the root α in the Weyl group of A.

3.4. **Main result.** One formulation is as follows.

Theorem. *Suppose $\mathfrak{g}$ is classical. Then*

1. *$(V_\delta^*)^M$ is irreducible as a W-module for δ relevant.*
2. *$\overline{X}(\nu)$ is unitary if and only if the form is definite on the relevant K-types.*

Because of part one, we say an irreducible representation of W is relevant if it is V_δ^M of a relevant K-type.

Formula (3.3.4) is more general. Given root data $(X, R, Y, \check{R},)$ one can define the *graded affine Hecke algebra* first introduced by Lusztig and Drinfeld independently in a different context. Let $\mathfrak{a}$ be the complexification of the underlying real vector space corresponding to the root system. Let $\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}$ where $\mathbb{A} := S(\mathfrak{a})$. The *graded affine Hecke algebra* is the algbra with underlying vector space $\mathbb{H}$ with the usual product on on $\mathbb{C}[W] \otimes 1$ and $1 \otimes \mathbb{A}$ and relations

$$t_\alpha \omega = s_\alpha(\omega) t_\alpha + \langle \omega, \check{\alpha} \rangle, \qquad \alpha \text{ a simple root.} \tag{3.4.1}$$

Reductive linear groups are parametrized by root data. For a split group G one takes the dual root system.

The analog of $X(\nu)$ is $\mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu$ with action on the left. The intertwining operators a_{s_i} corresponding to simple roots are given by multiplication on the right by the r_i of the form

$$r_i := (1 - t_i \alpha_i) \frac{1}{1 - \alpha_i}. \tag{3.4.2}$$

One can show that the r_i satisfy the braid relations. Thus if $w \in$ has a reduced decomposition $w = s_1 \circ \cdots \circ s_k$, we can define $a_w = a_{s_1} \dots a_{s_k}$. Then $X(\nu) \cong \mathbb{C}[W]$ as a left W-module so it decomposes as

$$\sum V_\delta \otimes V_\delta. \tag{3.4.3}$$

The a_w simply act on the right factor (the α are evaluated on $\mathbb{C}_\nu$ and become scalars). Formula (3.3.4) holds for all V_δ.

Example. Consider the case of $So(2,1)$. The dual root system is type C_2. Let $\alpha_1 = (1,-1)$ and $\alpha_2 = (0,1)$ be the simple roots. Write $\omega_1 = (1,0)$ and $\omega_2 = (0,1)$. The relations in the Hecke algebra are

$$\begin{aligned} t_1\omega_1 &= \omega_2 t_1 + 1, \qquad t_1\omega_2 = \omega_1 - 1, \\ t_2\omega_1 &= \omega_1 t_2, \qquad t_2\omega_2 = -\omega_2 t_2 + 1. \end{aligned} \tag{3.4.4}$$

The long element is $w_0 = s_1s_2s_1s_2$. Suppose $\mathbb{C}_\nu$ is such that ω_1 acts by ν_1 and ω_2 acts by ν_2. Then multiplication on the right by r_{w_0} in $X(\nu)$ is given by the formula

$$[1 + (\nu_1 - \nu_2)t_1][1 + 2\nu_1 t_2][1 + (\nu_1 + \nu_2)t_1][1 + 2\nu_2 t_2] \tag{3.4.5}$$

divided by $(1+\nu_1-\nu_2)(1+2\nu_1)(1+\nu_1+\nu_2)(1+2\nu_2)$. There are two spherically relevant W-modules. One is the reflection representation the other is 1-dimensional. In the reflection representation the action can be written as

$$t_1 \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad t_2 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.4.6}$$

The fact that this gives a symmetric matrix is tied up with the fact that $s_1s_2s_1s_2 = s_2s_1s_2s_1$. The matrix is symmetric only in a basis where the t_i are orthogonal.

In the other representation,

$$t_1 \leftrightarrow 1, \qquad t_2 \leftrightarrow -1. \tag{3.4.7}$$

Remark. The affine graded Hecke algebra captures the representation theory of Iwahori spherical modules for the corresponding split p-adic group. In particular this approach allows one to treat both the p-adic and the real group at the same time.

4. Description of the spherical dual

The parametrization of the spherical dual is done in terms of the dual Lie algebra. A connected linear reductive group G is determined by a root datum. This is an quadruplet $(X, R, Y, \check{R})$ where,

- **A:** X and Y are lattices in a vector space V endowed with an inner product so that X and Y are in perfect duality,
- **B:** R is a (reduced) root system and $\check{R}$ are the dual roots.

