WHAT IS A UNITARY DUAL

FOURIER SERIES: $f: S^1 \to \mathbb{C}$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta$$

$$f(\theta) = \sum \hat{f}(n)e^{in\theta}$$

L^2- convergence & other types of convergence

FOURIER ANALYSIS: $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$

f can be recovered from $\hat{f}$.

PONTRYAGIN: G loc. cpct topological abelian group

$$\hat{G} = \{\chi : G \to \mathbb{C} \mid \chi(g^{-1}) = \overline{\chi(g)}\}$$

MAIN RESULT: $\hat{\hat{G}} = G$

Can think of these results as

$$L^2(S^1) = \hat{\bigoplus} \chi_n \qquad L^2(\mathbb{R}) = \int_{\hat{G}} \chi_\xi \, d\mu(\xi)$$

DOES NOT WORK FOR nonabelian GROUPS

$X = S^{n-1}$. $\Delta = \sum \partial_{x_i}^2$ acts on $C^\infty(S^{n-1})$

Find "eigenvalues".

- $SO(n)$ acts, S^{n-1} has an $SO(n)$-inv. measure.

Decompose $L^2(S^{n-1} = SO(n)/SO(n-1))$ similar to $L^2(S^1)$.

SO(n) has no nontrivial characters

But any (π, V) that occurs is unitary

i.e. $\exists \langle\, ,\rangle \gg 0$ satisfying $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w\rangle$

FACTS ABOUT COMPACT GROUPS

(π, V) irreducible means if $W \subseteq V$ is an invariant subspace, $W = (0)$ or $W = V$

- Any irred. rep'n on a Hilbert space is finite dimensional
- $(\pi, \mathcal{H})$ a rep'n on a (countable) Hilbert space, $\mathcal{H} = \bigoplus_{\pi \in \hat{G}} m_\pi \pi \qquad m_\pi < \infty$
- Irreducible rep's of groups like $SO(n)$

are well understood.

- (Plancherel Formula) $L^2(G)$ has an action of $G\times G$:
$$L^2(G) = \bigoplus_{\pi\in\hat{G}} \pi \otimes \pi^*$$

π^* is the dual rep'n:
$$V^* = \{\ell : V \to \mathbb{C}\} \quad (\pi^*(g)\ell)(v) = \ell(\pi(g^{-1})v)$$

CRUCIAL FACT: G has a bi-invariant measure d_G such that $\mathrm{vol}(G)<\infty$.

CONSEQUENCE: Any representation is unitarizable

$\langle\, ,\rangle$ any $>>0$ inner product. Define
$$(v,w) = \int_G \langle \pi(g)v, \pi(g)w\rangle\, dg$$

Invariant bilinear form means
$$\exists\ 2 : V \to V^h \qquad 2\circ\pi(g) = \pi(g)^h \circ 2$$
$$V^h = \{\ell : V \to \mathbb{C} \mid \ell(\lambda v) = \bar{\lambda}\ell(v)\}$$
$$\pi(g)^h \ell(v) = \underline{\ell(\pi(g^{-1})v)}$$

Hermitian: $2^h : (V^h)^h \to V^h \quad 2^h|_V = 2$

• $L^2(S^{n-1}) = \bigoplus m_\pi V_\pi, \quad m_\pi = V_\pi^{SO(n-1)}$

NONCOMPACT GROUPS

$SO(p,q), U(p,q), GL(n,\mathbb{R}), GL(n,\mathbb{C}), \ldots$

DECOMPOSE: $L^2(G)$ or $L^2(\mathcal{X})$ where $\mathcal{X}$ is a space with a G action & a G-invariant measure.

Example: $G = SL(2,\mathbb{R})$ $\Gamma \subseteq G$ discrete subgroup satisfying $\mathrm{vol}(\Gamma\backslash G) < \infty$

• If $\Gamma\backslash G$ is cpct $L^2(\Gamma\backslash G) \simeq \bigoplus m_\pi V_\pi, m_\pi < \infty$

Otherwise a mix of $L^2(S^1)$ & $L^2(\mathbb{R})$

GENERAL STRATEGY

(I) Classify Irreducible Modules

(II) Classify hermitian ones

(III) Find the Unitary Ones

REMARK: An irreducible (π, V) has at most one invariant form up to a scalar (Schur's Lemma)

REDUCTION TO AN ALGEBRAIC PROBLEM

$\left\{\begin{matrix}\text{unitary irreducible} \\ \text{modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unitary admissible} \\ (\mathfrak{g}_c, K) \text{ modules}\end{matrix}\right\}$

- $K \subseteq G$ maximal compact subgroup

G	K
$GL(n,\mathbb{R})$	$O(n)$
$GL(n,\mathbb{C})$	$U(n)$
$SO(p,q)$	$S[O(p)\times O(q)]$

G	K
$SU(p,q)$	$S[U(p)\times U(q)]$
etc	

<u>Admissible:</u> $V|_K = \bigoplus_{\mu\in\hat{K}} m_\mu V_\mu \quad m_\mu < \infty$

Analogous to taking $C^\infty(X) \subseteq L^2(X)$
Lie algebra acts by differentiation

SOLVE THE ALGEBRAIC PROBLEM

There are algorithms for computing signatures but very complicated.

<u>Spherical Case</u> $\quad V^K \neq (0)$

- Most complete answer known
- Answer is a finite union of complexes "fairly simple"
- same answer for p-adic groups

SPHERICAL DUAL FOR $(\mathfrak{sl}(2,\mathbb{R}), SO(2))$

Parameter Space: $\nu \in \mathbb{C}$

Hermitian: $\bar{\nu} = \nu$ or $\bar{\nu} = -\nu$

$\bar{\nu} = -\nu$ unitary

$\bar{\nu} = \nu$: Standard module

$$W = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

$$[W, X] = 2X, \quad [W, Y] = -2Y, \quad [X, Y] = W$$

$X(\nu)$ has a basis $\{v_n\}$ n even so that

$$W \cdot v_n = n v_n, \quad X \cdot v_n = \frac{1}{2}(\nu + n + 1) v_{n+2}, \quad Y v_n = \frac{1}{2}(\nu - n + 1) v_{n-2}$$

- Hermitian: $\langle x \cdot v, w \rangle + \langle v, \bar{x} w \rangle = 0$

$\bar{W} = -W$, $\bar{X} = Y$

assume $\langle v_0, v_0 \rangle = 1$ $\left(\langle v_n, v_m \rangle = 0 \text{ if } n \neq m \right)$

$$\langle v_n, v_n \rangle = \left[\frac{1}{2}(\nu + n - 1)\right]^{-2} \langle X v_{n-2}, X v_{n-2} \rangle$$

$$= -\left[\frac{1}{2}(\nu + n - 1)\right]^{-2} \langle YX v_{n-2}, v_{n-2} \rangle =$$

$$-\frac{\frac{1}{2}(\nu - n + 1)}{\frac{1}{2}(\nu + n - 1)} \langle v_{n-2}, v_{n-2} \rangle = \frac{n - 1 - \nu}{n - 1 + \nu} \langle v_{n-2}, v_{n-2} \rangle$$

$$\langle v_0, v_0\rangle = 1, \quad \langle v_{2m}, v_{2m}\rangle = \prod_{1\le i\le m} \frac{2i-1-\nu}{2i-1+\nu}$$

UNITARY FOR $0 \le \nu \le 1$

NOT UNITARY FOR $1 < \nu$

$$\langle v_0, v_0\rangle = 1 \qquad \langle v_2, v_2\rangle = \frac{1-\nu}{1+\nu}$$

$\nu < 0$ gives the same rep's