Spherical Unitary dual for quasisplit real groups

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(joint work with Dan Ciubotaru)

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Notation

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- G is the real points of a linear connected reductive group.
- $\mathfrak{g}_0 := Lie(G)$, θ Cartan involution, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}}$, K maximal compact subgroup, $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$,
- $P = MAN$ minimal parabolic subgroup, $M := C_K(A)$.
- $W := N_K(A)/M$ the Weyl group.
- $\lambda \in \widehat{K}$ a $K-$type, then W acts on V_λ^M.

Problem

Compute the representation of W on V_λ^M

More generally if $\chi \in \widehat{M}$, compute the representation of W_χ (the centralizer of χ in W) on $\mathrm{Hom}_M[\chi, V_\lambda]$.

Motivation

(1) For $G = GL(n, \mathbb{C})$, $K = U(n)$ and M is the diagonal torus, and $W = S_n$. Kostka-Foulkes polynomials encode information about V_λ^M.

(2) Spherical unitary dual.

Spherical unitary dual

Let $\chi \in \widehat{MA}$. The spherical principal series is

$$X(\chi) := Ind_P^G(\chi \otimes \delta_P^{-1/2} \otimes \mathbb{1}), \qquad (1)$$

where χ is an unramified character, ($i.e.$ $\chi\,|_M = triv$), and δ_P is the modulus function of P.

- $\mathrm{Hom}_K[Triv : X(\chi)] = 1$, $L(\chi) :=$ the unique irreducible subquotient containing the trivial $K-$type,
- Every spherical irreducible module is an $L(\chi)$ for some χ.
- $L(\chi) \cong L(\chi')$ if and only if there exists $w \in W$ such that $w\chi = \chi'$.
- $L(\chi)$ is hermitian if and only if there is $w \in W$ such that $w\chi = \overline{\chi^{-1}}$.

- For every $w \in W$ there is an intertwining operator $A_w(\chi) : X(\chi) \longrightarrow X(w\chi)$,
- A_w gives rise to

  $$a_w(\chi, \lambda) : \mathrm{Hom}_K[V_\lambda, X(\chi)] \cong V_\lambda^M \longrightarrow \mathrm{Hom}_K[V_\lambda : X(w\chi)] \cong V_\lambda^M,$$

- A_w is normalized so that $a_w(\chi, triv) = id$; this makes A_w analytic for the region for which $\langle Re\chi, \alpha\rangle \geq 0$ for all roots of N.
- In the hermitian case $a_w(\chi, \lambda)$ gives rise to a hermitian form. $L(\chi)$ is unitary if and only if $a_w(\chi, \alpha)$ positive semidefinite for all λ.
- If $w = s_1 \ldots s_k$ is a reduced decomposition,

  $$a_w = a_{s_1} \cdot \cdots \cdot a_{s_k},$$

  and each a_{s_i} is induced from a corresponding operator on a real rank one group.

- A $K-$type will be called **petite**, if $a_w(\chi,\lambda)$ only depends on the Weyl group representation V_λ^M. More precisely we put a condition on the form of the $a_{s_i}(\chi,\lambda)$. For example when a_{s_i} comes from $SL(2,\mathbb{R})$, it has the form

$$a(2m, s_\alpha, \chi) = \begin{cases} Id & \text{if } m = 0, \\ \prod_{0<j\leq m} \frac{2j-1-<\chi,\check{\alpha}>}{2j-1+<\chi,\check{\alpha}>}\, Id & \text{if } m \neq 0. \end{cases}$$

  ($2m$ parametrizes a representation of $SO(2)$) We require that $m = 0, 1$ only. For other real rank one groups there are similar conditions motivated by formulas of Johnson and Wallach.

There are analogous results when we replace χ by an arbitrary character, or $\mathbb{R}$ by a p-adic field.

The p-adic case

G is assumed split, with Borel subgroup $B = AN$, and let $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$ be the field with its ring of integers and maximal ideal. The character χ is assumed unramified for the moment, *i.e.* $\chi\mid_{A\cap K} = triv$. In this case $K = G(\mathcal{R})$. Let ${}^\vee G$ be the complex dual group. Then $\{L(\chi) \text{ unramified}\} \longleftrightarrow \{s \in {}^\vee G \text{ semisimple}\}/{}^\vee G$. The element s decomposes into an elliptic and a hyperbolic part $s = s_e s_h$. The orbit of s will be called the infinitesimal character. We can collect the infinitesimal characters according to the elliptic part, $Unit_{sph}(G) = \bigsqcup Unit_{sph,s_e}(G)$. An interesting result is that in the adjoint case,

$$Unit_{sph,s_e}(G) \cong Unit_{sph,1}(G(s_e)),$$

where $G(s_e)$ is the split group dual to ${}^\vee G(s_e)$. Related/similar results were described by A. Pantano in her talk.

