UNIPOTENT REPRESENTATIONS AND UNITARITY

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1. Introduction

According to conjectures of Arthur, the residual spectrum of $L^2(G/\Gamma)$ should be parametrized by maps

$$\mathcal{W}_{\mathbb{A}} \times SL(2,\mathbb{C}) \longrightarrow {}^L G, \tag{1.1}$$

where ${}^L G$ is the dual group. For a local field $\mathbb{F}$ this implies that homomorphisms

$$\mathcal{W}_{\mathbb{F}} \times SL(2,\mathbb{C}) \longrightarrow {}^L G, \tag{1.2}$$

should parametrize interesting unitary representations. In particular for $\mathbb{F} = \mathbb{R}$,

$$\mathcal{W} \cong \mathbb{C}^{\times} \times \{1, \tau\}, \qquad \tau z \tau^{-1} = \overline{z},\ \tau^2 = 1. \tag{1.3}$$

Definition 1.0.1. *An Arthur parameter is called special unipotent if in the case $\mathbb{F}$ real, $\psi|_{\mathbb{C}^\times} = triv$ and in case $\mathbb{F}$ p-adic, $\psi|_{\mathcal{W}_I} = triv$.*

Recall that in the p-adic case $\mathcal{W}_{\mathbb{F}}$ contains another copy of an $SL(2)$. In [ABV] a definition of packets is given which attaches a family of representations to such a parameter which has the properties conjectured by Arthur.

2. Unipotent Representations

For $G(\mathbb{C})$ viewed as a real group we can define the following set of parameters. Let

$$\psi : SL(2,\mathbb{C}) \longrightarrow \check{G} \tag{2.1}$$

correspond to an even nilpotent orbit $\check{\mathcal{O}}$. Write

$$\check{e} := \psi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \check{h} := \psi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \check{f} := \psi \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.2}$$

Let $C(\check{e})$ be the centralizer of $\check{e}$ and $A(\check{e})$ be the component group. To each (semisimple) $s \in [A(\check{e})]$ we can attach an infinitesimal character $\lambda_s \in G_{\mathfrak{g}}(\check{e})$ with the following properties.

1. $||\lambda_s||$ is minimal subject to $e^{2i\pi\lambda_s}$ maps to s. In particular, $\lambda_e = \frac{1}{2}\check{h}$.
2. There is a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ with the following properties.

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a: Let I_s be the maximal primitive ideal with infinitesimal character λ_s. Then

$$\mathcal{U}(\lambda_s) := \{\pi \mid Ann\pi = I_s\} \longleftrightarrow \widehat{A(\mathcal{O})}.$$

so that *"nice"* character formulas hold (analogous to [BV]).

b: $\mathcal{U}(\mathcal{O})$ is formed of unitary representations. This is true for classical groups by [B89].

c: $\pi_\chi|_K \cong R(\mathcal{O}, \mathcal{S}_\chi)$. For classical groups, this was also conjectured independently by McGovern. For the case of λ_e it is proved in [BO-00]

3. Spherical Representations

Consider the case when $\psi|_{\mathcal{W}} = Triv$. There is a distinguished representation attached to this, namely the spherical module $X(\check{\mathcal{O}})$ with parameter $\frac{1}{2}\check{h}$ for the split rational form.

Problems.

1. Show that $X(\check{\mathcal{O}})$ is unitary.
2. Describe the spherical unitary dual in terms of the $X(\check{\mathcal{O}})$.

For classical groups (any local field), (1) follows from [B90], and [BM94]. (2) is done in [B90], [BM94] and [B00]. The answer is consistent with the *"Lefschetz principle"* that results should be the same independent of the local field.