The dual group $\check{G}$ is the group corresponding to $(Y, \check{R}, X, R)$. A maximal torus $H \subset G$, gives rise to a set (X_*, X^*) where X_* is the lattice of 1-parameter subgroups and X^* is the lattice of characters. Then there is a maximal torus $\check{H}$ so that $X_*(\check{H}) = X^*$ and $X^*(\check{H}) = X_*$. In particular the Lie algebra $\check{\mathfrak{h}}$ of $\check{H}$ coincides with $\mathfrak{h}^*$. Suppose we are in the situation of $G = G(\mathbb{R})$ a split form. Then $A \cong (\mathbb{R}^*)^n$ and a character ν is given by an $(\nu_1, \dots, \nu_n) \in \mathbb{C}^n \cong \mathfrak{a}_c^*$. This is then identified with an element in $\check{\mathfrak{a}}_c \subset \check{\mathfrak{g}}$. The set of cuspidal unitary spherical representations turns out to be in 1-1 correspondence with nilpotent orbits $\check{\mathcal{O}}$ which are cuspidal in the sense that they do not meet any proper Levi component of $\check{\mathfrak{g}}$. The parameter $\nu(\check{\mathcal{O}})$ is obtained as follows. By the Jacobson-Morozov theorem, orbits $\check{\mathcal{O}}$ are in 1-1 correspondence with Lie triples; these are $\check{G}$-conjugacy classes of maps

$$\psi_{\check{\mathcal{O}}} : sl(2, \mathbb{C}) \longrightarrow \check{\mathfrak{g}}. \tag{4.0.8}$$

Then let

$$\check{e} = \psi(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), \quad \check{h} = \psi(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}), \quad \check{f} = \psi(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}). \tag{4.0.9}$$

Then $\nu(\check{\mathcal{O}}) = \frac{1}{2}\check{h}$.

Remark. There are only finitely many nilpotent orbits. In the case of $sl(n)$ they correspond to Jordan canonical forms, similarly for the other classical algebras. For the other simple algebras there are lists.

When the orbit is not cuspidal, $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ gets a complementary series attached to it. The centralizer $\mathfrak{z}(\check{\mathcal{O}})$ of the image of $\psi_{\check{\mathcal{O}}}$ is

$$\prod sp(n_i, \mathbb{C}) \times \prod so(n_j, \mathbb{C}).$$

The complementary series corresponding to $\check{\mathcal{O}}$ is simply the one corresponding to C_G and the trivial nilpotent. For these we have the following.

Theorem. *The complementary series attached to $\check{\mathcal{O}}$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$. These are:*

B: $0 \le \nu_1, \dots, \nu_1 < \dots < \nu_k, \dots, \nu_k < 1/2$.

C, D: $0 \le \nu_1, \dots, \nu_1 < \dots < \nu_k, \dots, \nu_k \le 1/2 < \nu_{k+1} < \dots < \nu_{k+l} < 1$ *so that $\nu_i + \nu_j \ne 1$ for $i \ne j$ and there are an even number of ν_i such that $1 - \nu_{k+1} < \nu_i \le 1/2$ and an odd number of ν_i such that $1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$.*

Example. Consider the case of $G = Sp(2, \mathbb{R})$. The dual algebra is $\check{\mathfrak{g}} = so(5, \mathbb{C})$. The nilpotent orbits are parametrized by partitions of 5, only (5), (311), (221) and (1111) occur. We get

$\check{\mathcal{O}}$	$\nu(\check{\mathcal{O}})$	$\mathfrak{z}(\check{\mathcal{O}})$	Complementary Series
(5)	$(2,1)$	$-$	cuspidal
(311)	$(1,0)$	$so(1)\times so(2)$	none
(221)	$(1/2,-1/2)$	$so(1)\times sp(2)$	$-1/2<\nu<1/2$
(1^5)	$(0,0)$	$so(5)$	$0<\nu_2\leq\nu_1<1/2.$

References

[B1] D. Barbasch *The unitary dual for complex classical Lie groups*, Invent. Math., vol 96, 1989, pp 103-176

[B2] *Unipotent representations for real reductive groups*, Proceedings of ICM 1990, Springer Verlag, Tokyo, 1991, pp. 769-777

[B3] *Orbital integrals of nilpotent orbits*, article in volume in honor of Harish-Chandra, to appear

[BM1] *A reduction theorem for the unitary dual of a p-adic group*, Inv. Math, 1989

[BM2] *The unitary spherical dual for split classical groups*, preprint

[BV1] D. Barbasch, D. Vogan *The local structure of characters*, J. Funct. Analysis, vol. 34 no. 1, 1980, pp. 27-55

[BV2] D. Barbasch, D. Vogan *Unipotent representations of complex semisimple Lie groups*, Ann. of Math., 1985, vol 121, pp. 41-110

[BV3] D. Barbasch, D. Vogan *Weyl group representations and nilpotent orbits*, Representation theory of reductive groups, Birkhäuser-Boston, Progress in mathematics vol 40, 1983, pp. 21-35

[KV] A. Knapp, D. Vogan *Cohomological induction and unitary representations*, Princeton University Press, Princeton NJ, 1995

[L] G. Lusztig, *Characters of reductive groups over a finite field* , Annals of Math. Studies, vol. 107, 1984, Princeton University Press

[MW] C. Moeglin, J.-L. Waldspurger *Modèles de Whittaker dégénéres pour les groupes p-adiques*, Math. Z., vol. 196, 1987, pp. 427-452

[SV1] W. Schmid, K. Vilonen *Two geometric character formulas for reductive Lie groups* , J. Amer. math. Soc., vol. 11 no. 4, 1998, pp. 799-867

[SV2] W. Schmid, K. Vilonen *The B-V conjectures*, preprint

[V] D. Vogan *Associated varieties and unipotent representations*, Harmonic Analysis on reductive groups, Progress in Mathematics vol. 101, Birkhäuser, Boston-Basel-Berlin, 1991, pp. 315-388