Main Result

Recently, joint with Dan Ciubotaru we have extended the previous result to

- arbitrary χ for split groups of any kind, (using results of Roche)
- blocks (in the sense of Bernstein) of unipotent representations for p-adic groups studied by Lusztig,
- blocks associated to unramified characters of quasisplit groups.

Main Topic of this talk

(again, joint with Dan Ciubotaru)

Let G be quasisplit. Then associated to it there is an (outer) automorphism ${}^\vee\tau$ of ${}^\vee G$. Then form ${}^L G := {}^\vee G \rtimes \{{}^\vee\tau\}$, and let ${}^\vee G\,{}^\vee\tau$ be the connected component of ${}^\vee\mathcal{T}$. In this case,

$$\{L(\chi) \text{ unramified } \} \leftrightarrow \{s \in {}^\vee G^{{}^\vee\mathcal{T}} \text{ semisimple } \}/{}^\vee G.$$

A semisimple element decomposes $s = s_h s_e$ with $s_e \in {}^\vee G^{{}^\vee\mathcal{T}}$. Let $G(s_e)$ be as before. Then there is an inclusion

$$Unit_{sph,s_e}(G) \subset Unit_{sph,1}(G(s_e))$$

In the cases $U(n+1,n)$, $U(n,n)$, and $O(n+2,n)$ this is an equality. For type E_6 the inclusion is into the spherical unitary dual for p-adic F_4.

Sketch of some proofs

I. Need some facts for split groups, namely a set of **relevant** $K-$types which detect unitarity for spherical representations. We use the double cover of the real group.

Type $\mathbf{A_{n-1}}$. Spherical factors of $V_\omega \otimes V_{\omega'}$ (ω, ω' are fundamental representations). The Weyl group representations are (kl) with $k + l = n$.

Type $\mathbf{B_n}$, $\mathbf{D_n}$. Spherical factors of $[V_\omega \otimes V_{\omega'}] \otimes [V_\rho \otimes V_{\rho'}]$. $\omega, \omega', \rho, \rho'$ are fundamental representations.)The Weyl group representations are $(k) \times (l)$ and $(kl) \times (0)$ with $k + l = n$.

Type $\mathbf{C_n}$. Spherical factors of $\wedge^k \mathbb{C}^{2n} \otimes \wedge^k (\mathbb{C}^{2n})^*$. Same Weyl group representations as before.

Type $\mathbf{F_4}$. The maximal compact subgroup is $Sp(2) \times Sp(6)$.

$K-type$	$W-type$
$(0 \mid 0,0,0)$	1_1,
$(0 \mid 1,1,0)$	2_1,
$(4 \mid 0,0,0)$	2_3,
$(1 \mid 2,1,0)$	8_1,
$(1 \mid 1,1,1)$	4_2,
$(2 \mid 2,0,0)$	9_1.

Combined with results of Dan Ciubotaru for F_4, this gives an embedding of the spherical unitary dual of the split F_4 into the spherical unitary dual of the split p-adic F_4.

With A. Pantano we have computed much larger lists, all types.

(II). For each $\sigma \in \widehat{W}$ on the list, we need a $V_{\lambda(\sigma)}$ which is petite, and such that $V^M_{\lambda(\sigma)}$ contains σ. Let $\tau \in Aut(G)$ satisfy

- τ and θ commute,
- G_τ is split of type dual to $G({}^\vee\tau)$.

Then $K \supset K_\tau$ has a Cartan subgroup $H = MT$, with $T \subset K_\tau$ a Cartan subgroup. Let $\mu(\sigma)$ be a relevant $K-$type of G_τ such that $V^{M_\tau}_{\mu(\sigma)}$ contains σ. We look for a relevant V_λ satisfying

- $V_\lambda\mid_{K_\tau}$ contains V_μ, and is petite.
- $V_\lambda\mid_M$ contains $triv$.

λ such that

(1) $\lambda\mid_M = triv$,

(2) $\lambda\mid_T = \mu$

works for the clasical groups.

In E_6 this is not good enough. In one of the cases, a weight of V_λ different from the highest one satisfies property (1) above.

Papers giving more details can be found on the same web site where you found these notes, under preprints.