Fix $B = AN$ a Borel subgroup in the split group G. To each character $\nu \in \widehat{A}$, we can attach the principal series

$$X(\nu) := Ind_B^G[\nu\delta_B^{-1/2}]_{K-finite}. \tag{3.1}$$

We denote by $\overline{X}(\nu)$ the irreducible spherical subquotient. If $w \in N(A)/C(A)$ is an element of the Weyl group, then there is an intertwining operator

$$A_w(\nu) : X(\nu) \longrightarrow X(w\nu). \tag{3.2}$$

If $Re\nu$ is dominant, then $X(\nu)$ has a unique irreduvible quotient namely $\overline{X}(\nu)$. If $Re\nu$ is anti-dominant, then $\overline{X}(\nu)$ is the unique irreducible submodule of $X(\nu)$. So if w_0 is the long Weyl group element, then $\overline{X}(\nu)$ is the image of $A_{w_0}(\nu)$.

Properties.

1. if $w = s_1 \cdot \dots \cdot s_k$ is a reduced decomposition, then $A_w = A_{s_1} \circ \dots \circ A_{s_k}$.
2. $\overline{X}(\nu)$ is hermitian if and only if there is $w \in W$ such that $w\nu = \overline{\nu}$.
3. $X(\nu)^h \cong X(-\overline{\nu})$ so if (2) holds, the hermitian form on $\overline{X}(\nu)$ is

$$\langle v, w\rangle = (v, A_w(\nu)w). \tag{3.3}$$

Implicit in this statement is the fact that the radical is the largest proper submodule of $X(\nu)$.

Since it is an intertwining operator, $A_w(\nu)$ it induces a map

$$A_w(\nu) : Hom_K[V_\nu : X(\nu)] \longrightarrow Hom_K[V_\nu : X(\nu)] \tag{3.4}$$

for any K-type $\mu \in \widehat{K}$. Let $M := C(A) \cap K$. Using Frobenius reciprocity this can be interpreted as a map

$$a_w(\nu) : (V_\mu^*)^M \longrightarrow (V_\mu^*)^M. \tag{3.5}$$

This is hermitian and its signature determines whether $\overline{X}(\nu)$ is unitary or not.

Now $(V_\mu^*)^M$ has a structure of W module. There are some K-types which I will call *spherically relevant* where $a_{w,\mu}(\nu)$ is completely determined by this W-action. The condition is roughly $|\langle\nu, \check{\alpha}\rangle| \leq 1$ for any root. Indeed for such a K-type

$$a_{w,\nu}(\nu) = \begin{cases} -\langle\nu, \alpha\rangle - 1 & \text{on the } +1 \text{ eigenspace of } s_i, \\ \langle\nu, \alpha\rangle - 1 & \text{on the -1 eigenspace of } s_i. \end{cases} \tag{3.6}$$

Theorem 3.0.2. *(real or p-adic) Suppose $\mathfrak{g}$ is classical. Then*

1. *$(V_\mu^*)^M$ is irreducible as a W-module for μ relevant.*
2. *$\overline{X}(\nu)$ is unitary if and only if the form is definite on the relevant K-types.*

The point of this discussion is that to detect nonunitarity it is enough to treat the p-adic case. Then the results are even stronger. The calculation is entirely in the group algebra $\mathbb{C}[W]$. According to [BM89] and [BM93] we need only consider the case ν real and replace $X(\nu)$ by $\mathcal{H} \otimes_{\mathcal{A}} \mathbb{C}_\nu$ where $\mathcal{H}$ is the affine graded Hecke algebra, $\mathcal{A} \subset \mathcal{H}$ is isomorphic to $S(\mathfrak{a})$ with $\mathfrak{a} = Lie(A)$. Then $X(\nu)$ is isomorphic to $\mathbb{C}[W]$ with basis $\{t_x\}_{x \in W}$ and action of W on the left. The intertwining operator A_i corresponding to a simple root is given by multiplication on the right by $r_i := -t_i\langle\nu, \alpha_i\rangle - 1$. We can easily verify that

$$(1 + t_i)r_i = (-\langle\nu, \alpha_i\rangle - 1)(1 + t_i) \tag{3.7}$$
$$(1 - t_i)r_i = (\langle\nu, \alpha_i\rangle - 1)(1 - t_i). \tag{3.8}$$

So using the fact that $\mathbb{C}[W] \cong \sum V_\sigma \otimes V_\sigma$, we find that A_i is of the form $\mathbb{1} \otimes a_i$ and a_i coincides with formula (3.6) derived for the real case. The more general operators A_w are defined by taking products of the a_i according to reduced decompositions. The fact that the result is independent of the choices implies that $a_{\sigma,w_0}(\nu)$ is hermitian. Its signature determines the unitarity.

4. Description of the spherical dual

Each $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ gets a complementary series attached to it. To describe it recall that $\check{\mathcal{O}} \leftrightarrow \{\check{e}, \check{h}, \check{f}\}$. The centralizer

$$C_G(\check{e}, \check{h}) \cong \prod Sp(n_i, \mathbb{C}) \times \prod SO(n_j, \mathbb{C}).$$

The complementary series corresponding to $\check{\mathcal{O}}$ is simply the one corresponding to C_G and the trivial nilpotent. For these we have the following.

Theorem 4.0.3. *The complementary series attached to $\check{\mathcal{O}}$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$. These are:*
B: $0 \le \nu_1, \dots, \nu_1 < \dots < \nu_k, \dots, \nu_k < 1/2.$
C, D: $0 \le \nu_1, \dots, \nu_1 < \dots < \nu_k, \dots, \nu_k \le 1/2 < \nu_{k+1} < \dots < \nu_{k+l} < 1$ *so that $\nu_i + \nu_j \neq 1$ for $i \neq j$ and there are an even number of ν_i such that $1 - \nu_{k+1} < \nu_i \le 1/2$ and an odd number of ν_i such that $1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$.*

References

[B1] D. Barbasch *The unitary dual for complex classical Lie groups* Invent. Math., vol 96, 1989, pp 103-176

[B2] *Unipotent representations for real reductive groups* Proceedings of ICM 1990, Springer Verlag, Tokyo, 1991, pp. 769-777

[B3] *Orbital integrals of nilpotent orbits* article in volume in honor of Harish-Chandra, to appear

[BM1] *A reduction theorem for the unitary dual of a p-adic group* Inv. Math, 1989

[BM2] *the unitary spherical dual for split p-adic groups* ???

[BV1] D. Barbasch, D. Vogan *The local structure of characters* J. Funct. Analysis, vol. 34 no. 1, 1980, pp. 27-55

[BV2] D. Barbasch, D. Vogan *Unipotent representations of complex semisimple Lie groups* Ann. of Math., 1985, vol 121, pp. 41-110

[BV3] D. Barbasch, D. Vogan *Weyl group representations and nilpotent orbits* Representation theory of reductive groups, Birkhäuser-Boston, Progress in mathematics vol 40, 1983, pp. 21-35

[KV] A. Knapp, D. Vogan *Cohomological induction and unitary representations* Princeton University Press, Princeton NJ, 1995

[L] G. Lusztig, *Characters of reductive groups over a finite field* Annals of Math. Studies, vol. 107, 1984, Princeton University Press

[MW] C. Moeglin, J.-L. Waldspurger *Modèles de Whittaker dégénéres pour les groupes p-adiques* Math. Z., vol. 196, 1987, pp. 427-452

[SV1] W. Schmid, K. Vilonen *Two geometric character formulas for reductive Lie groups* J. Amer. math. Soc., vol. 11 no. 4, 1998, pp. 799-867

[SV2] W. Schmid, K. Vilonen *The B-V conjectures* preprint

[V] D. Vogan *Associated varieties and unipotent representations* Harmonic Analysis on reductive groups, Progress in Mathematics vol. 101, Birkhäuser, Boston-Basel-Berlin, 1991, pp. 315-388

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